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Addition to my paper “Generalized ordinary differential equations and continuous dependence on a parameter”

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ADDITION TO MY PAPER "GENERALIZED ORDINARY
DIFFERENTIAL EQUATIONS AND CONTINUOUS
DEPENDENCE ON A PARAMETER"

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This paper contains an improved treatment of section 3 of [1]. It
follows that the results of section 4 of [1] are valid in a more general
form. Further the results of section 5,1 [1] are formulated and proved
in a correct way.

We shall use definitions and notations introduced in [1].

1. We shall give new proofs of Theorems 3.1 and 3.2 [1]. In these proofs
weaker assumptions concerning the monotony properties of \( \psi(\eta) \) are needed.

Let \( \psi(\eta) \) be defined for \( 0 \leq \eta \leq \sigma \), \( (\sigma > 0) \), \( \psi(0) = 0 \), \( \psi(\eta) \geq 0 \). Let

\[
\sum_{j=1}^{\infty} 2^j \psi \left( \frac{\eta}{2^j} \right)
\]

converge uniformly. (This assumption is specially fulfilled if \( \psi(\eta) \) is nondecreasing
and \( \sum_{j=1}^{\infty} \frac{2^j}{\eta} \psi \left( \frac{\eta}{2^j} \right) < \infty \).) Let us put

\[
\Psi(\eta) = \sum_{j=1}^{\infty} \frac{2^j}{\eta} \psi \left( \frac{\eta}{2^j} \right) \text{ for } 0 < \eta \leq \sigma , \quad \Psi(0) = 0 .
\]

If \( \frac{\sigma}{2^{k+1}} \leq \eta \leq \frac{\sigma}{2^k} \) \( (k = 0, 1, 2, \ldots) \), \( \zeta = 2^k \eta \), then \( \Psi(\eta) = \sum_{j=k+1}^{\infty} \frac{2^j}{\zeta} \psi \left( \frac{\zeta}{2^j} \right) . \)

As (1) converges uniformly, it follows, that \( \Psi(\eta) \to 0 \) for \( \eta \to 0 + . \)

Let \( \tau_* < \tau \leq \tau_* + \sigma . \) Let \( Q \) be the square \( \tau_* \leq t \leq \tau^* , \tau_* \leq \tau \leq \tau^* . \)

Let \( V(\tau, t) \) be defined on \( Q \), \( \tau_* \leq \lambda_1 < \lambda_2 \leq \tau^* . \) Let us denote

\[
S_i(V; \lambda_1, \lambda_2) = S_i = \sum_{j=0}^{2^i-1} [V(\zeta_j, \zeta_{j+1}) - V(\zeta_j, \zeta_j)], \tag{1a}
\]

\[
Z_i(V; \lambda_1, \lambda_2) = Z_i = \sum_{j=0}^{2^i-1} [V(\zeta_j, \zeta_{j+1}) - V(\zeta_{j+1}, \zeta_j)]. \tag{1b}
\]
where
\[ \zeta_j = \lambda_1 + j \left( \frac{1}{2i} \left( \frac{1}{2i} \right) \right), \quad j = 0, 1, 2, \ldots, 2^i. \]

**Lemma 1.** If \( \frac{\partial V}{\partial t} = v(\tau, t) \) is continuous and if \( |v(\tau, t) - v(\tau', t)| \leq C|\tau - \tau'| \)

for \( \tau, \tau', t \in \langle \tau_0, \tau^* \rangle \), then \( \int_{\lambda_1}^{\lambda_2} DV \) exists and

\[ \int_{\lambda_1}^{\lambda_2} DV = \int_{\lambda_1}^{\lambda_2} v(\tau, \tau) \, d\tau, \quad (2) \]

\[ \int_{\lambda_1}^{\lambda_2} DV = \lim_{i \to \infty} S_i = \lim_{i \to \infty} Z_i, \quad (3) \]

