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THE FRATTINI SUBGROUPS OF ABELIAN GROUPS

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In the present paper the form of the Frattini subgroup of an abelian group is described. Further, there is proved that every abelian group is the Frattini subgroup of suitable groups; in the class of all abelian groups with this property there exists a minimal one, unique up to isomorphism (the \( \Phi \)-closure of the given group). The paper concludes with some applications concerning the \( \Phi \)- and \( \varphi \)-series of an abelian group and the study of generating systems of abelian groups.

1. INTRODUCTION

The Frattini subgroup \( \Phi(G) \) of a (in general non-abelian) group \( G \) is defined as the intersection of all maximal proper subgroups of the group \( G \), if \( G \) has maximal subgroups; otherwise one puts \( \Phi(G) = G \). We can also characterize the Frattini subgroup \( \Phi(G) \) in terms of the "non-generators" of \( G \), that is of those elements which can be omitted from any generating system of the group \( G \) without loss of the property of being a generating system of the whole group: \( \Phi(G) \) is just the subgroup of all such non-generators.

Many papers have been dedicated to the investigation of the Frattini subgroups; the papers of G. A. MILLER [8] and W. GASCHütZ [5] also discuss finite abelian groups. The present article extends the results of the latter to arbitrary abelian groups. In § 2 there is described the form of the Frattini subgroup of an abelian group: The Frattini subgroup of an abelian group \( G \) is the subgroup of those elements \( g \in G \) for which the equations

\[
(1,1) \quad p \cdot x = g
\]

are solvable in the group \( G \) for any prime \( p \) (Theorem 1). In contradistinction to the non-commutative case, the Frattini subgroup of an abelian group is fully invariant (compare e.g. [9]). In § 3 there is proved that every abelian group is the Frattini subgroup of some abelian group; in the class of all abelian groups the Frattini subgroups of which are isomorphic to a given abelian group, there exists a minimal one, unique up to isomorphism (Theorem 4, Appendix to
Theorem 4): the so-called $\varphi$-closure of the group $G$.\(^1\) Theorem 4.4 in B. H. NEUMANN's paper [9] implies that there exist non-abelian groups which cannot be Frattini subgroups of any group whatsoever; from the results of § 3 it follows that the assumption of non-commutativity cannot be weakened in this theorem (for finite groups this has already been shown by Gaschütz [5]). In the case of abelian groups we obtain from the latter a positive answer to a problem of B. H. Neumann from the end of his paper [9]: If a finite abelian group $G$ is the Frattini subgroup of some group, then there even exists a finite group whose Frattini subgroup is isomorphic to $G$. The final section of this paper, § 4, is devoted to some applications of previous results.

By a group we shall mean throughout an abelian group written additively. The symbols $G_1 + G_2$ resp. $\bigoplus_{\delta \in \Delta} G_\delta$ will denote the direct sum of groups $G_1$ and $G_2$ resp. $G_\delta$ ($\delta \in \Delta$), $G/H$ the quotient group $G$ modulo $H$ and $pG$ the subgroup of a group $G$ of all elements $p \cdot g$ with $g \in G$. For any non-void subset $\mathcal{M}$ of $G$, $\{\mathcal{M}\}$ is used to denote the subgroup of $G$ generated by the elements of $\mathcal{M}$; thus $\{\emptyset\} = G$ means that $\emptyset$ is a generating system of the group $G$. By the symbols $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ we shall denote the set-theoretical union, intersection and difference of sets $\mathcal{A}$ and $\mathcal{B}$, respectively. $\mathcal{A} \subseteq \mathcal{B}$ means that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} + \mathcal{B}$. The set of all primes shall be denoted by $\mathbb{P}$, the greatest common divisor of integers $m$ and $n$ by $(m, n)$ and the power of a set $\mathcal{M}$ by $\tau(\mathcal{M})$.

The concept of the rank of a group $G$ as the cardinal number of a maximal linearly independent set of the group $G$ is well-known (see e. g. [7]); let us denote it by $\rho(G)$. A set $\mathcal{G} = (g_\delta)_{\delta \in \Delta}$ of non-zero elements of $G$ is called $D$-independent if for any finite subset $\{g_{\delta_1}, \ldots, g_{\delta_n}\}$ of $\mathcal{G}$ a relation

\[
k_1 \cdot g_{\delta_1} + k_2 \cdot g_{\delta_2} + \ldots + k_n \cdot g_{\delta_n} = 0\]

with integers $k_1, k_2, \ldots, k_n$ implies $k_i \cdot g_{\delta_i} = 0$ ($i = 1, 2, \ldots, n$). By the $D$-rank of a group $G$ we understand the cardinal number of a maximal $D$-independent set of the group $G$ containing only elements of infinite or prime power orders; we shall denote it by $\rho_D(G)$ (for properties of the $D$-rank of an abelian group thus defined see [2]). Thus, especially, the equality $\rho_D(G) = \rho(G)$ follows for a torsion-free $G$.

By an elementary group we understand a group in which all non-zero elements are of a prime order. A group $G^*$ is said to be divisible if $pG^* = G^*$ holds for any $p \in \mathbb{P}$, or alternately, if every equation of the form (1,1) admits a solution in $G^*$ for any $g \in G$ and any $p \in \mathbb{P}$ (see [6]). A group having no non-zero divisible subgroup is called reduced. Every abelian group can be embedded in a so-called divisible closure $\overline{G}$, i. e. in a minimal divisible group $\overline{G}$ containing $G$, which is unique up to isomorphic extensions of the identical isomorphism of the group $G$ (see e. g. A. G. KUROŠ [7]). For any non-zero element

\(^1\) However, the $\varphi$-closure thus defined does not have the property $\varphi(\varphi(G)) = \varphi(G)$ usually demanded in the abstract concept of a closure relation (compare e. g. BIRKHOFF [1]).
There exists a positive integer \( n \) such that \( n \cdot g = 0 \), \( n \cdot g \in G \) (see e. g. Hilfssatz 2 in [3]); thus, especially, \( r_D(G) = r_D(\langle g \rangle) \). On the other hand, any divisible group \( G^* \) containing \( G \) each element of which has the mentioned property is obviously a divisible closure of the group \( G \); especially, if \( r_D(G) \) is finite and \( G^* \) is a divisible group for which the relations \( G \subseteq G^* \) and \( r_D(G) = r_D(G^*) \) hold, then \( G^* \) is a divisible closure of the group \( G \).

