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UNICITY OF SOLUTIONS OF GENERALIZED DIFFERENTIAL EQUATIONS

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This paper contains two unicity theorems concerning generalized differential equations introduced in [1]. Theorem 1 gives a new criterion for the unicity of solutions of classical differential equations at the same time.

0. We shall use the notations of [1]. Let us say that the function $\psi(\eta)$, $0 \leqq \eta \leqq \sigma$ fulfils the condition (A) if $\psi(\eta) \geqq 0$, $\eta^{-1}\,\psi(\eta)$ is non-decreasing and $\sum_{i=1}^{\infty} 2^i\,\psi\left(\frac{\sigma}{2^i}\right) < \infty$. In this case we put $\Psi(\eta) = \sum_{i=1}^{\infty} \psi\left(\frac{\eta}{2^i}\right) \cdot \frac{2^i}{\eta}$.

1. Theorem 1. *Let $F(x, t) \in F(G, \omega_2, \omega_2, \sigma)$ and let $\psi(\eta) = \omega_2^2(\eta)$ fulfil the condition* (A)*. Let us further suppose that $x(\tau) = c$ for $\tau_1 \leqq \tau \leqq \tau_2$ is a solution of*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{1}$$

and let $\tau_0 \in \langle \tau_1, \tau_2 \rangle$. Then $x(\tau)$ is the unique regular solution of (1) *satisfying $x(\tau_0) = c$.*

Theorem 1 has the following precise meaning: Let $y(\tau)$, $\tau \in \langle \tau_3, \tau_4 \rangle$ be a regular solution of (1), $y(\tau_0) = c$ for a $\tau_0 \in \langle \tau_1, \tau_2 \rangle \cap \langle \tau_3, \tau_4 \rangle$. Then $y(\tau) = c$ for $\tau \in \langle \tau_1, \tau_2 \rangle \cap \langle \tau_3, \tau_4 \rangle$.

Note 1. The solution $y(\tau)$ is called regular, if there exists such a number $\sigma_1 > 0$ that $\|y(\tau_5) - y(\tau_6)\| \leqq 2\omega_2(|\tau_5 - \tau_6|)$ for $|\tau_5 - \tau_6| \leqq \sigma_1$, $\tau_5, \tau_6 \in \langle \tau_3, \tau_4 \rangle$. Definition 4, 2, 1 of [1] is obviously equivalent.

Note 2. Let $\omega_3(\eta)$ be a continuous increasing function, $\omega_3(0) = 0$ and let $\psi_3(\eta) = \omega_3(\eta)\,\omega_2(\eta)$ fulfil the condition (A). Let the solution $y(\tau)$ of (1) fulfil the inequality $\|y(\tau_5) - y(\tau_6)\| \leqq 2\omega_3(|\tau_5 - \tau_6|)$ for $|\tau_5 - \tau_6| \leqq \sigma_2$, $\tau_5, \tau_6 \in \langle \tau_3, \tau_4 \rangle$ where σ_2 is positive. It follows from Lemma 4,1,1, [1] that $y(\tau)$ is a regular solution.

Note 3. The unicity of solutions of generalized linear differential equations ([1], section 5,1) is a consequence of Theorem 1.

Proof of Theorem 1. Let the interval $\langle \tau_1, \tau_2 \rangle \cap \langle \tau_3, \tau_4 \rangle$ be non-degenerate. Let us find such a positive number $\sigma_3 < \frac{1}{2}\sigma$ that

$$\omega_2(\sigma_3) < \frac{1}{4}, \quad \eta \Psi(\eta) \leqq \frac{1}{4n} \omega_2(\eta) \quad \text{for} \quad 0 \leqq \eta \leqq 2\sigma_3 \,{}^{1)} \tag{2}$$

and that $(z, \tau) \in G$ for $\tau \in \langle \tau_3, \tau_4 \rangle$, $\|z - y(\tau)\| \leqq 2\omega_2(\sigma_3)$.