**Proof.** Let \( h(t) = \int_{\lambda_1}^{t} v(\tau, \tau) \, d\tau. \) As

\[ |h(t) - h(\tau) - V(\tau, t) + V(\tau, \tau)| = \left| \int_{\tau}^{t} v(\xi, \xi) \, d\xi - \int_{\tau}^{t} v(\tau, \tau) \, d\xi \right| \leq C \frac{|t - \tau|^2}{2}, \]

it follows, that \( h(t) + \varepsilon t \) (resp. \( h(t) - \varepsilon t \)), \( \varepsilon > 0 \) is an upper (lower) function of \( V \), the integral \( \int_{\lambda_1}^{\lambda_2} DV \) exists and (2) holds (cf. [1], section 1.1). Further

\[ \left| \int_{\lambda_1}^{\lambda_2} DV - S_i \right| = \sum_{j=0}^{2^i-1} \left| \int_{\zeta_j}^{\zeta_{j+1}} v(\tau, \tau) \, d\tau - \int_{\zeta_j}^{\zeta_{j+1}} v(\zeta_j, \tau) \, d\tau \right| \leq \sum_{j=0}^{2^i-1} \int_{\zeta_j}^{\zeta_{j+1}} C(\tau - \zeta_j) \, d\tau = \frac{C}{2} \left( \frac{\lambda_2 - \lambda_1}{2^i} \right). \]

Similarly

\[ \left| \int_{\lambda_1}^{\lambda_2} DV - Z_i \right| \leq \frac{C}{2} \left( \frac{\lambda_2 - \lambda_1}{2^i} \right) \]

and (3) holds.

**Theorem 1.** Let \( U(\tau, t) \) be defined and continuous on \( Q \) and let

\[ |U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)| \leq \psi(\eta) \]

if \( 0 < \eta \leq \sigma \) and if \( (\tau + \eta, t + \eta), (\tau + \eta, \tau), (\tau, t + \eta), (\tau, t) \in Q \). Then \( \int_{\tau}^{\tau^*} \int_{\lambda_1}^{\lambda_2} DV \)
exists and
\[
|\int DU - U(\lambda_1, \lambda_2) + U(\lambda_2, \lambda_1)| \leq \frac{1}{2}(\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1), \tag{4}
\]
\[
|\int DU - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1)| \leq \frac{1}{2}(\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1) \tag{5}
\]
for \(\tau^* \leq \lambda_1 < \lambda_2 \leq \tau^*\).

Proof. By a usual approximating process we find such a sequence of functions \(U_k(\tau, t)\) on \(Q\) that

i) \(U_k(\tau, t)\) has continuous derivatives of the second order,

ii) \(U_k(\tau, t) \to U(\tau, t)\) uniformly,

iii) for every \(\vartheta > 0\) there is such a \(K(\vartheta),\) that

\[
|U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)| \leq \varphi(\eta)
\]

if

\[
k > K(\vartheta), \quad \tau_* + \vartheta \leq \tau < \tau + \eta \leq \tau^* - \vartheta,
\]

\[
\tau_* + \vartheta \leq t < t + \eta \leq \tau^* - \vartheta.
\]

Let

\[
\vartheta > 0, \quad \tau_* + \vartheta \leq \lambda_1 < \lambda_2 \leq \tau^* - \vartheta.
\]

According to Lemma 1 \(\int DU_k\) exists and \(\int DU_k = \lim_{l \to \infty} S_l(U_k; \lambda_1, \lambda_2)\).

\[
S_{l+1}(U_k; \lambda_1, \lambda_2) - S_l(U_k; \lambda_1, \lambda_2) =
\]

\[
= \sum_{j=0}^{2^{l-1}} [U_k(\xi_j, \xi_j + \eta) - U_k(\xi_j, \xi_j) - U_k(\xi_j - \eta, \xi_j) + U_k(\xi_j - \eta, \xi_j)],
\]

\[
\xi_j = \lambda_1 + \frac{j}{2^l} (\lambda_2 - \lambda_1) + \frac{\lambda_2 - \lambda_1}{2^{l+1}} , \quad \eta = \frac{\lambda_2 - \lambda_1}{2^{l+1}}.
\]

If \(k > K(\vartheta),\) then

\[
|S_{l+1}(U_k; \lambda_1, \lambda_2) - S_l(U_k; \lambda_1, \lambda_2)| \leq 2^l \varphi \left(\frac{\lambda_2 - \lambda_1}{2^l}\right)
\]

and

\[
\left|\int DU_k - U_k(\lambda_1, \lambda_2) + U_k(\lambda_2, \lambda_1)\right| = \lim_{l \to \infty} S_l - S_0 \leq
\]

\[
\leq \sum_{l=0}^{\infty} |S_{l+1} - S_l| \leq \sum_{l=0}^{\infty} 2^l \varphi \left(\frac{\lambda_2 - \lambda_1}{2^{l+1}}\right) = \frac{1}{2} (\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1). \tag{6}
\]