Let us, moreover, add that by the characteristic \( \chi(g) \) of an element \( g \in G \) in the group \( G \) we shall understand the sequence

\[
(1.2) \quad \chi(g) = (k_1, k_2, \ldots, k_i, \ldots),
\]

where \( k_i \) (\( i = 1, 2, \ldots \)) is the maximal non-negative integer \( n \) for which the equation

\[
(1.3) \quad p_i^n \cdot x = g
\]

is solvable in \( G \), if such an \( n \) exists and \( k_i = \infty \) otherwise, i. e. if the equation \( (1.3) \) admits a solution in \( G \) for any positive integer \( n \) \((p_1 < p_2 < \ldots < p_i < \ldots \) are assumed to be all primes).

2. THE FRATTINI SUBGROUP OF AN ABELIAN GROUP

First of all let us prove several lemmas from which Theorem 1 will immediately follow.

**Lemma 1.** Let \( G \) be an abelian group; then the relation

\[
(2.1) \quad \Phi(G) \subseteq pG
\]

holds for every prime \( p \).

**Proof.** The relation \( (2.1) \) is trivial if \( pG = G \). In the contrary case the quotient group \( \tilde{G} = G/pG \) is a non-zero elementary group; therefore evidently \( \Phi(\tilde{G}) = 0 \). Thus there exist maximal subgroups \( \tilde{M}_\delta \) of the group \( \tilde{G} \) \((\delta \in \Delta)\) such that

\[
(2.2) \quad \bigcap_{\delta \in \Delta} \tilde{M}_\delta = 0.
\]

The corresponding subgroups \( M_\delta, pG \subseteq M_\delta \subseteq G \), are maximal in \( G \) and by \( (2.2) \) we have \( \bigcap_{\delta \in \Delta} M_\delta = pG \). This implies the relation \( (2.1) \).

From Lemma 1, one can easily deduce

**Lemma 2.** For an abelian group \( G \) there holds the relation

\[
\Phi(G) \subseteq \bigcap_{\nu \in \Pi} pG.
\]

We proceed to prove

**Lemma 3.** For an abelian group \( G \) there holds the relation

\[
\Phi(G) \supseteq \bigcap_{\nu \in \Pi} pG.
\]
Proof. Suppose
\[ g_0 \in \bigcap_{p \in \Pi} pG \]
and let \( M \) be an maximal subgroup of \( G \) not containing the element \( g_0 \). Hence
\[ \{M \cup \{g_0\}\} = G \]
and there exists a prime \( q \) such that
\[ q \cdot g_0 \in M. \]
According to (2,3) we can find an element \( g^* \) such that
\[ q \cdot g^* = g_0, \]
which by (2,4) can be written in the form
\[ g^* = m + k \cdot g_0, \quad m \in M, \quad 0 < k < q. \]
Thus by (2,6)
\[ g_0 = q \cdot m + kq \cdot g_0; \]
according to (2,5) we immediately deduce that \( g_0 \in M \) which contradicts with our assumption; and the lemma follows.\(^3\)

Lemma 2 and 3 imply

**Theorem 1.**\(^3\) The Frattini subgroup of an abelian group \( G \) is of the form
\[ \Phi(G) = \bigcap_{p \in \Pi} pG. \]

An immediate consequence of Theorem 1 is the following corollary which characterises divisible groups:

**Corollary 1.**\(^3\) An abelian group is divisible if and only if \( \Phi(G) = G \).

Remark 1. If we take into account the alternative definition of the Frattini subgroups by means of generating systems mentioned in § 1 we obtain easily the following result of [4]: A necessary and sufficient condition for a non-zero group \( G \) to be divisible is that every generating system \( \mathcal{G} \) of \( G \) is strongly reducible (i. e. \( \mathcal{G} \setminus \{g\} \) is again a generating system of \( G \) for any element \( g \in \mathcal{G} \)).

The following implications are easily verified:
\[ G = \sum_{\delta \in \Delta} G_{\delta} \Rightarrow pG = \sum_{\delta \in \Delta} pG_{\delta} \Rightarrow \bigcap_{p \in \Pi} pG = \sum_{\delta \in \Delta} \left( \bigcap_{p \in \Pi} pG_{\delta} \right); \]
obtaining the following result (which is, however, easily proved directly):

\[^{3}\) For the prime \( q \) there holds simply \( qG \subseteq M \), since the quotient group \( G/M \) is of the order \( q \), and therefore \( \bigcap_{p \in \Pi} pG \subseteq qG \subseteq M \).

\[^{3}\) The paper was already in the press when the author found that the assertions of Theorem 1 and Corollary 1 were mentioned already in the monograph of L. Fuchs, Abelian groups, Budapest 1958.
Theorem 2. Let $G = \sum_{\delta \in \Delta} G_{\delta}$ be a direct decomposition of a group $G$. Then $\Phi(G) = \sum_{\delta \in \Delta} \Phi(G_{\delta})$.

From Theorems 1 and 2 we also obtain

Corollary 2. If $G$ is a $p$-primary group, then $\Phi(G) = pG$. If $G$ is a torsion group, and $G = \sum_{p \in \Pi} G_p$ its direct decomposition with $p$-primary components $G_p$, then $\Phi(G) = \sum_{p \in \Pi} pG_p$ (some of the direct summands can, of course, be trivial).

