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ON THE PRINCIPAL FREQUENCY OF A CONVEX MEMBRANE
AND RELATED PROBLEMS

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An upper bound of the principal frequency of a convex membrane is established. Neglecting a factor depending only on purely physical quantities it contains only the area and perimeter of the membrane.

1. In his book *Patterns of Plausible Inference*¹⁾ G. PÓLYA presents many numerical data about membranes of diverse shapes, giving the length L of their perimeters, their areas A and the pitches λ of their principal tones (principal frequency of their vibrations if the boundary is fixed, or the minimum of the quotient

$$\left[\frac{\iint_D (\nabla u)^2 d\sigma}{\iint_D u^2 d\sigma} \right]^{\frac{1}{2}} \quad (1)$$

where u may be any continuous function with piecewise continuous first derivatives vanishing on the boundary B of D , that is on the boundary of the membrane). From these data it is possible to calculate the quantity c defined by the equation $\lambda = c \cdot L/A$ and it is found that in every instance quoted there c is a number between 1 and about 1.3. Other data show that if ε is an arbitrarily little positive quantity c may be as high as $\pi/2 - \varepsilon$. (In the case if a very elongated rectangle.)

Now we will prove the following

Theorem 1. *Let D be a convex plane domain of area A , perimeter L . Then the square root of the least eigenvalue λ^2 of the differential equation $\Delta u + \lambda u = 0$ (Δ is the two dimensional Laplace operator, u vanishing on the boundary of D) or in other words the minimum λ of the quotient (1) (u continuous, its first derivatives piecewise continuous in the interior of D , u vanishing on the boundary) satisfies the inequality $\lambda \leq \sqrt{3L/A}$.*

This statement will be proved if we can find a function u satisfying the conditions of the variational problem, for which the expression (1) is not greater

¹⁾ Princeton University Press 1954, pp. 9 and 11.

than $\sqrt[3]{3L/A}$. It will be seen that such a function is the point function $d(P)$ which is the distance of the point P from that point of the boundary B which is nearest to P :

$$d(P) = \min_{Q \in B} \overline{PQ}.$$

2. Clearly it is enough to prove the statement for the case when the boundary is a polygon.²⁾ The sides of the polygon (and their length) will be denoted by a_1, a_2, \dots, a_n . This polygon can be divided into domains D_1, D_2, \dots, D_n where the interior of D_i consists of the set of those points of D which are nearer to the side a_i than to any point of any other side of the polygon. D_i is included in the triangular domain bounded by the side a_i and the two bisectors of the angles formed by a_i and the two adjacent sides.

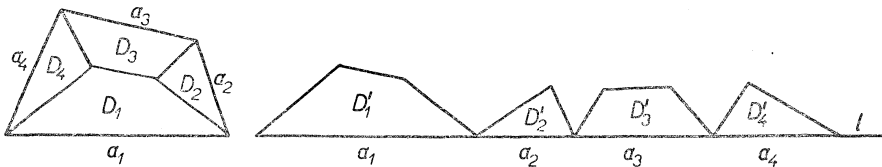


Fig. 1

The function $d(P)$ vanishes on the boundary, it is continuous and its first derivatives are continuous in the interior of the domains D_i , so it is an admissible function from the point of view of the variational problem. Moreover in the interior of D_i , $(\nabla d)^2 = 1$ and so $\int \int_D (\nabla d)^2 d\sigma = A$.

The integral $\int \int_{D_i} d^2 d\sigma$ too has a simple meaning. It represents the axial moment of inertia of the domain D_i with respect to the side a_i of the polygon.

Imagine now the domain D being cut up along the lines separating the subdomains D_i . Let these domains be rearranged so, that the sides a_i lie on a common straight line l . (See fig. 1). After the rearrangement the domain D_i will be called D'_i . We imagine that the domains D'_i are all on one side of the line l and the „bases“ a_i cover a coherent piece of length L on the line l . It is clear that the domains D'_i are not overlapping.

The integral $\int \int_D d^2 d\sigma$ may now be interpreted as the sum of the axial moments of inertia of the domains D'_i with respect to l . This is certainly greater than the axial moment of inertia of a rectangle of base L , area A with respect to its side L :

$$\int \int_D d^2 d\sigma > \frac{1}{3} \frac{A^3}{L^2}.$$

²⁾ Cf. COURANT-HILBERT: Methoden der math. Physik vol. I, second ed., pp. 365—366.

We conclude that

$$A^2 \leq \frac{\iint_D (\nabla d)^2 d\sigma}{\iint_D d^2 d\sigma} < \frac{A}{\frac{1}{3} \frac{A^3}{L^2}} = 3 \left(\frac{L}{A} \right)^2. \quad ^3)$$

3. G. PÓLYA and G. SZEGÖ⁴⁾ gave another upper estimate of A . According to this

$$A^2 \leq j^2 B_a (2A)^{-1} \quad (2)$$

when the domain is star-shaped with respect to some interior point a . Here $j = 2.40\dots$ is the first positive root of the Bessel function $J_0(x)$ and the quantity B_a is equal to $\int_B h^{-1} ds$. In this formula h denotes the length of the perpendicular drawn from a to the tangent at a variable point of B where ds is the line element and the integral is extended over the whole boundary B of D .