Similarly

\[
\left|\int DU_k - U_k(\lambda_2, \lambda_2) + U_k(\lambda_2, \lambda_1)\right| \leq \frac{1}{2} (\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1). \tag{7}
\]

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Let $0 < \delta \leq \sigma$ and let $A = \{x_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}$ be such a subdivision of $\langle \lambda_1, \lambda_2 \rangle$ (i.e. $\lambda_1 = x_0 < \alpha_1 < \ldots < \alpha_s = \lambda_2$, $\alpha_0 \leq \tau_1 \leq \alpha_1 \leq \ldots \leq \alpha_{s-1} \leq \tau_s \leq \alpha_s$) that
\[\tau_j - \alpha_{j-1} < \delta, \quad \alpha_j - \tau_j < \delta.\]  
(8)
Let us put
\[R(V, A) = \sum_{j=1}^{s} [V(\tau_j, \alpha_j) - V(\tau_j, \alpha_{j-1})].\]

Then
\[\left| \int_{\lambda_1}^{\lambda_2} DU_k - R(U, A) \right| = \right| \int_{\lambda_1}^{\lambda_2} DU_k - U_k(\tau_1, \tau_1) + \int_{\tau_1}^{\alpha_1} DU_k - U_k(\tau_1, \alpha_1) + U_k(\tau_1, \tau_1) \right| \leq \sum_{j=1}^{s} \left[ (\tau_j - \alpha_{j-1}) \Psi(\tau_j - \alpha_{j-1}) + (\alpha_j - \tau_j) \Psi(\alpha_j - \tau_j) \right] \leq \frac{\lambda_2 - \lambda_1}{2} \sup_{0 < t \leq \delta} \Psi(t).\]

If $A_1, A_2$ are subdivisions of $\langle \lambda_1, \lambda_2 \rangle$ which fulfill (8), then
\[|R(U, A_2) - R(U, A_1)| \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \Psi(t).\]  
(9)
As $U_k \to U$ it follows that $R(U_k, A_1) \to R(U, A_1), R(U_k, A_2) \to R(U, A_2)$ and
\[|R(U, A_2) - R(U, A_1)| \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \Psi(t).\]  
(10)
Consequently $\int_{\lambda_1}^{\lambda_2} DU$ exists as $\Psi(\delta) \to 0$ with $\delta \to 0.1)$

1) Summarizing the results of [1], section 1 we obtain, that $\int_{\lambda_1}^{\lambda_2} DV$ exists if and only if for every $\varepsilon > 0$ there exists such a positive function $\delta(t)$ that $\lambda_1$
\[|R(V, A_2) - R(V, A_1)| < \varepsilon\]  
(*)
if the subdivisions $A_1 = \{x_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s\}, A_2 = \{x_0, \tau_1', \alpha_1', \ldots, \tau_s', \alpha_s'\}$ of $\langle \lambda_1, \lambda_2 \rangle$ fulfill the conditions
\[\tau_j - \alpha_{j-1} < \delta(\tau_j), \quad \alpha_j - \tau_j < \delta(\tau_j), \quad j = 1, 2, \ldots, s,\]
\[\tau_j - \alpha_{j-1} < \delta(\tau_j), \quad \alpha_j - \tau_j < \delta(\tau_j), \quad j = 1, 2, \ldots, r.\]  
(**)
In this case $\int_{\lambda_1}^{\lambda_2} DU - R(V, A_1) \leq \varepsilon$.

Let $\delta_1$ be such a positive constant, that $(\lambda_2 - \lambda_1) \Psi(\delta) < \varepsilon$ for $0 < \delta < \delta_1$. We proved that (*) is fulfilled for $V = U$ (cf., (10)) if $\delta(\tau) = \delta_1$. As in this case $(\delta(\tau) = \delta_1 = \text{const})$ (**), is equivalent to the usual conditions (the respective subdivisions $A_1, A_2$ are fine enough) of the Riemann theory of integration, we may say, that $\int_{\lambda_1}^{\lambda_2} DU$ exists in the sense of Riemann.