Remark 2. Theorem 1 can also be formulated in the following equivalent forms: a) The Frattini subgroup $\Phi(G)$ of a group $G$ is the subgroup of those elements $g \in G$ for which the equation (1,1) is solvable in $G$ for any $p \in \Pi$.

b) The Frattini subgroup $\Phi(G)$ of a group $G$ is the subgroup of those elements $g \in G$ the characteristics (1,2) whose fulfil $k_i \neq 0$ for all $i = 1, 2, \ldots$

Remark 3. From the form of the Frattini subgroup $\Phi(G)$ of an abelian group $G$ we see readily that $\Phi(G)$ is fully invariant in the group $G$ (in contrary to the non-commutative case, see B. H. Neumann [9]). In general, even for a non-abelian group $G$ the relation

$$\Phi(G) \eta \subseteq \Phi(G \eta)$$

subsists for every homomorphism $\eta$ of that group $G$; equality is not always true (compare W. Gaschütz [5]). An example of the infinite cyclic group $G(\infty) = \{u\}$ and the natural homomorphism of this group with the kernel $\{p^2 \cdot u\}$ shows that equality $\Phi(G) \eta = \Phi(G \eta)$ is not generally true even if $G$ is abelian. Similarly we can see that the proper inclusion $G \subset K$ does not imply $\Phi(G) \subset \Phi(K)$ but only $\Phi(G) \supset \Phi(K)$. Of course, one can immediately deduce that $\Phi(G/\Phi(G)) = 0$ (compare Gaschütz [5]).

3. THE $\varphi$-CLOSURE OF AN ABELIAN GROUP

It is the purpose of this section to prove the main theorem asserting that every abelian group $G$ is the Frattini subgroup of a suitable abelian group; moreover, we shall prove that among all groups whose Frattini subgroups are isomorphic to $G$ exists a minimal group, unique up to isomorphism, the $\varphi$-closure of the group $G$. First of all we shall prove the following lemmas:

Lemma 4. Let $G$ be an abelian group and $p \in \Pi$. Then there exists a group $H_p$ which satisfies the following conditions:

(I) $pH_p = G$.

(II) For every non-zero element $h \in H_p$ there exists a positive integer $n^4$ such that $n \cdot h \neq 0$ and $n \cdot h \in G$.

4) In fact, either $n = p$ or $n = 1$. 

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Proof. Let us embed $G$ in a divisible closure $\overline{G}$: $G \subseteq \overline{G}$ (see § 1). The set of elements $h \in G$ with the property $p . h \in G$ obviously forms a subgroup of the group $G$; let us denote it by $H_p$. Now, $H_p$ satisfies the conditions (I) (every equation of the form $(1,1)$ with $g \in G$ is solvable in $\overline{G}$) and (II) (indeed this is true for every element of $\overline{G}$).

Lemma 5. Let $G \subseteq A$ be abelian groups and $\overline{A}$ a divisible closure of $A$; let $p \in \Pi$. Denote by $A_p^*$ the subgroup of the group $\overline{A}$ consisting of all elements $\bar{a} \in \overline{A}$ such that $p . \bar{a} \in G$.

a) If $pA \supseteq G$, then $A_p^* \subseteq A$.
b) If $pA = G$, then $A_p^* = A$.

Proof. First we shall prove the proposition a). Let $a^*$ be an arbitrary element of the group $A_p^*$; then, $p . a^* = g^*, g^* \in G$. According to our assumption there exists an element $a \in A$ such that $p . a = g^*$. Thus $p . (a^* - a) = 0$; since $a^* - a \in \overline{A}$, necessarily there exists a positive integer $n$ with

$$n \cdot (a^* - a) = 0, \quad n \cdot (a^* - a) \in A.$$  

Hence it follows that $(n, p) = 1$ and consequently $a^* - a \in A$. We obtain $a^* \in A$, i.e. $A_p^* \subseteq A$.

Now, the proposition b) follows immediately, since $pA = G$ implies $A \subseteq A_p^*$.

Remark 4. Let $\overline{G}$ be a divisible closure of a group $G$. On the basis of Lemma 5 one can easily see that the group $H$ satisfying $H \subseteq G$ and $pH = G$ is unique.

Lemma 6. Let $G$ and $H_p$ be abelian groups satisfying the equality (I). Then the condition (II) is equivalent to

(II') There exists no proper subgroup $H'_p \subset H_p$ such that $pH'_p = G$.

Proof. (II) $\Rightarrow$ (II'). Assume (II); any divisible closure $\overline{H}_p$ of the group $H_p$ is also a divisible closure of the group $G$ itself. The condition (II') now follows by Lemma 5 (resp. Remark 4).

(II') $\Rightarrow$ (II). We shall give an indirect proof. Let $h_0 \neq 0$ be an element of $H_p$ such that $\{h_0\} \cap G = 0$. Embed the group $H_p$ in a divisible closure $\overline{H}_p$:

$$G \subseteq H_p \subseteq \overline{H}_p.$$  

Let $\overline{G}$ be a divisible closure of the group $G$, so that

$$G \subseteq \overline{G} \subseteq H_p;$$

clearly, $h_0 \notin \overline{G}$. The set $H'_p$ of those elements $\overline{g} \in \overline{G}$, for which $p \cdot \overline{g} \in G$, forms a group satisfying obviously the relation (3,1); further, Lemma 5 implies that $H'_p$ is a (proper) subgroup of $H_p$.

By similar arguments as in the proof of (II') $\Rightarrow$ (II), Lemma 6, we easily verify
Lemma 7. Let $G$ and $A$ be abelian groups such that $pA \supseteq G$, $p \in \Pi$. Then there exists a subgroup $H_p \subseteq A$ satisfying (I) and (II).

