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A THEOREM ON NORMAL SEMIGROUPS

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This paper contains a proof of a theorem concerning compact semigroups S, the idempotents of which are contained in the centre of S.

Let S be a Hausdorff compact semigroup. If a is an element ϵS we shall denote by $\Gamma(a) = \Gamma_1(a)$ the closure of the set $\{a, a^2, a^3, \ldots\}$, and by $\Gamma_n(a)$, $n \geqq 2$, the closure of $\{a^n, a^{n+1}, a^{n+2}, \ldots\}$. It is known that $\Gamma(a)$ contains a unique idempotent e. In fact, $\bigcap\limits_{n=1}^{\infty} \Gamma_n(a)$ is a (closed commutative) group containing e as a unity element. We shall say that *a belongs to e*. The set of all elements ϵS belonging to e will be denoted $K(e)$. Since every element ϵS belongs to one and only one idempotent e_α, S can be written as a class sum of disjoint sets $S = \bigcup\limits_{e_\alpha \epsilon E} K(e_\alpha)$, where $E = \{e_\alpha \mid \alpha \epsilon \Lambda\}$ is the set of all idempotents ϵS. In general $K(e_\alpha)$ need not be a semigroup. To each e_α there exists a unique maximal group $G(e_\alpha)$ such that $G(e_\alpha)$ is the greatest subgroup of S containing e_α as unity element. Clearly $G(e_\alpha) \subseteqq K(e_\alpha)$. The group $G(e_\alpha)$ is closed and we have $e_\alpha K(e_\alpha) = K(e_\alpha)\, e_\alpha = G(e_\alpha)$. An element $a \epsilon K(e_\alpha)$ is contained in $G(e_\alpha)$ if and only if $ae_\alpha = e_\alpha a = a$ holds. The detailed proofs of these statements can be found in the paper [1].

In the same paper I proved the following theorem:

If S is commutative and if a belongs to e_α and b belongs to e_β, then ab belongs to $e_\alpha e_\beta$.

In a letter to the author Professor Allen Shields (Ann Arbor, Michigan) raised the question whether this statement can be proved for a wider class of semigroups, especially for the so called normal semigroups. This note contains an affirmative answer to this question. In fact we shall prove somewhat more, namely, that the statement holds if the set of idempotents is contained in the centre of S.

Definition. *A semigroup S is called to be normal if for every $x \epsilon S$ we have $xS = Sx$.*

Lemma 1. *In a normal semigroup S the set of all idempotents is contained in the centre of S.*

Proof. Let x be any element $\in S$ and e an idempotent $\in S$. The relation $eS = Se$ implies that there exists an element $u \in S$ with $ex = ue$ and an element $v \in S$ with $xe = ev$. Now $ex = ue$ implies $exe = (ue)e = ue = ex$ and $xe = ev$ implies $exe = e(ev) = ev = xe$. Hence $ex = xe$ which proves our Lemma.

Remark. The converse of Lemma 1 need not hold. The semigroup $S = \{0, a_1, a_2, a_3\}$ with the multiplication table

	0	a_1	a_2	a_3
0	0	0	0	0
a_1	0	0	0	0
a_2	0	0	0	0
a_3	0	0	a_1	a_1

has a unique idempotent 0 which commutes with every element $\in S$, but $Sa_2 \neq a_2S$.

Lemma 2. *Let S be a Hausdorff compact semigroup and $a \in S$. Let $X = \{x_1, x_2, x_3, \ldots\}$, $Y = \{y_1, y_2, y_3, \ldots\}$ be two sequences of elements such that $a = x_n y_n$ holds for every integer $n \geq 1$. Then to every $\xi \in \overline{X}$ there exists an $\eta \in \overline{Y}$ such that $a = \xi\eta$.*

Proof. The proof follows indirectly. Suppose that such an η does not exist. This means: For every $\eta_\nu \in \overline{Y}$ we have $a \neq \xi\eta_\nu$. Find to every η_ν two neighbourhoods $U_\nu(a)$ and $V(\xi\eta_\nu)$ with $U_\nu(a) \cap V(\xi\eta_\nu) = \emptyset$. With respect to the continuity of the multiplication we can further find neighbourhoods $U_\nu(\xi)$, $U(\eta_\nu)$ such that $U_\nu(\xi)\,U(\eta_\nu) \subseteqq V(\xi\eta_\nu)$ holds. Hence $U_\nu(a) \cap U_\nu(\xi)\,U(\eta_\nu) = \emptyset$ and a non $\in U_\nu(\xi)\,U(\eta_\nu)$. Consider now the set $\bigcup_{\eta_\nu} U(\eta_\nu)$, where η_ν runs through all elements $\in \overline{Y}$. This constitutes a covering of the compact set $\overline{Y}$ by means of open sets. There exists therefore a finite covering such that $\overline{Y} \subseteqq \bigcup_{i=1}^{k} U(\eta_{\nu_i}) = Q$, and we have a non $\in \bigcup_{i=1}^{k} U_{\nu_i}(\xi) \,.\, U(\eta_{\nu_i})$. If now $U(\xi)$ is a neighbourhood of ξ such that $U(\xi) \subseteqq U_{\nu_1}(\xi) \cap \ldots \cap U_{\nu_k}(\xi)$ we have a non $\in U(\xi) \,.\, Q$. On the other side $U(\xi)$ contains at least one element $\in X$, say x_m. The element y_m (with the same index m) is contained in Q and we have therefore $a = x_m y_m \in U(\xi)\,Q$. This contradiction proves our Lemma.