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It follows from (9) and (10) that
\[ \int_{\lambda_2}^{\lambda_1} DU_k - R(U_k, A_1) \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \psi(t), \]
\[ \int_{\lambda_2}^{\lambda_1} DU - R(U, A_1) \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \psi(t). \]
As \( \delta \) is arbitrary and \( R(U_k, A_1) \to R(U, A_1), \int_{\lambda_2}^{\lambda_1} DU_k \to \int_{\lambda_2}^{\lambda_1} DU. \)

Passing to the limit for \( k \to \infty \) in (6) and (7) we obtain (4) and (5) with the additional assumption \( \tau_* < \lambda_1 < \lambda_2 < \tau^* \).

Let \( \zeta \in (\tau_*, \tau^*) \). As \( DU \) is uniformly continuous in \( \lambda \) on \( (\tau_*, \tau^*) \) and \( U(\tau, t) \) is continuous on \( Q, \int_{\lambda_1}^{\lambda_2} DU \) exists if \( \tau_* \leq \lambda_1 < \lambda_2 \leq \tau^* \) (cf. Theorem 1,3,5, [1]) and (4) and (5) hold for \( \tau_* \leq \lambda_1 < \lambda_2 \leq \tau^* \). Theorem 1 is proved.

Let \( S \) be the set of such \( (\tau, t) \) that \( \tau_* \leq \tau \leq \tau^*, \tau_* \leq t \leq \tau^*, |\tau - t| \leq \sigma \).

**Theorem 2.** Let the functions \( U_k(\tau, t), k = 0, 1, 2, \ldots \) be defined and continuous on \( S \) and let
\[ |U_k(\tau + \eta, t + \eta) - U_k(\tau + \eta, t) - U_k(\tau, t + \eta) + U_k(\tau, t)| \leq \psi(\eta) \]
if
\[ 0 < \eta \leq \sigma, (\tau + \eta, t + \eta), (\tau + \eta, t), (\tau, t + \eta), (\tau, t) \in S. \]

Let \( U_k(\tau, t) \to U(\tau, t) \) uniformly on \( S \) with \( k \to \infty \). (11)

Then
\[ \int_{\lambda_2}^{\lambda_1} DU_k \to \int_{\lambda_2}^{\lambda_1} DU \text{ with } k \to \infty \text{ uniformly for } \tau_* \leq \lambda_1 \leq \lambda_2 \leq \tau^*. \]
(12)

**Proof.** From (2) and (3) we obtain in a similar manner as in the proof of the preceding theorem that
\[ \int_{\lambda_2}^{\lambda_1} DU_k - B(U_k, A) \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \psi(t) \]
in the subdivision \( A \) of \( \langle \lambda_1, \lambda_2 \rangle \) fulfills (8), \( 0 < \delta \leq \sigma \) and (12) follows from (11), as \( B(U_k, A) \to B(U, A) \) with \( k \to \infty \) (cf. Theorem 1,3,4, [1]).

**Note 1.** The results of section 4, [1] (specially Theorems 4.1,1, 4.1,2, 4.2,1, Lemma 4.1,1) are valid, if we omit the assumption that \( \eta^{-1}\psi(\eta) \) is nondecreasing (we assume of course, that \( \sum_{j=1}^{\infty} 2^j \psi \left( \frac{\sigma}{2^j} \right) < \infty; \psi(\eta) = \omega_1(\eta) \omega_2(\eta) \) is nondecreasing, as \( \omega_1(\eta) \) and \( \omega_2(\eta) \) are nondecreasing).}

**Note 2.** Suppose that the values of the function \( U(\tau, t) \) \( (\tau, t \in (\tau_*, \tau^*)) \) belong to a Banach space \( B \). For \( X \in B \) let \( |X| \) be the norm of \( X \). Let \( \langle \lambda_1, \lambda_2 \rangle \subset \)
\( \langle \tau_*, \tau^* \rangle \). If \( A = \{ \alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s \} \) is a subdivision of \( \langle \lambda_1, \lambda_2 \rangle \), put

\[
R(U, A) = \sum_{j=1}^{s} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})].
\]