Theorem 3. Let $G$ be an abelian group and $p \in \Pi$. Then the following propositions hold:

a) Groups satisfying the conditions (I) and (II) are minimal in the class of groups with (I). If $A$ is a group such that $pA = G$, then there exists a subgroup $H_p \subseteq A$ satisfying the conditions (I) and (II), i.e. $pH_p \equiv G$ already holds for every proper subgroup $H'_p \subseteq H_p$.

b) A group is defined by the conditions (I) and (II) uniquely up to isomorphism. If $H_p$ and $K_p$ are different groups with the properties (I) and (II) (where, eventually, instead of $H_p$ write $K_p$), then there exists an isomorphism between $H_p$ and $K_p$ which is an extension of the identical isomorphism of the group $G$.

Proof. The proposition a) is an easy consequence of Lemmas 6 and 7.

Thus, we shall only prove the proposition b). Divisible closures $\bar{H}_p$ and $\bar{K}_p$ of the groups $H_p$ and $K_p$ respectively, are also divisible closures of the group $G$. Now, an isomorphism exists between the groups $\bar{H}_p$ and $\bar{K}_p$ which is an extension of the identical isomorphism of the group $G$; according to Lemma 5 this isomorphism carries $H_p$ onto $K_p$, as desired.

Remark 5. If $G$ is a torsion-free group one can readily see that there exists a torsion-free group $H_p$ satisfying (I); condition (I) determines such a group uniquely (up to isomorphism), since (I) implies (II) for torsion-free groups. Furthermore, it is easily shown that if $G$ is finite then the group $H_p$ of Lemma 4 is also finite. In general, there holds the equality $r_p(H_p) = r_p(G)$ for groups $G$ and $H_p$ with properties (I) and (II).

Now we can formulate the main theorem.

Theorem 4. Let $G$ be an abelian group. Then there exists a group $H$ satisfying the conditions (II) (with $H$ instead of $H_p$) and

$$\Phi(H) = G.$$ (III)

Proof. In fact, let us embed the group $G$ in a divisible closure $\bar{G}$; let $H_p$ be the uniquely defined subgroups of $\bar{H}$ for which

$$pH_p = G \quad (\text{for every } p \in \Pi).$$

Putting $H = \{H_p\}_{p \in \Pi}$ we have obviously $G \subseteq H \subseteq \bar{G}$. Since $G \subseteq H_p \subseteq H$ for every $p \in \Pi$, we obtain $G \subseteq pH$ for every $p \in \Pi$, i.e.

$$\Phi(H) = \bigcap_{p \in \Pi} pH \supseteq G.$$ (3.2)

Now we shall prove the converse inclusion. In fact, the quotient group $H/G$ is a direct sum of the $p$-primary (elementary, in fact) groups $H_p/G$ and therefore the relation

$$H_p \cap \{H_p\}_{p \in \Pi} = G, \quad \text{where} \quad \Pi_0 = \Pi \setminus \{p_0\}$$ (3.3)
holds for every $p_0 \in \Pi$. Every element $h \in H$ can be expressed formally in the form of an infinite sum

$$h = \sum_{q \in \Pi} h_q, \quad h_q \in H_q,$$

where $h_q = 0$ for all but a finite number of $q \in \Pi$. The expression (3.4) is not unique; if

$$h = \sum_{q \in \Pi} h'_q, \quad h'_q \in H_q,$$

is another expression of the element $h \in H$, then (3.3) implies

$$h'_q - h_q \in G \quad \text{for every} \quad q \in \Pi.$$

Thus, an element $h^*$ of the group $pH$ can be written in the form

$$h^* = p \cdot h = \sum_{q \in \Pi} p \cdot h_q = \sum_{q \in \Pi} h_q^* \in H_q \quad \text{and} \quad h^*_p \in G$$

for every $p \in \Pi$. The relations (3.5) and (3.6) then immediately show that every element of $\Phi(H) = \bigcap_{p \in \Pi} pH$ belongs to $G$; this assertion and the relation (3.2) together imply the desired equality (III). The validity of (II) is, of course, obvious.

Lemma 8. Let $G \subseteq A$ be abelian groups and $A$ a divisible closure of $A$. Let us denote by $A^*$ the subgroup of the group $A$ of those elements $a \in A$ that $n \cdot a \in G$ for a suitable square-free positive integer $n$.

a) If $\Phi(A) \supseteq G$, then $A^* \subseteq A$.

b) If there exists for each element $a \in A$ a positive integer $t$ such that $t \cdot a \in G$ and if $\Phi(A) = G$, then $A^* = A$.

Proof. First, one can readily see that $A^*$ is indeed a subgroup of the group $A$. In order to prove the proposition a) we recall the following relation

$$G \subseteq \Phi(A) \subseteq pA \subseteq A$$

which holds by Theorem 1 for every $p \in \Pi$. Thus, in view of Lemma 5, every element $a \in A$ such that $p \cdot a \in G$ for some $p \in \Pi$ belongs to $A$.

Now, let $a^*$ be an arbitrary element of $A^*$; consequently, there exists a square-free positive integer $n$ such that $n \cdot a^* = g, g \in G$. Let

$$n = p_1 p_2 \ldots p_m,$$

where $p_i$ are different primes. Put

$$r_i = p_1 p_2 \ldots p_{i-1} p_{i+1} \ldots p_m \quad \text{for} \quad i = 1, 2, \ldots, m.$$ 

It follows that $(r_1, r_2, \ldots, r_m) = 1$, i.e. there exist integers $s_1, s_2, \ldots, s_m$ such that $\sum_{i=1}^m s_i r_i = 1$. Since, according to the above consideration, $n \cdot a^* = p_i \cdot (r_i \cdot a^*) \in G$ and therefore

$$r_i \cdot a^* = a_i \quad a_i \in A \quad \text{for} \quad i = 1, 2, \ldots, m,$$

Especially, if $\Phi(A) = G$ and if $A$ satisfies the condition (II) (with $A$ in place of $H_p$) then $A^* = A$. 

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we obtain the equality
\[ a^* = \sum_{i=1}^{m} s_i r_i \cdot a^* = \sum_{i=1}^{m} s_i \cdot a_i; \]
consequently \( a^* \in A \), i. e. \( A^* \subseteq A \).