Let us suppose that

$$\|y(\tau_6) - y(\tau_5)\| \leqq \frac{1}{2^{k-1}} 2\omega_2(|\tau_6 - \tau_5|) \tag{3}$$

holds for $\tau_5, \tau_6 \in \langle \tau_0 - \sigma_3, \tau_0 + \sigma_3 \rangle \cap \langle \tau_1, \tau_2 \rangle \cap \langle \tau_3, \tau_4 \rangle = \langle \tau_7, \tau_8 \rangle$, k natural. Then

$$\|F(y(\tau + \eta), \tau + \eta) - F(y(\tau + \eta), \tau) - F(y(\tau), \tau + \eta) + F(y(\tau), \tau)\| \leqq \frac{1}{2^{k-1}} 2\omega_2^2(\eta)$$

for τ, $\tau + \eta \in \langle \tau_7, \tau_8 \rangle$ according to (3) and according to the properties of $F(x, t)$.

It follows from Theorem 3,1 of [1] (we use this Theorem for each component separately) that

$$y(\tau_6) - y(\tau_5) = \int_{\tau_5}^{\tau_6} \mathrm{D}F(y(\tau), t) = F(y(\tau_5), \tau_6) - F(y(\tau_5), \tau_5) + R \tag{4}$$

where $\|R\| \leqq n \frac{1}{2^{k-1}} 2|\tau_6 - \tau_5| \Psi(|\tau_6 - \tau_5|)$, $\tau_5, \tau_6 \in \langle \tau_7, \tau_8 \rangle$. As $x(\tau) = c$ is a solution of (1) it follows that

$$0 = c - c = \int_{\tau_5}^{\tau_6} \mathrm{D}F(c, t) = F(c, \tau_6) - F(c, \tau_5)$$

and

$$\begin{aligned} &\|F(y(\tau_5), \tau_6) - F(y(\tau_5), \tau_5)\| = \\ &= \|F(y(\tau_5), \tau_6) - F(y(\tau_5), \tau_5) - F(c, \tau_6) + F(c, \tau_5)\| \leqq \\ &\leqq \|y(\tau_5) - c\| \, \omega_2(|\tau_6 - \tau_5|) = \|y(\tau_5) - y(\tau_0)\| \, \omega_2(|\tau_6 - \tau_5) \leqq \\ &\leqq \frac{1}{2^{k-1}} 2\omega_2(\sigma_3) \, \omega_2(|\tau_6 - \tau_5|) \leqq \frac{1}{2^k} \omega_2(|\tau_6 - \tau_5|) \end{aligned} \tag{5}$$

according to (3) and (2). From (4), (5) and (2) we obtain

$$\|y(\tau_6) - y(\tau_5)\| \leqq \frac{1}{2^k} \omega_2(|\tau_6 - \tau_5|) + \frac{1}{2^{k-1}} 2n|\tau_6 - \tau_5| \, \Psi_2(|\tau_6 - \tau_5|) \leqq$$
$$\leqq \frac{1}{2^k} 2\omega_2(|\tau_6 - \tau_5|) \, .$$

[1]) It is supposed that $\omega_2(\eta) \geqq c\eta$, $c > 0$ (cf. [1], section 4); n is the dimension of the space.

As $y(\tau)$ is a regular solution, (3) holds for $k = 1$ and consequently for every natural k. It follows that $y(\tau) = c$ for $\tau \in \langle \tau_7, \tau_8 \rangle$ and the proof may be finished without difficulties.

Example. Let us examine the (classical) differential equation

$$\frac{dx}{dt} = f(x, t) \tag{6}$$

where $x \in E_1$ and $f(x, t)$ is defined in the following manner:

Let us choose a number β, $\frac{1}{2} < \beta < 1$ and a sequence of numbers φ_k, $k = 1, 2, 3, \dots$ We put $f(x, t) = 0$ if $x \geqq 1$ or $x = 0$,

$$f\left(\frac{1}{2^k}, t\right) = \left(\frac{1}{2}\right)^{\beta k} \cos(2^k t + \varphi_k), \quad k = 1, 2, 3, \dots$$

If

$$\frac{1}{2^k} < x < \frac{1}{2^{k-1}}, \quad x = \lambda \frac{1}{2^k} + \mu \frac{1}{2^{k-1}}, \; \lambda \geqq 0, \quad \mu \geqq 0, \quad \lambda + \mu = 1,$$

we put

$$f(x, t) = \lambda f\left(\frac{1}{2^k}, t\right) + \mu f\left(\frac{1}{2^{k-1}}, t\right)$$

and if $x < 0$, we define $f(x, t) = f(-x, t)$.