Suppose that for every \( \varepsilon > 0 \) there exists such a function \( \delta(\tau) > 0 \) that

\[
|R(U, A) - R(U, A')| < \varepsilon
\]

if the subdivisions \( A = \{ \alpha_0, \tau_1, \alpha_1, \ldots, \tau_s, \alpha_s \} \), \( A' = \{ \alpha_0', \tau_1', \alpha_1', \ldots, \tau_r', \alpha_r' \} \) of \( \langle \lambda_1, \lambda_2 \rangle \) fulfill the conditions

\[
\tau_i - \alpha_{i-1} < \delta(\tau_i), \quad \alpha_i - \tau_i < \delta(\tau_i), \quad i = 1, 2, \ldots, s, \\
\tau_i' - \alpha_{i-1}' < \delta(\tau_i'), \quad \alpha_i' - \tau_i' < \delta(\tau_i'), \quad i = 1, 2, \ldots, r. \tag{1*}
\]

Then there exists such a \( W \in B \) that

\[
|W - R(U, A)| \leq \varepsilon
\]

if the subdivision \( A \) of \( \langle \lambda_1, \lambda_2 \rangle \) fulfills \( (1*) \). In this case we define

\[
\int_{\hat{\lambda}_1}^{\hat{\lambda}_2} DU(\tau, t) = W.
\]

We shall show that Theorem 1 remains valid if the values of \( U \) belong to \( B \). Let us put

\[
\overline{S}(U; \lambda_1, \lambda_2) = \lim_{\epsilon \to 0} S_i(U; \lambda_1, \lambda_2), \tag{2*}
\]

\[
\overline{Z}(U; \lambda_1, \lambda_2) = \lim_{\epsilon \to 0} Z_i(U; \lambda_1, \lambda_2), \tag{3*}
\]

where \( S_i, Z_i \) are defined by the formulas \( (1a) \), \( (1b) \). In the same manner as we deduced \( (6) \) we obtain that the limits in \( (2*) \) and \( (3*) \) exist and that

\[
|\overline{S}(U; \lambda_1, \lambda_2) - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1)| \leq \frac{1}{2}(\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1), \tag{4*}
\]

\[
|\overline{Z}(U; \lambda_1, \lambda_2) - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1)| \leq \frac{1}{2}(\lambda_2 - \lambda_1) \Psi(\lambda_2 - \lambda_1). \tag{5*}
\]

Obviously

\[
\varphi S_i(U; \lambda_1, \lambda_2) = S_i(\varphi U; \lambda_1, \lambda_2), \quad \varphi Z_i(U; \lambda_1, \lambda_2) = Z_i(\varphi U; \lambda_1, \lambda_2)
\]

where \( \varphi \) is a linear functional on \( B \). It follows that

\[
\varphi \overline{S}(U; \lambda_1, \lambda_2) = \int_{\hat{\lambda}_1}^{\hat{\lambda}_2} D_{\varphi} U(\tau, t) = \varphi \overline{Z}(U; \lambda_1, \lambda_2).
\]

Hence

\[
\overline{S}(U; \lambda_1, \lambda_2) = \overline{Z}(U; \lambda_1, \lambda_2) \tag{6*}
\]

and

\[
\overline{S}(U; \lambda_1, \lambda_2) + \overline{S}(U; \lambda_2, \lambda_3) = \overline{S}(U; \lambda_1, \lambda_3). \tag{7*}
\]

Let \( \delta > 0 \) and let \( A \) be a subdivision of \( \langle \lambda_1, \lambda_2 \rangle \), \( \alpha_i - \tau_i < \delta, \tau_i - \alpha_{i-1} < \delta, \)

\[i = 1, 2, \ldots, s.\]

It follows from \( (4*) - (7*) \) that

\[
|\overline{S}(U; \lambda_1, \lambda_2) - R(U, A)| \leq (\lambda_2 - \lambda_1) \sup_{0 < t \leq \delta} \Psi(t).
\]

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Consequently $\int_{\lambda_1}^{\lambda_2} DU(\tau, t)$ exists and

$$\int_{\lambda_1}^{\lambda_2} DU(\tau, t) = \bar{S}(U; \lambda_1, \lambda_2).$$

From (4*), (5*) and (6*) we obtain (4) and (5). Theorem 1 holds.

Theorem 2 follows from Theorem 1 in the same manner as in the scalar case.