In order to prove the statement b) it is therefore sufficient to show that \( A^* \supseteq A \). Assume the contrary, that this inclusion is not true, i. e. that there exists an element \( a \in A \) and a positive integer \( t \) such that
\[ t = q^2 \cdot t_0 \quad \text{for some} \quad q \in \Pi \]
and
\[ t \cdot a \in G, \quad \text{but} \quad qt_0 \cdot a \notin G. \]

We shall now prove that the element \( qt_0 \cdot a \) belongs to \( \Phi(A) \): First, obviously,
\[ qt_0 \cdot a \in QA. \]
Let \( p \) be an arbitrary prime, \( p \neq q \); then there exist integers \( r, s \) such that
\[ rp + sq = 1. \]
Further, by (3,7) there exists an element \( a_p \in A \) for which
\[ p \cdot a_p = st \cdot a. \]
It follows then for the element \( rqt_0 \cdot a + a_p \in A \) by (3,12), (3,8) and (3,11) that
\[ p \cdot (rqt_0 \cdot a + a_p) = prqt_0 \cdot a + sq^2 t_0 \cdot a = prqt_0 \cdot a + (1 - pr) qt_0 \cdot a = qt_0 \cdot a, \]
and hence
\[ qt_0 \cdot a \in pA \quad \text{for every} \quad p \in \Pi, \quad p \neq q. \]
The relation (3,13) together with (3,10) imply in view of Theorem 1 that the element \( qt_0 \cdot a \) belongs to \( \Phi(A) \), in contradiction to (3,9) (for \( G = \Phi(A) \) by our assumption). This completes the proof of the lemma.

Remark 6. The assumption concerning the existence of the integer \( t \) in Lemma 8 b) cannot be omitted (in the related proposition b) of Lemma 5 the existence of such an integer already follows from the assumption \( \pi A = G \): Let \( G = R^+ \) and \( A = R^+ + G(\infty) \), where \( R^+ \) is the additive group of all rational numbers and \( G(\infty) \) the infinite cyclic group. In fact, \( \Phi(A) = R^+ = G \), but \( A^* = R^+ \neq A \).

Remark 7. Let \( \overline{G} \) be a divisible closure of a group \( G \). In view of Lemma 8 b) it is easy to see that the group \( H \) satisfying \( H \subseteq \overline{G} \) and \( \Phi(H) = G \) is unique. The group \( H \) is, in fact, precisely the subgroup of those elements \( \overline{g} \in \overline{G} \) for which there exist square-free integer \( n \) such that \( n \cdot \overline{g} \in G \).

Lemma 9. Let \( G \) and \( H \) be abelian groups satisfying the equality (III). Then the condition (II) (with \( H \) instead of \( H_p \)) is equivalent to the following
(3,14) There exists no proper subgroup \( H' \subset H \) such that 
\[ \Phi(H') = G. \]

Proof. \((\Pi) \Rightarrow (\Pi^{(III)})\). This follows immediately on the basis of Lemma 8 (resp. Remark 7) in the same way as the assertion \((\Pi) \Rightarrow (\Pi)\) of Lemma 6.

\((\Pi^{(III)}) \Rightarrow (\Pi)\). Let us follow again a similar line as in the proof of Lemma 6. Let \( h_0 \in H \) be a non-zero element such that \( \{h_0\} \cap G = 0 \). Embed the group \( H \) in a divisible closure \( H \) and consider a divisible closure \( \bar{G} \) of the group \( G \) such that \( G \subseteq \bar{G} \subseteq H \). Repeating the construction from the proof of Theorem 4 we obtain the group \( \bar{G} \) satisfying \( \bar{G} \subseteq \bar{G} \) and \( \Phi(\bar{G}) = G. \)

Since \( h_0 \) non \( \epsilon \) \( H \) and thus \( H' \) is in view of Lemma 8 a proper subgroup of the group \( H \) with property \((3,14)\), as desired.

Similar consideration yield

Lemma 10. Let \( G \) and \( A \) be abelian groups such that \( \Phi(A) \supsetneq G \). Then there exists a subgroup \( H \subseteq A \) satisfying \((\Pi)\) and \((\Pi)\) (with \( H \) instead of \( H_p \)).

Now, the assertions of Lemmas 8, 9 and 10 (resp. Remark 7) immediately imply the following appendix to Theorem 4.

Appendix to Theorem 4. a) Groups satisfying the conditions \((\Pi)\) and \((\Pi)\) are minimal in the class of groups with \((\Pi)\). If \( A \) is a group such that \( \Phi(A) = G \), then there exists a subgroup \( H \subseteq A \) satisfying the conditions \((\Pi)\) and \((\Pi)\) (with \( H \) instead of \( H_p \)), i.e. \( \Phi(H') = G \) holds already for each proper subgroup \( H' \subset H \).

b) A group is defined by the conditions \((\Pi)\) and \((\Pi)\) uniquely up to isomorphism. If \( H \) and \( K \) are different groups with the properties \((\Pi)\) and \((\Pi)\) (with \( H \) or \( K \) instead of \( H_p \)), then there exists an isomorphism between \( H \) and \( K \) which is an extension of the identical isomorphism of the group \( G \).

The group defined by a given group \( G \) in this manner (unique up to isomorphism) is said to be the \( \Phi \)-closure of the group \( G \); we shall denote it by \( \Phi(G) \).

Theorem 4 implies directly

Corollary 3. The \( \Phi \)-closure of a torsion (resp. torsion-free) group is a torsion (resp. torsion-free) group also. The \( \Phi \)-closure of a finite group is finite. The \( D \)-rank of a group \( G \) and that of its \( \Phi \)-closure \( \Phi(G) \) are equal, \( r_D(\Phi(G)) = r_D(G) \).