$f(x, t)$ is obviously continuous. By means of Theorem 1 we shall prove that $x(t) = 0$ is the unique solution of (6) satisfying $x(t_0) = 0$. According to the results of section 2, [1] it will do to prove that $x(\tau) = 0$ is the only solution of

$$\frac{dx}{d\tau} = DF(x, t) \tag{7}$$

(where $F(x,t) = \int_0^t f(x,\tau)\, d\tau$) satisfying $x(t_0) = 0$.

We shall prove that $F(x, t) \in F(E_2, \eta, 16\eta^\beta, 1)$. As $|f(x, t)| \leqq 1$ we have $|F(x, t + \eta) - F(x, t)| \leqq \eta$. Let us write $F(x, t) = \sum_{-\infty}^{\infty} F_k(x, t)$ where $F_0(x, t) = 0$,

$$F_k\left(\frac{1}{2^k}, t\right) = \left(\frac{1}{2}\right)^{(\beta+1)k} [\sin(2^k t + \varphi_k) - \sin \varphi_k], \; F_k(x, t) = 0$$

if $x \geqq \frac{1}{2^{k-1}}$ or $x \leqq \frac{1}{2^{k+1}}$, $F_k(x, t)$ is linear in x on the intervals $\left\langle \frac{1}{2^{k+1}}, \frac{1}{2^k} \right\rangle$, $\left\langle \frac{1}{2^k}, \frac{1}{2^{k-1}} \right\rangle$,

$$F_{-k}(x, t) = F_k(-x, t) \quad \text{for} \quad k = 1, 2, 3, \dots.$$

Let us prove that

$$|F_k(x_2, t_2) - F_k(x_2, t_1) - F_k(x_1, t_2) + F_k(x_1, t_1)| \leqq 4|x_2 - x_1| \cdot |t_2 - t_1|^\beta . \quad (8)$$

Let us suppose that $k > 0$ and that $x_1 = \frac{1}{2^{k+1}}$, $x_2 = \frac{1}{2^k}$. If $|t_2 - t_1| \leqq \frac{1}{2^k}$, then

$$|F_k(x_2, t_2) - F_k(x_2, t_1) - F_k(x_1, t_2) + F_k(x_1, t_1)| =$$

$$= |F_k(x_2, t_2) - F_k(x_2, t_1)| = (\tfrac{1}{2})^{(\beta+1)k} |\sin (2^k t_2 + \varphi_k) - \sin (2^k t_1 + \varphi_k)| \leqq$$

$$\leqq \left(\frac{1}{2}\right)^{\beta k} |t_2 - t_1| \leqq \left(\frac{1}{2}\right)^{\beta k} \left(\frac{1}{2^k}\right)^{1-\beta} |t_2 - t_1|^\beta = 2|x_2 - x_1| \cdot |t_2 - t_1|^\beta ;$$

if $|t_2 - t_1| > \frac{1}{2^k}$ then

$$|F_k(x_2, t_2) - F_k(x_2, t_1)| \leqq 2\left(\frac{1}{2}\right)^{(\beta+1)k} = \frac{4}{2^{k+1}} \cdot \left(\frac{1}{2^k}\right)^\beta < 4|x_2 - x_1| \, |t_2 - t_1|^\beta,$$

so that (8) holds in this case. Hence it follows without difficulties that (8) holds for all x_1, x_2, t_1, t_2 and all k. If we consider that for fixed x_1, x_2 there exist at most four indices k in such a way that $|F_k(x_1, t)| + |F_k(x_2, t)| \neq 0$, we obtain that

$$|F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1)| \leqq 16|x_2 - x_1| \cdot |t_2 - t_1|^\beta$$

for arbitrary x_1, x_2, t_1, t_2.

We have proved that $F(x, t) \in F(E_2, \eta, 16\eta^\beta, 1) \subset F(E_2, 16\eta^\beta, 16\eta^\beta, 1)$. As $\beta > \frac{1}{2}$, $\psi_2(\eta) = 256\eta^{2\beta}$ fulfils the condition (A) and we may use Theorem 1.