2. Some considerations in section 5,1, [1] are not correct (especially p. 444, formulas in the 3rd and 5th line from below and p. 445 formula in the 6th line from above). The aim of section 5,1, [1] was to prove that the solutions of generalized linear differential equations are unique and to establish the variation-of-constants formula. Here we shall give the correct proofs. The author intends to use the variation-of-constants formula for an investigation of linear differential equations. Therefore we shall work with general moduli of continuity $\omega_3, \omega_4$ while in section 5,1 [1] it was assumed that the respective moduli of continuity are powers of $\eta$.

Let $\omega_3(\eta), \omega_4(\eta)$ be nondecreasing on $\langle 0, \sigma \rangle$, $\omega_3(\eta) \geq c\eta, \omega_4(\eta) \geq c\eta, c > 0, \omega_5(\eta) = \max(\omega_3(\eta), \omega_4(\eta))$, $\psi_4(\eta) = \omega_4^2(\eta), \psi_5(\eta) = \omega_5(\eta)\omega_4(\eta)$.

Suppose that $\sum_{i=1}^{\infty} 2^i \psi_5 \left(\frac{\sigma}{2^i}\right) < \infty$ and put $\Psi_i(\eta) = \sum_{i=1}^{\infty} \frac{2^i}{\eta} \psi_i \left(\frac{\eta}{2^i}\right), i = 4,5$.

Let $A(t), t \in (-\infty, \infty)$ be an $n \times n$-matrix and let $B(t)$ be an $n$-vector and suppose that

$$\|A(t_2) - A(t_1)\| \leq \omega_4(|t_2 - t_1|) \quad \text{for} \quad |t_2 - t_1| \leq \sigma, \quad (13)$$

$$\|B(t_2) - B(t_1)\| \leq \omega_3(|t_2 - t_1|) \quad \text{for} \quad |t_2 - t_1| \leq \sigma. \quad (14)$$

Lemma 2. Let $c \in E_n$. There exists at most one regular solution $x(\tau)$ of

2) Let $F(x, t) \in F(G, \omega_1, \omega_2, \sigma)$ (cf. [1], section 4,1). Let $x(\tau), \tau \in \langle \tau_1, \tau_2 \rangle$ be a solution of

$$\frac{dx}{d\tau} = DF(x, t).$$

$x(\tau)$ is a regular solution, if there is such a $\sigma' > 0$, that $\|x(\tau_3) - x(\tau_4)\| \leq 2\omega_1(|\tau_3 - \tau_4|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$, $|\tau_3 - \tau_4| \leq \sigma'$. This definition is equivalent to Definition 4,2,1 [1], as the interval $\langle \tau_1, \tau_2 \rangle$ is compact.

Let $y(\tau)$ be a solution of

$$\frac{dy}{d\tau} = D(A(t) y + B(t))$$

on an interval $I$ (if $\tau$ may be closed, open or $I = (-\infty, \infty)$). Let $\langle \tau_1, \tau_2 \rangle$ be a compact subinterval of $I$. $y(\tau)$ is continuous on $I$ (cf. Theorem 1.3.6 and Definition 2.1.1 [1] and therefore there exists such a bounded open subset $G$ of $E_{n+1}$ $(n + 1)$-dimensional Euclidean space), that contains all the points $(y(\tau), \tau)$ for $\tau \in \langle \tau_1, \tau_2 \rangle$. Obviously $A(t) y + B(\tau) \in F(G, K_4, \omega_2(\eta), K_5, \omega_3(\eta), \sigma)$ if $K_4$ is great enough. $y(\tau), \tau \in \langle \tau_1, \tau_2 \rangle$ is regular [with respect to $F(G, K_4, \omega_2(\eta), K_5, \omega_3(\eta), \sigma)$], if there is such a $\sigma' > 0$, that $\|y(\tau_4) - y(\tau_3)\| \leq 2K_4\omega_2(|\tau_4 - \tau_3|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle$, $|\tau_4 - \tau_3| \leq \sigma'$. We shall say that $y(\tau)$ is a regular solution of (16), if for every compact subinterval $\langle \tau_1, \tau_2 \rangle$ of $I$ there exist such positives $K_4$ and $\sigma'$ that $\|y(\tau_3) - (\tau_3)\| \leq K_4\omega_3(|\tau_4 - \tau_3|)$ for $\tau_3, \tau_4 \in \langle \tau_1, \tau_2 \rangle, |\tau_4 - \tau_3| \leq \sigma'$. 570
\[ \frac{\text{d}x}{\text{d}\tau} = DA(t) x, \] 

which fulfills \( x(t_0) = c \).