If \( r_D(G) < \aleph_0 \) and \( \Phi(H) = G \) with \( r_D(H) = r_D(G) \) is fulfilled for a group \( H \), then \( H = \Phi(G) \).

We shall conclude this paragraph with the following theorem concerning the \( \Phi \)-closures of abelian groups.

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4) Alternatively, \( H' \) is the subgroup of \( \bar{G} \) of all those elements \( \bar{g} \in \bar{G} \) that are solutions of equations \( n \cdot x = g \), where \( g \in G \) and \( n \) is a square-free integer (see Remark 7).

7) In contradistinction to the Frattini subgroup \( \Phi(G) \) of a group \( G \) being an unique subgroup of \( G \), \( \Phi(G) \) is determined up to isomorphism only; thus, the equality \( H = \Phi(G) \) means precisely that \( H \) satisfies the conditions \((\Pi)\) and \((\Pi)\) (with \( H \) instead of \( H_p \)).
Theorem 5.  

a) \( \Phi(\varphi(G)) = G \).  

b) There holds \( \varphi(\Phi(G)) \subseteq G \) up to isomorphism.  

c) If \( H \leq G \), then \( \varphi(H) \subseteq \varphi(G) \) up to isomorphism.  

d) If \( G = \sum_{\delta \in A} G_{\delta} \), then \( \varphi(G) \cong \sum_{\delta \in A} \varphi(G_{\delta}) \).  

Proof. The statement a) is trivial. On the basis of Lemma 10 and Appendix to Theorem 4 we deduce easily also b) and c).  

We shall prove d). Let \( \bar{G} = \sum_{\delta \in A} \bar{G}_{\delta} \) be a divisible closure of the group \( G \), where \( G_{\delta} \) are divisible closures of the groups \( G_{\delta} \). Let \( H_{\delta} \) be the \( \varphi \)-closure of the group \( G_{\delta} \) for which \( H_{\delta} \subseteq \bar{G}_{\delta}, \delta \in A \). In view of Theorem 2, \( G \) is the Frattini subgroup of the group \( \sum_{\delta \in A} H_{\delta} \). Since \( \sum_{\delta \in A} H_{\delta} \subseteq G \), according to Appendix to Theorem 4 (resp. Remark 7) we conclude that \( \sum_{\delta \in A} H_{\delta} \) is a \( \varphi \)-closure of the group \( G \).

4. SOME APPLICATIONS

A) By the descending Frattini series (\( \Phi \)-series) of a group \( G \) we shall mean the descending series  

\[ G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_x \supseteq \ldots \supseteq G_\tau, \]

where \( G_x = \Phi(G_{x-1}) \) for isolated ordinals \( x \), \( G_x = \bigcap_{\delta < x} G_{\delta} \) for a limit \( x \) and \( \tau \) is the least ordinal number such that \( \Phi(G_\tau) = G_\tau \). According to Corollary 1 \( G_\tau \) is divisible (\( G_\tau \) is obviously the maximal divisible subgroup of the group \( G \)); thus, the \( \Phi \)-series of \( G \) ends at the trivial group if and only if \( G \) is reduced. We shall refer to the ordinal \( \tau \) as the \( \Phi \)-length of the group \( G \). Let us observe that \( \Phi(G_x/G_{x+1}) = 0 \) for every \( x < \tau \). An element \( g \in G \) belongs to \( G_n \) (for a positive integer \( n \)) if and only if \( k_i \mid n \) is fulfilled in its characteristic \((1,2)\) for each index \( i \); further, \( g \in G_\omega \) \(^{10}\) is equivalent to the proposition that \( k_i = \infty \) in \((1,2)\) for each \( i \).  

Consequently, the \( \Phi \)-length of a torsion-free group \( G \) is \( \leq \omega \); a necessary and sufficient condition for \( G_\omega = 0 \) is that \( G \) is reduced. Especially, for torsion-free groups \( R \) of rank 1 (i.e. for non-zero subgroups of the additive group \( R^+ \) of all rational numbers) we easily deduce:  

a) \( \Phi(R) = 0 \) if and only if the type corresponding to the group (i.e. the class of all characteristics \((1,2)\) differing one from another only for a finite number of }
components $k_i$ that are different from $\infty$) consists of characteristics $(1,2)$ with $k_i = 0$ for an infinite number of indices $i$.

$\beta$) If $\Phi(R_1) \cong \Phi(R_2)$, then $R_1 \cong R_2$.

$\gamma$) Let us denote by $\mathcal{K}$ the class of all (non-isomorphic) torsion-free groups of rank 1 and by $\mathcal{K}_0$ the subclass of all groups of $\Phi$-length 1. Thus, by $\alpha$) and $\beta$) $\Phi$ is a correspondence which carries the set $\mathcal{K} \cup \{0\}$ onto itself; moreover, the correspondence $\Phi$ is one-to-one between $(\mathcal{K} \setminus \mathcal{K}_0) \cup \{0\}$ and $\mathcal{K} \cup \{0\}$.

$\delta$) The $\Phi$-length of a group $R$ is equal to zero if and only if $R \cong R^\perp$. A necessary and sufficient condition for a group $R$ to have a finite non-zero $\Phi$-length is that the type $r$ corresponding to the group $R$ satisfies the following condition: There exists a positive integer $N$ such that $k_i \leq N$ is true for an infinite number of indices $i$ in each characteristic $(1,2)$ belonging to $r$.

$\varepsilon$) In the contrary case the $\Phi$-length of a group $R$ is equal to $\omega$:

$$R = R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots \supset R_\infty = 0.$$ 

All the groups $R_n (n = 1, 2, \ldots)$ are isomorphic if and only if the type corresponding to the group $R$ consists of characteristics $(1,2)$ with $k_i = \infty$ for only a finite number of indices $i$.

For $p$-primary reduced groups, of course, the concepts of $\Phi$-series and the $\Phi$-length coincide with the concepts of Ulm’s series and the length of the $p$-primary group (see e. g. I. Kaplansky [6]).