In the following we retain the notations introduced at the beginning of the paper. Before proving the main theorem we first prove:

Lemma 3. *Let S be a Hausdorff compact semigroup in which every idempotent is contained in the centre of S. Let be $x \in K(e_1)$, $y \in K(e_2)$. Then xye_1e_2 is contained in the maximal group $G(e_1e_2)$.*

Proof. According to the assumption we have $e_1 \in \Gamma(x)$, $e_2 \in \Gamma(y)$, hence $e_1e_2 \in \Gamma(x) \cdot e_2$. If B is any subset of S and $b \in S$ it follows from the continuity of the multiplication $b \cdot \overline{B} \subseteqq \overline{bB}$. We have therefore $\Gamma(x)\, e_2 = \overline{\{x, x^2, x^3, \ldots\}}\, e_2 \subseteqq \subseteqq \overline{\{xe_2, x^2e_2, x^3e_2, \ldots\}} = \overline{\{(xe_2), (xe_2)^2, (xe_2)^3, \ldots\}} = \Gamma(xe_2)$. Hence $e_1e_2 \in \Gamma(xe_2)$, i. e. $xe_2 \in K(e_1e_2)$. By the same argument we get $ye_1 \in K(e_1e_2)$. Now, as remarked above, we have $K(e_1e_2) \cdot e_1e_2 = G(e_1e_2)$. Therefore the elements $xe_2 \cdot e_1e_2 = = xe_1e_2$ and $ye_1 \cdot e_1e_2 = ye_1e_2$ are elements of the group $G(e_1e_2)$ and so is their product $xe_1e_2 \cdot ye_1e_2 = xye_1e_2$. This proves Lemma 3.

Theorem. *Let S be a Hausdorff compact semigroup in which every idempotent is contained in the centre of S (in particular, a Hausdorff compact normal semigroup). If x belongs to the idempotent e_1, y belongs to the idempotent e_2, then xy belongs to the idempotent e_1e_2.*

Proof. We have to show that $x \in K(e_1)$, $y \in K(e_2)$ imply $xy \in K(e_1e_2)$. We prove it in three steps.

a) Suppose that $xy \in K(e)$, i. e. $e \in \Gamma(xy)$. We then have $e_1e_2e \in e_1e_2\Gamma(xy) \subseteqq \subseteqq \Gamma(xye_1e_2)$. According to Lemma 3 we have $\Gamma(xye_1e_2) \subseteqq G(e_1e_2)$. Since e_1e_2e is an idempotent and $G(e_1e_2)$ contains the unique idempotent e_1e_2, we have $e_1e_2e = e_1e_2$.

b) Consider the group $J = \bigcap_{n=1}^{\infty} \Gamma_n(xy) \subset G(e)$. Since $xy \cdot e \in J$, we have $xyJ = xy(eJ) = (xye)\, J = J$, hence $xyJ = J$. There exists therefore an element $a \in J$ with $xya = e$. Denote $ya = b$. The relation $e = xb$ implies $e = e^2 = = e(xb) = xeb = x(xb)\, b = x^2b^2$, and by induction $e = x^nb^n$ for every integer $n \geqq 1$. Now we use Lemma 2. According to this Lemma there exists to the element $e_1 \in \Gamma(x) = \overline{\{x, x^2, x^3, \ldots\}}$ an element $B \in \Gamma(b)$ such that $e = e_1B$. But then $ee_1 = (e_1B)\, e_1 = e_1B = e$, hence $ee_1 = e$.

Analogously $Jxy = J$ implies the existence of an element $c \in J$ with $cxy = e$. Denoting $d = cx$ we have $dy = e$, and $e = e^2 = e(dy) = dey = d(dy)\, y = d^2y^2$. By induction: $e = d^ny^n$ for every integer $n \geqq 1$. According to Lemma 2 we can find an element $D \in \Gamma(d)$ such that $e = De_2$. This gives finally $ee_2 = = (De_2)\, e_2 = De_2 = e$, hence $ee_2 = e$.

c) The relations $e_1e_2e = e_1e_2$ and $e = e_2e = e_1e$ imply together $e_1e_2 = = e_1e_2e = (e_1e)\,(e_2e) = e \cdot e = e$. This proves our Theorem.

Corollary 1. In a Hausdorff compact semigroup S in which every idempotent is contained in the centre of S the elements xy and yx belong to the same idempotent.

Corollary 2. Under the same assumption every set $K(e)$ is a semigroup.

References

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Резюме

ОБ ОДНОЙ ТЕОРЕМЕ, КАСАЮЩЕЙСЯ НОРМАЛЬНЫХ ПОЛУГРУП

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Пусть S — хаусдорфова бикомпактная полугруппа. Скажем, что $a \in S$ принадлежит к идемпотенту e, если e является (единственным) идемпотентом замыкания множества $\{a, a^2, a^3, \ldots\}$.

Полугруппу S называем нормальной, если для всякого $x \in S$ $xS = Sx$.

Целью настоящей заметки является доказательство следующего утверждения:

Если S — бикомпактная хаусдорфова полугруппа, идемпотенты которой содержатся в центре S (в частности, если S является нормальной), и если x принадлежит к идемпотенту e_1 и y принадлежит к идемпотенту e_2, то xy принадлежит к идемпотенту e_1e_2.