2. In this section we shall prove other results concerning the unicity.[2])

Theorem 2. *Let* $F(x, t) \in F(G, \omega_1, \omega_2, \sigma)$, *let* $\psi(\eta) = \omega_1(\eta) \cdot \omega_2(\eta)$ *fulfil the condition* (A) *and let for a positive* λ

$$\lim_{\eta \to 0+} \Psi(\eta) \frac{\exp (\lambda \eta^{-1} \omega_2(\eta))}{\eta^{-1} \omega_2(\eta)} = 0 . \quad (9)$$

If $(x_0, t_0) \in G$, *then there is at most one regular solution* $x(\tau)$ *of*

$$\frac{dx}{d\tau} = DF(x, t) \quad (10)$$

satisfying $x(t_0) = x_0$.

Proof. Let $x(\tau)$, $y(\tau)$ be regular solutions of (10) for $\tau \in \langle t_0, t_0 + \zeta \rangle$ (or $\tau \in \langle t_0 - \zeta, t_0 \rangle$) satisfying $x(t_0) = y(t_0) = x_0$, where $0 < \zeta < \min (\sigma, \lambda)$ and

[2]) Corollary 2 and Theorem 3 are due to Jan Mařík.

$(x(\tau), t) \in G$, $(y(\tau), t) \in G$, if $\tau \in \langle t_0, t_0 + \zeta\rangle$ and $t \in \langle t_0, t_0 + \zeta\rangle$. Let k be a natural number, $0 \leqq \xi \leqq \zeta$. Then

$$x\left(t_0 + \frac{l+1}{k}\xi\right) - y\left(t_0 + \frac{l+1}{k}\xi\right) = x\left(t_0 + \frac{l}{k}\xi\right) - y\left(t_0 + \frac{l}{k}\xi\right) +$$

$$+ \int_{t_0+\frac{l}{k}\xi}^{t_0+\frac{l+1}{k}\xi} \mathrm{D}F(x(\tau), t) - \int_{t_0+\frac{l}{k}\xi}^{t_0+\frac{l+1}{k}\xi} \mathrm{D}F(y(\tau), t) = x\left(t_0 + \frac{l}{k}\xi\right) - y\left(t_0 + \frac{l}{k}\xi\right) +$$

$$+ F\left(x\left(t_0 + \frac{l}{k}\xi\right),\ t_0 + \frac{l+1}{k}\xi\right) - F\left(x\left(t_0 + \frac{l}{k}\xi\right),\ t_0 + \frac{l}{k}\xi\right) -$$

$$- F\left(y\left(t_0 + \frac{l}{k}\xi\right),\ t_0 + \frac{l+1}{k}\xi\right) + F\left(y\left(t_0 + \frac{l}{k}\xi\right),\ t_0 + \frac{l}{k}\xi\right) + r_1 - r_2\,,$$

where $\|r_1\|, \|r_2\| \leqq \frac{\zeta}{k}\Psi\left(\frac{\zeta}{k}\right)$. Hence

$$\left\|x\left(t_0 + \frac{l+1}{k}\xi\right) - y\left(t_0 + \frac{l+1}{k}\xi\right)\right\| \leqq$$

$$\leqq \left\|x\left(t_0 + \frac{l}{k}\xi\right) - y\left(t_0 + \frac{l}{k}\xi\right)\right\| \left(1 + \omega_2\left(\frac{\zeta}{k}\right)\right) + 2\frac{\zeta}{k}\Psi\left(\frac{\zeta}{k}\right),$$

$$\|x(t_0 + \xi) - y(t_0 + \xi)\| \leqq 2\frac{\zeta}{k}\Psi\left(\frac{\zeta}{k}\right)\left(1 + \left(1 + \omega_2\left(\frac{\zeta}{k}\right)\right) + \ldots +\right.$$

$$\left. + \left(1 + \omega_2\left(\frac{\zeta}{k}\right)\right)^{k-1}\right) \leqq 2\frac{\zeta}{k}\Psi\left(\frac{\zeta}{k}\right)\left(1 + \omega_2\left(\frac{\zeta}{k}\right)\right)^k \left[\omega_2\left(\frac{\zeta}{k}\right)\right]^{-1} \leqq \tag{11}$$

$$\leqq 2\Psi\left(\frac{\zeta}{k}\right) \exp\left\{\zeta \,.\, \frac{k}{\zeta}\,\omega_2\left(\frac{\zeta}{k}\right)\right\}\left[\frac{k}{\zeta}\omega_2\left(\frac{\zeta}{k}\right)\right]^{-1}.$$

As $\zeta \leqq \lambda$, the proof of Theorem 2 is finished by passing to the limit for $k \to \infty$ in (11).