B) Consider the ascending series

$$(4,1) \quad G = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_n \subseteq \cdots \subseteq G_\omega,$$

where $G_n = \varphi(G_{n-1})$ and $G_\omega = \bigcup_{n<\omega} G_n$. If $(1,2)$ is the characteristic of an element $g \in G_n$ in the group $G_n$, then evidently

$$(k_1 + m, k_2 + m, \ldots, k_i + m, \ldots)$$

is the characteristic of this element $g$ in the group $G_{n+m}$ (of course, $\infty + m = \infty$). Hence, $G_\omega$ is necessarily a divisible group, i. e. $\varphi(G_\omega) = G_\omega$. It is easy to see that $G_\omega$ is the divisible closure of the group $G$. We deduce further that the $\varphi$-series $(4,1)$ of a group $G$ is either strictly ascending (and its $\varphi$-length is $\omega$) or $G$ is divisible (and then, clearly, $G = G_n = G_\omega$ for $n = 1, 2, \ldots$).

C) Let us apply this to the study of generating systems of abelian groups. If $\mathcal{G}$ is an irreducible generating system of a group $G$ (for terminology we refer to [4]), then necessarily $\mathcal{G} \cap \Phi(G) = \emptyset$. Moreover, any linear combination of the form

$$k_1 \cdot g_1 + k_2 \cdot g_2 + \cdots + k_n \cdot g_n$$

where $g_i \in \mathcal{G}$ ($i = 1, 2, \ldots, n$) and $k_i = \pm 1$ at least for one index $i$, does not lie in $\Phi(G)$. 

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On the basis of the preceding consideration we can easily prove, that a group 
\( G = D + A \), where \( D \) is a divisible group and \( A \) a group possessing a generating system \( \mathfrak{A} \) with the property

\[(4,2) \quad m(\mathfrak{A}) < m(D)\]

has no irreducible generating system. In fact, we see immediately that 
\( \Phi(G) \supset D \). If \( \mathfrak{G} \) is an arbitrary (infinite) generating system of the group \( G \), then there exists a proper subset \( \mathfrak{G}_0 \subset \mathfrak{G} \) such that \( \{\mathfrak{G}_0\} \supset A \) holds. Let \( g^* \) be an element of \( \mathfrak{G} \setminus \mathfrak{G}_0 \):

\[ g^* = d^* + a^*, \quad d^* \in D, \quad a^* \in A. \]

For suitable integers \( k_i \) and elements \( g_i^{(0)} \in \mathfrak{G}_0 \) \((i = 1, 2, \ldots, n)\) we thus have the relation

\[ a^* = k_1 \cdot g_1^{(0)} + k_2 \cdot g_2^{(0)} + \ldots + k_n \cdot g_n^{(0)}, \]

i.e.

\[ d^* = g^* - k_1 \cdot g_1^{(0)} - k_2 \cdot g_2^{(0)} - \ldots - k_n \cdot g_n^{(0)}; \]

consequently, according to our preceding consideration, \( \mathfrak{G} \) is not irreducible (for the coefficient by \( g^* \) equals 1 and \( d^* \in \Phi(G) \)).

The assumption (4,2) cannot be weakened: In the paper [4] there is shown that the \( p \)-primary group \( G(p^n) + \sum_{i=1}^{\infty} G_i(p) \), where \( G(p^n) \) is Prüfer's group of the type \( p^n \) and \( G_i(p) \) are cyclic groups of the order \( p \) \((i = 1, 2, \ldots)\), possesses an irreducible generating system. In a similar manner we can prove that the torsion-free group \( W = R^+ + \sum_{i=1}^{\infty} \{u_i\} \), where \( R^+ \) is the additive group of all rational numbers and \( \sum_{i=1}^{\infty} \{u_i\} \) the free abelian group with the basis \( u_1, u_2, \ldots, u_n, \ldots \), has an irreducible generating system: Let

\[ r_2, r_3, \ldots, r_n, \ldots, (n + 1) \cdot r_{n+1} = r_n \quad (n = 2, 3, \ldots) \]

be the familiar generating system and the defining relations of the group \( R^+ \); in the given group consider the set

\[ \mathfrak{WR} = (w_i)_{i=1,2,\ldots}, \]

where

\[ w_{3k-2} = r_{2k} + 3 \cdot u_{2k-1}, \]

\[ w_{3k-1} = 2 \cdot u_{2k-1} - (2k + 1) \cdot u_{2k} \]

and

\[ w_{2k} = r_{2k+1} + 2 \cdot u_{2k} \]

for \( k = 1, 2, \ldots \) It is easily shown that for \( k = 1, 2, \ldots \)

\[ u_{2k} = -6k(k + 1)(2k + 3) \cdot w_{3k+3} - 12k(k + 1) \cdot w_{3k+2} + \]

\[ + 8k(k + 1) \cdot w_{3k+1} + (3k + 2) \cdot w_{3k} + 3 \cdot w_{3k-1} - 2 \cdot w_{3k-2}, \]

\[ u_{2k-1} = (2k + 1) \cdot w_{3k} + 2 \cdot w_{3k-1} - w_{3k-2} \]
and 
\[ r_{2k} = w_{2k-2} - 3 \cdot u_{2k-1} \quad \text{(resp. } r_{2k+1} = w_{2k} - 2 \cdot u_{2k}). \]

Hence \( B \) is a generating system of the group \( W \) which is, evidently, irreducible (if the element \( w_{2k-2} \) resp. \( w_{2k-1}, w_{2k} \) is omitted, then the remaining set generates a proper subgroup of \( W \) not containing the element \( u_{2k-1} \) resp. \( u_{2k} \)).

