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Czechoslovak Mathematical Journal, Vol. 9 (1959), No. 4, 574–578

Persistent URL: http://dml.cz/dmlcz/100383

THE POLYNOMIAL APPROXIMATION OF CONTINUOUS FUNCTIONS DEFINED ON $(-\infty, \infty)$

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(Received March 19, 1959)

The object of this note is to establish a theorem concerning the polynomial approximation of continuous functions defined on the whole axis $(-\infty, \infty)$. Our theorem consists of introducing a type of explicit approximating polynomials. Finally, a generalization to the case of functions of several variables is indicated.

Let Θ be a fixed real number with $0 < \Theta < \frac{1}{2}$ and define

$$L_n(x; f) = \sqrt{\frac{n}{\pi}} \int_{-1}^{1} f(n^{\Theta}t)[1 - (t - n^{-\Theta}x)^2]^n \, dt \,, \tag{1}$$

where $f(u)$ is a real function for which the integral of (1) exists, and $n = 1, 2, 3, \ldots$. Clearly $L_n(x; f)$ is a polynomial in x, of degree $2n$, and it will reduce to the Landau singular integral in case Θ is taken to be zero etc. Evidently the classical Landau integral is incapable of approximation to a continuous function defined on the infinite interval $(0, \infty)$ or $(-\infty, \infty)$.

We are now going to establish the following result:

Theorem. *For any continuous function $f(x)$ defined and bounded on the whole interval $(-\infty, \infty)$, we have*

$$\lim_{n\to\infty} L_n(x; f) = f(x) \,, \quad (-\infty < x < \infty) \,. \tag{2}$$

Moreover, the limit relation holds uniformly on any finite interval.

Proof. Choose a number δ such that

$$\max\,(0, \tfrac{1}{4} - \Theta) < \delta < \tfrac{1}{2} - \Theta \,, \tag{3}$$

and denote for brevity

$$\Phi_n(x) = n^{-\Theta}x \,, \quad \varepsilon(n) = \left(\frac{1}{n}\right)^{\Theta+\delta} . \tag{4}$$

Let us split the integral of (1) (with $f(u) \equiv 1$) as a sum of three integrals:

$$\int_{-1}^{1} [1-(t-\Phi_n(x))^2]^n \,dt = \int_{-1}^{\Phi_n(x)-\varepsilon(n)} + \int_{\Phi_n(x)-\varepsilon(n)}^{\Phi_n(x)+\varepsilon(n)} + \int_{\Phi_n(x)+\varepsilon(n)}^{1} = J_1(n) + J_2(n) + J_3(n)\,.$$

Given any finite interval $[\alpha, \beta]$, we shall now show that the relations

$$\lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\, J_1(n) = 0\,, \quad \lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\, J_2(x) = 1\,, \quad \lim_{n\to\infty} \sqrt{\frac{n}{\pi}}\, J_3(n) = 0 \tag{5}$$

hold uniformly with respect to x ($\alpha \leqq x \leqq \beta$). Notice that $\left(1-\dfrac{1}{X}\right)^X \uparrow e^{-1}$ for $X \to \infty$. We clearly have, for large n and for $-1 \leqq t \leqq \Phi_n(x) - \varepsilon(n)$,

$$\begin{aligned}|1-(t-\Phi_n(x))^2|^n &\leqq \max\{|1-(1+\Phi_n(x))^2|^n,\ [1-(\varepsilon(n))^2]^n\} \leqq \\ &\leqq \max\left\{[2|\Phi_n(x)| + (\Phi_n(x))^2]^n,\ \left[1-\left(\frac{1}{n}\right)^{2(\Theta+\delta)}\right]^n\right\} \leqq \\ &\leqq \max\{|3n^{-\Theta}x|^n,\ (e^{-1})^{n^{1-2(\Theta+\delta)}}\} \leqq e^{-n^{1-2(\Theta+\delta)}}\,;\end{aligned}$$

so that

$$\begin{aligned}\sqrt{\frac{n}{\pi}}\, |J_1(n)| &\leqq \sqrt{\frac{n}{\pi}} \int_{-1}^{\Phi_n(x)-\varepsilon(n)} |1-(t-\Phi_n(x))^2|^n \,dt \leqq \\ &\leqq \sqrt{\frac{n}{\pi}}\, e^{-n^{1-2(\Theta+\delta)}} [\Phi_n(x) - \varepsilon(n) + 1] \to 0\,, \quad (n\to\infty)\end{aligned}$$

the right-hand side tending to zero uniformly with respect to x ($\alpha \leqq x \leqq \beta$). Hence we have proved that $\sqrt{\dfrac{n}{\pi}}\, J_1(n) \to 0$ ($n \to \infty$) uniformly for x. Similarly, we find that $\sqrt{\dfrac{n}{\pi}}\, J_3(n)$ also converges uniformly to zero for x ($\alpha \leqq x \leqq \beta$).