In order to prove Lemma 2 we may use the proof of Lemma 5.1 [1]. At the same time Lemma 2 is a consequence of Theorem 1 [2].

**Lemma 3.** The regular solutions of (15) are defined for \( \tau \in (-\infty, \infty) \).

Lemma 3 is a consequence of Lemma 2 and Theorem 4.2.1 [1], where we put \( F_1(x, t) = F_0(x, t) = A(t) x, \ v_0(\tau) = 0, \ v = 1, \ G = E(x, t; \|x\| < 1) \). It follows that for every \( T > 0 \) there is such a \( \delta > 0 \) that the regular solutions of (15) which fulfil \( \|x(0)\| < \delta \) are defined on \( \langle 0, T \rangle \) and fulfil \( \|x(\tau)\| < 1 \) on \( \langle 0, T \rangle \). By the substitution \( t' = t - t \) we obtain the solutions of (15) which fulfil \( \|x(0)\| < \delta' \) are defined on \( \langle -T, 0 \rangle \) and fulfil \( \|x(\tau)\| < 1 \) on \( \langle -T, 0 \rangle \).

The fundamental matrix of (15) is a \( n \times n \)-matrix \( \Phi(\tau), \Phi(0) = E^3 \) every column of which is a regular solution of (15). It follows from Lemmas 2 and 3 that \( \Phi(\tau) \) is defined uniquely for \( \tau \in (-\infty, \infty) \).

Our aim is to establish the variation-of-constants formula for the solutions of

\[ \frac{\text{d}x}{\text{d}\tau} = D[A(t) x + B(t)]. \]  (16)

Let \( A_k(t), B_k(t) \) be such matrices and vectors, that \( A_k(t) \rightarrow A(t), B_k(t) \rightarrow B(t) \) uniformly on every bounded interval, \( \frac{\text{d}}{\text{d}t} A_k(t) = a_k(t), \frac{\text{d}}{\text{d}t} B_k(t) = b_k(t) \) are continuous, \( A_k(t) \) fulfil (13), \( B_k(t) \) fulfil (14).

As the generalized equation

\[ \frac{\text{d}x}{\text{d}\tau} = D[A_k(t) x + B_k(t)] \]

is equivalent to the classical equation

\[ \frac{\text{d}x}{\text{d}t} = a_k(t) x + b_k(t) , \]

we have the variation-of-constants formula

\[ x_k(s) = \Phi_k(s)[z + \int_0^s \mathcal{E}_k(t) b_k(t) \text{d}t] = \]

\[ = \Phi_k(s)[z + \int_0^s D\mathcal{E}_k(\tau) B_k(t)] , \quad x_k(0) = z , \]  (17)

where \( \Phi_k(\tau) \) is the fundamental matrix of

\[ \frac{\text{d}x}{\text{d}\tau} = D[A_k(t) x] \]  (18)

and \( \mathcal{E}_k(\tau) = \Phi_k^{-1}(\tau) \).

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3) \( E \) is the unit matrix.
According to Lemma 2 and Theorem 4.2.1, [1] $\Phi_k(\tau) \rightarrow \Phi(\tau)$ uniformly on every bounded interval. As $\mathcal{E}^{(k)}_k(\tau)$ is the fundamental matrix of
\[
\frac{d x}{d \tau} = -DA^{*}(t)x ,
\]
(19)
it follows similarly, that $\mathcal{E}^{*}_k(\tau) \rightarrow \mathcal{E}^{*}(\tau)$ uniformly on every bounded interval, where $\mathcal{E}^{*}(\tau)$ is the fundamental matrix of
\[
\frac{d x}{d \tau} = -DA^{*}(t)x .
\]

Passing to the limit for $k \rightarrow \infty$ in $\Phi_k(\tau)\mathcal{E}_k(\tau) = E$ we obtain that $\Phi(\tau) \mathcal{E}(\tau) = E$ ($\mathcal{E} = \mathcal{E}^{**}$).