Finally, let us add the following

Remark 8. If \( G \) is a generating system of a group \( G \), then the set \( G \setminus \Phi(G) \) does not necessarily generate the whole group \( G \) (in the case that \( G \) is divisible the set \( G \setminus \Phi(G) \) is even void). Hence, it follows that Satz 18* in W. Specht [10] is not true; if the group \( G/\Phi(G) \) is finitely generated, then \( G \) need not have the same property (it is enough to consider the subgroup \( W_n = R^+ + \sum_{i=1}^{n} \{u_i \} \) of the group \( W \) for a natural number \( n \); obviously, \( \Phi(W_n) = R^+ \)).

**Bibliography**

гая характеристика подгруппы $\Phi(G)$ в терминах т. наз. „необразующих“, т. е. тех элементов группы $G$, каждый из которых может быть удален из любой системы образующих группы $G$ без нарушения свойства являться системой образующих всей группы; $\Phi(G)$ — это именно подгруппа этих необразующих.

После вступительных замечаний исследует автор в § 2 структуру подгруппы Фраттини данной абелевой группы.

Теорема 1. Подгруппа Фраттини абелевой группы $G$ имеет форму

$$\Phi(G) = \bigcap_{p \in \Pi} pG,$$

где $\Pi$ — множество всех простых чисел.

Отсюда вытекает, в частности, дальнейшая характеристика полных абелевых групп (Следствие 1), из которой непосредственно получаем характеристику при помощи систем образующих, приведенную в [4] (Заметка 1).

Следствие 1. Абелева группа $G$ является полной тогда и только тогда, если $\Phi(G) = G$.

Для прямой суммы тогда справедлива

Теорема 2. Пусть $G = \sum_{\delta \in \Delta} G_\delta$ — прямое разложение группы $G$. Тогда

$$\Phi(G) = \sum_{\delta \in \Delta} \Phi(G_\delta).$$

Следствие 2 специфицирует полученные результаты для специальных классов групп и замечание 2 дает эквивалентное выражение теоремы 1 при помощи понятия характеристики элемента группы. Из формы подгруппы Фраттини $\Phi(G)$ группы $G$ сразу вытекает, что $\Phi(G)$ — вполне характеристическая в $G$ и что $\Phi(G/\Phi(G)) = 0$.

Следующий § 3 посвящен вопросу существования группы, подгруппа Фраттини которой изоморфна данной группе $G$. Проблема решена при помощи конструкции группы $H_p$, обладающих свойством $pH_p = G$, где $p$ — простое число (Теорема 3). Автор вводит понятие $\Phi$-замыкания и формулирует решение в теореме 4 и добавлении к этой теореме.

Теорема 4. Пусть $G$ — абелева группа. Тогда существует группа, выполняющая следующие условия:

(II) Для каждого ненулевого элемента $h \in H$ существует натуральное число $n$ такое, что $n \cdot h = 0$ и $n \cdot h \in G$ и

(III) $\Phi(H) = G$.

1) В дальнейшем под группой всегда разумеется абелева группа с аддитивной записью.

2) Во время подготовки к печати автор заметил, что утверждения теоремы 1 и следствия 1 находятся уже в монографии Я. Фукса, Абелевы группы, Будапешт 1958.
Добавление к теореме 4. а) Группы, выполняющие условия (II) и (III), являются минимальными в классе групп со свойством (III). Если $A$ — группа такая, что $\Phi(A) = G$, то существует подгруппа $H \subseteq A$, выполняющая условия (II) и (III), т. е. для каждой собственной подгруппы $H' \subseteq H$ справедливо уже $\Phi(H') = G$.

б) Условиями (II) и (III) группа определена однозначно с точностью до изоморфизма. Если $H$ и $K$ — различные группы со свойствами (II) и (III), то между $H$ и $K$ существует изоморфизм, продолжающий тождественный автоморфизм группы $G$.

Группу, определенную к данной группе $G$ таким способом (однозначно с точностью до изоморфизма), назовем $\psi$-замыканием группы и обозначим ее через $\Phi(G)$.

В следствии 3 показаны некоторые соотношения между $D$-рангами данной группы и ее $\psi$-замыкания. Следующая теорема показывает некоторые свойства $\psi$-замыкания группы $G$.

**Теорема 5.** а) $\Phi(\Phi(G)) = G$.

б) $\psi(\Phi(G)) \leq G$ справедливо с точностью до изоморфизма.\(^3\)

c) Если $H \subseteq G$, то $\psi(H) \subseteq \Phi(G)$ с точностью до изоморфизма.\(^3\)

d) Если $G = \sum_{\delta \in \Delta} G_\delta$, то $\Phi(G) \cong \sum_{\delta \in \Delta} \Phi(G_\delta)$.

Последний § 4 посвящен применению полученных результатов к изучению убывающих цепей Фраттини (которые в случае $p$-группы совпадают с ульмовскими цепями), возрастающих цепей Фраттини и к изучению систем образующих данной группы. При помощи понятия подгруппы Фраттини доказывается, напр., следующее утверждение: Группа $G = D + A$, где $D$ — полная группа и $A$ — группа, обладающая системой образующих со свойством

\[
\text{(4,2)} \quad m(\mathbb{N}) < m(D),
\]

не имеет неприводимую систему образующих.\(^5\) Одновременно показано на примере группы $W = R^+ + \sum_{i=1}^{\infty} \{u_i\}$, где $R^+ — аддитивная группа рациональных чисел и $\sum_{i=1}^{\infty} \{u_i\} —$ свободная абелева группа счетного ранга, что предположения этого утверждения нельзя ослабить.

\(^3\) Т. е. $\psi(\Phi(G))$ (или же $\Phi(H)$) изоморфна подгруппе данной группы $G$ (или же $\Phi(G)$) и этот изоморфизм продолжает, сверх того, тождественный автоморфизм группы $\Phi(G)$ (или же $H$).

\(^4\) Символом $m(\mathbb{N})$ обозначена мощность множества $\mathbb{N}$.

\(^5\) Система образующих группы $G$ называется неприводимой, если всякое собственное подмножество не является уже системой образующих этой группы $G$ (см. [4]).