Actually, we have proved

$$\begin{aligned}&\lim_{n\to\infty} \sqrt{\frac{n}{\pi}} \int_{-1}^{\Phi_n(x)-\varepsilon(n)} |1-(t-\Phi_n(x))^2|^n \,dt = \\ &= \lim \sqrt{\frac{n}{\pi}} \int_{\Phi_n(x)+\varepsilon(n)}^{1} |1-(t-\Phi_n(x))^2|^n \,dt = 0\end{aligned} \tag{6}$$

uniformly in $[\alpha, \beta]$.

It remains to consider $J_2(n)$. As may be easily verified, we may write, for $\Phi_n(x) - \varepsilon(n) \leqq t \leqq \Phi_n(x) + \varepsilon(n)$,

$$\begin{aligned}[1-(t-\Phi_n(x))^2]^n &= \exp\{-n(t-\Phi_n(x))^2 + O(n(\varepsilon(n))^4)\} = \\ &= \exp\{-n(t-\Phi_n(x))^2\}\,.\left\{1 + O\left(\left(\frac{1}{n}\right)^{4\Theta+4\delta-1}\right)\right\},\end{aligned}$$

where $4\Theta + 4\delta - 1 > 0$ and the constant factor implied in the order estimation of $O\left(\left(\frac{1}{n}\right)^{4\,\Theta+4\delta-1}\right)$ is independent of t and x ($\alpha \leqq x \leqq \beta$), in view of (3) and (4). Therefore, making a change of variables $X = \sqrt{n}(t - \Phi_n(x))$, we have $\mathrm{d}t = \frac{1}{\sqrt{n}}\,\mathrm{d}X$, so that

$$\sqrt{\frac{n}{\pi}}\,J_2(n) = \left\{1 + O\left(\left(\frac{1}{n}\right)^{4\Theta+4\delta-1}\right)\right\}\sqrt{\frac{n}{\pi}}\int\limits_{-\sqrt{n}\varepsilon(n)}^{\sqrt{n}\varepsilon(n)} e^{-X^2}\frac{\mathrm{d}X}{\sqrt{n}} \to 1\,,$$

where the right-hand side tends to 1 uniformly with respect to x ($\alpha \leqq x \leqq \beta$), since $\sqrt{n}\varepsilon(n) = n^{\frac{1}{2}-\Theta-\delta} \to \infty$ ($n \to \infty$) in accordance with (3).

Finally, we have to deal with $L_n(x; f)$:

$$\sqrt{\frac{n}{\pi}}\left(\int\limits_{-1}^{\Phi_n(x)-\varepsilon(n)} + \int\limits_{\Phi_n(x)-\varepsilon(n)}^{\Phi_n(x)+\varepsilon(n)} + \int\limits_{\Phi_n(n)+\varepsilon(n)}^{1}\right) f(n^{\Theta}t)[1 - (t - n^{-\Theta}x)^2]^n\,\mathrm{d}t =$$

$$= \sqrt{\frac{n}{\pi}}\,J_1^*(n) + \sqrt{\frac{n}{\pi}}\,J_2^*(n) + \sqrt{\frac{n}{\pi}}\,J_3^*(n)\,.$$

Since $|f(x)|$ is bounded on the whole interval $(-\infty, \infty)$, say $|f(x)| < M$, we may infer from (5) that both $\sqrt{\frac{n}{\pi}}\,J_1^*(n)$ and $\sqrt{\frac{n}{\pi}}\,J_3^*(n)$ tend to zero uniformly with regard to x ($\alpha \leqq x \leqq \beta$) as $n \to \infty$. Also we have

$$\sqrt{\frac{n}{\pi}}\,|J_2^*(n) - f(x)\,J_2(n)| \leqq \sqrt{\frac{n}{\pi}}\int\limits_{\Phi_n(x)-\varepsilon(n)}^{\Phi_n(x)+\varepsilon(n)} |f(n^{\Theta}t) - f(x)|[1 - (t - \Phi_n(x))^2]^n\,\mathrm{d}t \leqq$$