Let $T > 0$. As the columns of $\mathcal{E}_k^{*}(\tau)$ are regular solutions of (19), there exist such $K_{6k}$ and $\sigma_{k}^{*}$ that
\[
||\mathcal{E}_k(t_2) - \mathcal{E}_k(t_1)|| \leq K_{6k}\omega_{4}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma_{k}^{*}.
\]

It follows that there exist such constants $K_{6k}$, that
\[
||\mathcal{E}_k(t_2) - \mathcal{E}_k(t_1)|| \leq K_{6k}\omega_{4}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma.
\]

According to Lemma 4.1.1 [1] there exist such $\sigma^{*} > 0$ ($\sigma^{*} \leq \sigma$) and $L > 0$ (independent on $k$) that
\[
||\mathcal{E}_k(t_2) - \mathcal{E}_k(t_1)|| \leq L_{k}\omega_{4}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma^{*} .
\]
(20)

Similarly
\[
||\Phi_k(t_2) - \Phi_k(t_1)|| \leq L'\omega_{4}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma^{*} .
\]
(21)

Theorem 2 together with (14) and (17) implies that
\[
\int_{0}^{s} \int_{0}^{t} D\mathcal{E}_k(\tau) B_k(t) \rightarrow \int_{0}^{s} \int_{0}^{t} D\mathcal{E}(\tau) B(t) \quad \text{uniformly with} \quad k \rightarrow \infty \quad \text{for} \quad s \in (-T, T).
\]
(22)

Consequently the uniform limit $x_k(t) = x(t)$ exists. From (13), (14), (17), (20), (21) and Theorem 1 we obtain that
\[
||\int_{0}^{t} D\mathcal{E}_k(\tau) B_k(t)|| \leq K_{7}\omega_{4}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma^{*} ,
\]
\[
||x_k(t_2) - x_k(t_1)|| \leq K_{8}\omega_{5}(|t_2 - t_1|) \quad \text{for} \quad t_1, t_2 \in (-T, T), \quad |t_2 - t_1| \leq \sigma^{*} .
\]

4) $\mathcal{E}^{*}$ is the conjugate transpose to $\mathcal{E}$.

5) The number $\sigma^{*}$ which occurs in Lemma 4.1.1 [1] depends only on $K, G, \omega_{1}, \omega_{2},$ not on the right-hand side of equation (4,1,04). Let us denote by $\xi_{j}^{*}(\tau)$ ($\xi_{j}^{*}(\tau)$) the $j$-th column of the matrix $\mathcal{E}^{*}_k(\tau)$ ($\mathcal{E}^{*}(\tau)$). In the present case we choose such an open and bounded set $G$, that contains all the points $(\xi_{k,j}^{*}(\tau), \tau)$, $(\xi_{j}^{*}(\tau), \tau)$, $k = 1, 2, 3, \ldots$, $j = 1, 2, \ldots, n$, $\tau \in (-T, T)$.
According to Theorem 4.1.1 [1] $x(\tau)$ is a solution of (16).\(^4\) Passing to the limit for $k \rightarrow \infty$ in (17) we obtain the required variation-of-constants formula

$$x(s) = \Phi(s)[z + \int_0^s D\xi(\tau) \, B(t)] , \quad x(0) = z . \quad (23)$$

As the difference of two regular solutions of (16) is a regular\(^2\) solution of (15), Lemma 2 implies that $x(\tau)$ is the only regular solution of (16), which fulfills $x(0) = z$.

REFERENCES


Резюме

ДОБАВЛЕНИЕ К МОЕЙ СТАТЬЕ „ОБОБЩЕННЫЕ ОБЫКНОВЕННЫЕ ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ И НЕПРERYВНАЯ ЗАВИСИМОСТЬ OT ПАРАМЕТРА“

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Дается улучшенное изложение третьего параграфа статьи [1]. В новых доказательствах теорем этого параграфа употребляются более слабые предположения, касающиеся монотонности функции $\psi(\eta)$. Следовательно, в этих более общих предположениях верны и основные результаты статьи [1], содержащиеся в четвертом параграфе. Далее, для обобщенных линейных уравнений выводится формула вариации постоянных (что представляет корректное изложение параграфа 5,1 статьи [1]).

\(^4\) $A_k(t) x + B_k(t) \in F(G, K_1 \omega_3(\eta), \omega_4(\eta), \sigma)$ where $G$ is a suitable bounded set and $K_1$ is great enough.

\(^2\) Cf. footnote 5).