The inequality (2) seems to be a sharper estimate than that of Theorem 1. However in special cases Theorem 1 may be refined so that it yields estimates hardly differing from that of formula (2). So if D is a convex polygon into which a circle can be inscribed in the elementary sense and a is the centre of the inscribed circle, ϱ is its radius, then from (2)

$$A \leq j \sqrt{\frac{L}{2\varrho A}} = j \sqrt{\frac{L^2}{2\varrho LA}} = j \sqrt{\frac{L^2}{4A^2}} \doteq 1.20 \frac{L}{A}.$$

On the other hand in this example the expression (1) can be computed explicitly if we put $u = d(P)$. For now each of the domains D_i is a triangle and

$$\sum \iint_{D_i} d^2 d\sigma = \sum \frac{1}{2} \varrho a_i \cdot \frac{\varrho^2}{6} = A \cdot \frac{\varrho^2}{6}.$$

From this

$$A^2 \leq \frac{\iint_D (\nabla d)^2 d\sigma}{\iint_D d^2 d\sigma} = \frac{6}{\varrho^2} = \frac{6}{4} \frac{L^2}{A^2} \doteq \left(1.22 \frac{L}{A} \right)^2. \quad (3)$$

4. The function $d(P)$ may be used also for calculating a lower estimate of the *torsional rigidity* P of a prism with cross section D . We use the term *torsional rigidity* in accordance with Pólya and Szegö⁵⁾ namely that P is the maximum of

³⁾ In exactly the same way one may prove the three dimensional analogy of this theorem:

Theorem 2. *The least eigenvalue A^2 of the differential equation $\Delta u + \lambda u = 0$ (Δ is the three dimensional Laplace operator, u vanishing on the surface S of a convex body of volume V), or the minimum of $(\iiint_V (\nabla u)^2 dV) / (\iiint_V u^2 dV)^{\frac{1}{2}}$ (u continuous, its first derivatives piecewise continuous in the interior of V , u vanishing on the surface) satisfies the inequality $A \leq \sqrt{3S/V}$.*

⁴⁾ PÓLYA-SZEGÖ: *Isoperimetric inequalities in Mathematical Physics*, Princeton University Press, 1951, pp. 14—15 and 91—94.

⁵⁾ Pólya-Szegö, l. c. p. 87.

$$4\left(\iint_D u \, d\sigma\right)^2 / \iint_D (\nabla u)^2 \, d\sigma \quad (4)$$

where u is subjected to the same conditions as in (1).

Now in the case treated above $\iint_D d(P) \, d\sigma$ is greater than the momentum of a rectangle with base L , area A , with respect to its side L :

$$\iint_D d(P) \, d\sigma > A \cdot \frac{A}{2L}$$

and from this follows

Theorem 3. *The torsional rigidity of a prism with convex cross section is not less than A^3/L^2 , where L is the length of the perimeter and A is the area of the cross section.*

5. It is easy to obtain a rough lower estimate for the principal frequency Λ of a convex membrane in the form $\Lambda > \gamma \frac{L}{A}$ with the help of Pólya-Szegő's inclusion lemma.⁶⁾ This lemma states that about any convex domain D of area A one can circumscribe a rectangle R with sides a, b having an area A_R such that $A_R \leq 2A$. Now if L_R is the perimeter and Λ_R the principal frequency belonging to R , then it is well known that $\Lambda_R \leq \Lambda$. On the other hand

$$\Lambda_R = \pi \sqrt{a^{-2} + b^{-2}} > 2(a^{-1}b^{-1}) = \frac{L_R}{A_R}$$

furthermore $L_R > L$ ⁷⁾ and so for any convex domain

$$\Lambda \geq \Lambda_R > \frac{L_R}{A_R} = \frac{1}{2} \cdot \frac{L}{A}.$$

Резюме

ОБ ОСНОВНОЙ ЧАСТОТЕ КОЛЕБАНИЙ ВЫПУКЛОЙ МЕМБРАНЫ И О РОДСТВЕННЫХ ЗАДАЧАХ

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Рассмотрим функционал (1), определенный на множестве функций $u(P)$, непрерывных в выпуклой плоской области D , имеющих там кусочно-непрерывные частные производные первого порядка и принимающих на

⁶⁾PÓLYA-SZEGŐ, l. c. p. 109.

⁷⁾ See e. g. ROUCHÉ-COMBEROUSSE: *Traité de Géométrie*, 7th ed. vol. I, p. 26, Gauthier-Villars, Paris, 1900.

границе B области значение нуль. Пользуясь функцией $u(P) = \min_{Q \in B} \overline{PQ}$, автор доказывает теорему:

Минимум Λ функционала (1) не превышает величины $\sqrt[3]{3L/A}$, где A — площадь области D , B — длина ее границы. Если D — многоугольник, в который можно вписать окружность, то эту оценку можно улучшить (неравенство (3)).

Аналогично доказывается, что функционал, образующий правую часть (4), не принимает значений, меньших A^3/L^2 . Отсюда следуют ограничения для т. наз. основной частоты Λ мембраны, натянутой на D , и для т. наз. жесткости при кручении P стержня поперечного сечения D .