$$\leqq \max_t |f(n^{\Theta}t) - f(x)|\sqrt{\frac{n}{\pi}}\,J_2(n)\,.$$

Here the "max" is taken over all t satisfying the condition $|t - n^{-\Theta}x| \leqq \varepsilon(n)$, so that $|n^{\Theta}t - x| \leqq n^{\Theta}\varepsilon(n) = \left(\frac{1}{n}\right)^{\delta} \to 0$ ($n \to \infty$) and consequently $\max\limits_t |f(n^{\Theta}t) - f(x)|$ tends to zero uniformly for all x ($\alpha \leqq x \leqq \beta$). We see thus that the difference $\sqrt{\frac{n}{\pi}}\,(J_2^*(n) - f(x)\,J_2(n))$ tends to zero uniformly with respect to x. Recall that $\sqrt{\frac{n}{\pi}}\,f(x)\,J_2(n) \to f(x)$ ($n \to \infty$) uniformly. Hence in conclusion we have shown that the relation

$$\lim_{n\to\infty}\sqrt{\frac{n}{\pi}}\,(J_1^*(n) + J_2^*(n) + J_3^*(n)) = f(x) \tag{7}$$

holds uniformly for x ($\alpha \leqq x \leqq \beta$). This completes the proof of our theorem, since $[\alpha, \beta]$ is an arbitrary interval.

It is not difficult to extend the theorem just proved to the case of functions of several variables. In fact, if we define (with $0 < \Theta < \frac{1}{2}$)

$$L_n(x_1, \ldots, x_k; f) =$$
$$= \left(\frac{n}{\pi}\right)^{k/2} \int_{-1}^{1} \ldots \int_{-1}^{1} f(n^{\Theta}t_1, \ldots, n^{\Theta}t_k) \prod_{\nu=1}^{k} [1 - (t_\nu - n^{-\Theta}x_\nu)^2]^n \, \mathrm{d}t_1 \ldots \mathrm{d}t_k \,,$$

then it can be shown in exactly the same manner that

$$\lim_{n\to\infty} L_n(x_1, \ldots, x_k; f) = f(x_1, \ldots, x_k) \tag{8}$$

holds for any continuous function $f(x_1, \ldots, x_k)$ defined and bounded on the k-dimensional Euclidean space R^k.

As a comparison of our theorem with Chlodovsky's generalization [1] of Bernstein polynomials, we may say that our result seems more satisfactory in view of the fact that Chlodovsky's theorems apply only to the functions defined on the half infinite interval $(0, \infty)$, which cannot be replaced by $(-\infty, \infty)$ anyway.

However, the author has been unable to decide whether the relation

$$\lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}x} L_n(x; f) = f'(x) \,, \quad (-\infty < x < \infty)$$

is true, in case only the continuity condition or other simple condition is imposed upon the derivative $f'(x)$ $(-\infty < x < \infty)$.

The author and his cooperator L. P. Hsu have also investigated other types of explicit approximating polynomials which may be regarded as the summation analogue to the type here presented, and possess some similarity with the Bernstein polynomial and its generalizations (see [2], [3], [4], [5]).

LITERATURE

[1] *Chlodovsky I.*: Sur le développement des fonctions dans un intervalle infini en séries de polynômes de M. S. Bernstein, Compositio Math., *4* (1937), 380—393.

[2] *Hsu, L. C.*: A new type of polynomials approximating a continuous or integrable function, Studia Mathematica *18* (1959), 43—48.

[3] *Hsu, L. C.* and *Hsu, L. P.*: On a new class of approximating polynomials for real functions, Science Record, new ser. (Academia Sinica), *3* (1959), 65—70.

[4] *Kantorovitch, L. V.*: Sur certains développement suivant les polynômes de la forme de S. Bernstein I, II, C. R. Acad. Sci. URSS (1930), 563—568, 595—560.

[5] *Lorentz, G. G.*: Bernstein polynomials (1953), Chapters I, II.

Резюме

О ПРИБЛИЖЕНИИ МНОГОЧЛЕНАМИ НЕПРЕРЫВНОЙ ФУНКЦИИ, ОПРЕДЕЛЕННОЙ НА ВСЕЙ ВЕЩЕСТВЕННОЙ ОСИ

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(Поступило в редакцию 19/III 1959 г.)

В работе указывается класс многочленов, обобщающих многочлены Ландау. Многочленами из этого класса можно аппроксимировать на интервале $(-\infty, \infty)$ любую непрерывную и ограниченную функцию, при этом на любом конечном промежутке эту функцию можно аппроксимировать равномерно.

Сделано замечание относительно обобщения этих результатов на случай функций многих переменных.