Corollary 1. *If $\omega_2(\eta) = K\eta$, $K > 0$ and if $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ satisfies the condition* (A), *then the assumptions of Theorem 2 are fulfilled.*

Corollary 2. *Let K_1, K_2 be positive. Suppose that*

$$0 < \alpha < 1\,, \quad \varepsilon > 0 \tag{12}$$

or

$$\alpha = 1\,, \quad 0 < \varepsilon \leqq 1\,. \tag{13}$$

Let

$$\omega_1(\eta) = K_1 \exp\{-\varepsilon|\log\eta|^\alpha\}\,, \quad \omega_2(\eta) = K_2\eta|\log\eta|^\alpha\,.$$

Then the assumptions of Theorem 2 are fulfilled.

Corollary 2 is a consequence of the following lemmas:

Lemma 1. *If $\psi(\eta)$, $0 \leqq \eta \leqq \sigma$ is non-decreasing, then* $\sum_{i=1}^{\infty} 2^i \psi\left(\frac{\sigma}{2^i}\right) < \infty$ *if and only if* $\int_0^\sigma t^{-2}\psi(t)\,dt < \infty$. *If* $\int_0^\sigma t^{-2}\psi(t)\,dt < \infty$, *then*

$$2\int_0^\eta t^{-2}\psi(t)\,dt \geqq \Psi(\eta) \geqq \int_0^{\eta/2} t^{-2}\psi(t)\,dt\,.$$

Lemma 2. *Let $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ ($\omega_1(\eta)$ and $\omega_2(\eta)$ are defined in Corollary 2). If $0 < \lambda < \frac{\varepsilon}{K_2}$, then*

$$\lim_{\eta\to 0+} \Psi(\eta)\,\frac{\exp\{\lambda\eta^{-1}\omega_2(\eta)\}}{\eta^{-1}\omega_2(\eta)} = 0\,. \tag{14}$$

Proof. Let us put $\varphi(\eta) = (-\log\eta)^\alpha$, $\varepsilon_1 = \lambda K_2$. As

$$\varphi'(\eta) = -\frac{\alpha}{\eta}(-\log\eta)^{\alpha-1} = -\frac{\alpha}{\eta}(\varphi(\eta))^{1-\frac{1}{\alpha}},$$

and as $\varphi(\eta)^{\frac{1}{\alpha}} \leqq \exp\{(\varepsilon - \varepsilon_1)\,\varphi(\eta)\}$ (for η small enough), we obtain

$$\eta^{-2}\psi(\eta) = K_1K_2\eta^{-1}\varphi(\eta)\exp(-\varepsilon\varphi(\eta)) \leqq$$
$$\leqq K_3\eta^{-1}\varphi(\eta)^{1-\frac{1}{\alpha}}\exp(-\varepsilon_1\varphi(\eta)) \leqq -K_4\varphi'(\eta)\exp(-\varepsilon_1\varphi(\eta)) =$$
$$= K_5(\exp\{-\varepsilon_1\varphi(\eta)\})'\,,$$
$$\int_0^\eta t^{-2}\psi(t)\,dt \leqq K_5\exp\{-\varepsilon_1\,\varphi(\eta)\}\,.$$

As $\exp\{\lambda\eta^{-1}\,\omega_2(\eta)\} = \exp\{\varepsilon_1\,\varphi(\eta)\}$, (14) holds according to Lemma 1.

Note 4. Lemma 2 holds, if α and ε are positive. Inequality (12) or (13) ensures that $\omega_1(\eta) \geqq c\eta$ ($c > 0$, $0 \leqq \eta \leqq 1$).

The following theorem shows that (9) cannot hold if

$$\lim_{\eta\to 0+} \omega_2(\eta)[\eta\,|\log\eta|]^{-1} = \infty\,.$$

Theorem 3. *Let $\omega_2(\eta) = \eta|\log\eta|\,\mu(\eta)$, $\mu(\eta) \geqq c_1 > 0$. Let (9) hold for a positive λ. Then $\mu(\eta)$ is bounded.*

Proof. We suppose that $\omega_1(\eta) \geqq c\eta$, $c > 0$. According to Lemma 1

$$\Psi(\eta) \geqq \int_0^{\frac{\eta}{2}} t^{-2}\psi(t)\,dt \geqq cc_1\int_0^{\frac{\eta}{2}}(-\log t)\,dt = cc_1\frac{\eta}{2}\left(1 + \left|\log\frac{\eta}{2}\right|\right) \geqq c_2\eta\,|\log\eta|$$

(for η small enough). From (9) it follows that

$$\eta|\log\eta|\,\frac{\exp\{\lambda|\log\eta|\,\mu(\eta)\}}{|\log\eta|\,\mu(\eta)} = \exp\{\lambda|\log\eta|\,\mu(\eta) + \log\eta - \log\mu(\eta)\}$$

tends to zero with $\eta \to 0+$. Consequently $\mu(\eta)$ is bounded.

LITERATURE

[1] *J. Kurzweil*: Generalized Ordinary Differential Equations and Continuous Dependence on a Parameter, Czech. mat. journal, *7* (*82*) 1957, 418—449.

Резюме

ОДНОЗНАЧНОСТЬ РЕШЕНИЙ ОБОБЩЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

ЯРОСЛАВ КУРЦВЕЙЛЬ (Jaroslav Kurzweil), Прага

(Поступило в редакцию 25/X 1957 г.)

Мы пользуемся определениями и обозначениями, введенными в [1]. Мы говорим, что функция $\psi(\eta)$, $0 \leqq \eta \leqq \sigma$ удовлетворяет условию (A), если $\eta^{-1}\psi(\eta)$ не убывает, $\psi(\eta) \geqq 0$, $\sum_{i=1}^{\infty} 2^i \psi\left(\frac{\sigma}{2^i}\right) < \infty$. В таком случае положим $\Psi(\eta) = \sum_{i=1}^{\infty} \psi\left(\frac{\eta}{2^i}\right) \frac{2^i}{\eta}$. Доказываются следующие главные результаты:

Теорема 1. *Пусть $F(x, t) \in F(G, \omega_1, \omega_2, \sigma)$ и пусть функция $\psi(\eta) = \omega_2^2(\eta)$ удовлетворяет условию* (A). *Пусть $x(\tau) = c$ для $\tau_1 \leqq \tau \leqq \tau_2$ является решением уравнения*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{1}$$

и пусть $\tau_0 \in \langle \tau_1, \tau_2 \rangle$. Тогда $x(\tau)$ является единственным регулярным решением уравнения (1), *выполняющим начальное условие $x(t_0) = x_0$.*

Показано, как эта теорема используется при решении дифференциального уравнения первого порядка (в классическом смысле).

Теорема 2. *Пусть $F(x, t) \in F(G, \omega_1, \omega_2, \sigma)$, пусть функция $\psi(\eta) = \omega_1(\eta)\, \omega_2(\eta)$ удовлетворяет условию* (A) *и пусть для некоторого положительного λ*

$$\lim_{\eta \to 0+} \psi(\eta) \frac{\exp(\lambda \eta^{-1} \omega_2(\eta))}{\eta^{-1}\omega_2(\eta)} = 0\,; \tag{9}$$

тогда каждое решение уравнения

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \tag{10}$$

однозначно определяется начальным условием.

Теорема 2 особенно полезна в следующих двух случаях:

1. $\omega_2(\eta) = c\eta$, $c > 0$, $\psi(\eta) = \omega_1(\eta)\,\omega_2(\eta)$ удовлетворяет условию (A),

2. $\omega_1(\eta) = K_1 \exp\{-\varepsilon\,|\log\ \eta|^\alpha\}$, $\omega_2(\eta) = K_2\eta\,|\log\eta|^\alpha$, где $K_1 > 0$, $K_2 > 0$, а числа ε, α удовлетворяют условиям

$$0 < \alpha < 1\,, \quad \varepsilon > 0 \tag{12}$$

или

$$\alpha = 1\,, \quad 0 < \varepsilon \leqq 1\,. \tag{13}$$

Наконец, доказывается, что (9) не может иметь места, если

$$\omega_2(\eta) = \eta\,|\log\eta|\,\mu(\eta)\,,$$

где $\mu(\eta) \geqq c_1 > 0$, $\limsup\limits_{\eta\to 0+} \mu(\eta) = \infty$.