AN EXAMPLE CONCERNING COUNTABLY COMPACT SPACES

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In the present note a Hausdorff topological space P is constructed such that P is not countably compact and every open point-finite covering of P contains a finite subcovering.

A topological space is said to be countably compact if it contains no infinite discrete closed subset. In [2] and [3] the following theorem is proved:

Theorem 1. *Every point-finite open covering of a countably compact space contains a finite subcovering. If a space P is regular and if every point-finite open covering of P contains a finite subcovering, then P is countably compact.*

The following simple example is known of a T_1 - space P such that P is not countably compact and every point-finite open covering of P is finite. For an uncountable set P a topology is defined such that closed sets are precisely P, Φ and all countable sets. In [2] an example is given of a Hausdorff space R possessing the following two properties:

(a) *R contains a discrete closed subset of potency* $2^{2^{\aleph_0}}$,

(b) *If $\{U\}$ is an open covering of R and if the family $\{\overline{U}\}$ is point-finite, then the covering $\{U\}$ contains a finite subcovering.*

In the present note an example is given solving a problem raised in [2] and [3].

Example. *There exists a Hausdorff space P such that*

(1) *P contains an infinite closed discrete subset (i. e., P is not countably compact).*

(2) *Every point-finite open covering of P contains a finite subcovering.*

Construction. Let J be the interval $\{x; 0 \leqq x < 1\}$ of real numbers. Denoting by T the set of all countable ordinals, we order the set $S = T \times J$ lexicographically, i. e. $(\gamma, x) > (\delta, y)$ if and only if either $\gamma > \delta$ or $\gamma = \delta$ and $x > y$. The ordered set S with the order topology is a topological space which will also be denoted by S. Let N be the set of all positive integers. We define a topology for the set $P = S \cup N$ in the following manner. The space S is an open subspace of P. It is sufficient to define local bases at points of N. Let $K_n (n \in N)$ be the set of all real numbers of the form $i/2^n$, where i is odd integer and $1 \leqq i \leqq 2^n$. For $n \in N$, $\gamma \in T$ and an open subset G of the open interval $(0, 1)$ such that $K_n \subset G$, we define

$$U_n(G, \gamma) = (n) \cup \bigcup \{(\delta) \times G; \delta \in T , \delta > \gamma\} .$$

The family $\{U_n(G, \gamma)\}_{G,\gamma}$ is by definition a local base at the point n. It is easy to show that the topological space P is a Hausdorff space.

Now we shall prove that the space P satisfies the conditions (1) and (2). The subspace N is discrete and closed in P and therefore the condition (1) is satisfied. First we state two lemmas.

Lemma 1. *Let $\{G_n\}$ be a sequence of open subsets of the open interval* $(0, 1)$ *such that for each $n \in N$ there is* a *$k \in N$ with $K_k \subset G_n$. Then the sequence $\{G_n\}$ is not point-finite, i. e., there exists* a *$x \in (0,1)$ such that the set*

$$\{n; n \in N , \quad x \in G_n\}$$

is infinite.

Lemma 2. *The space S is countably compact.*

The proof of lemma 1 is quite elementary and may be left to the reader. Now we shall prove lemma 2. Assume that R is an infinite subset of S. Putting

$$J(\gamma) = \{x; x \in S, \quad (\gamma, 0) \leqq x \leqq (\gamma + 1, 0)\}$$

for $\gamma \in T$, we see that $J(\gamma)$ and the closed interval $\langle 0, 1 \rangle$ of real numbers are homeomorphic to each other. It follows that if for some $\gamma \in T$ the set $R \cap J(\gamma)$ is infinite, then R has an accumulation point in $J(\gamma)$. If for each $\gamma \in T$ the set $R \cap J(\gamma)$ is finite, then there exists an infinite set $T' \subset T$ such that

$$\gamma \in T' \Rightarrow J(\gamma) \cap R \neq \Phi .$$

It is well-known that the space T is countably compact. It follows that there exists an accumulation point δ of the set T' in the space T. Clearly the point $(\delta, 0)$ is an accumulation point of the set R. This completes the proof.

Proof of condition (2). Let $\mathfrak{A}$ be an open point-finite covering of the space P. By lemma 2, the space S is countably compact and therefore according to theorem 1, there is a finite family $\mathfrak{A}' \subset \mathfrak{A}$ such that

$$S \subset \bigcup \{A; A \in \mathfrak{A}'\} .$$

Consequently, to prove the condition (2), it is sufficient to show that the set

$$\mathfrak{A}_1 = \{A; A \in \mathfrak{A}, \quad A \cap N \neq \Phi\}$$

is finite. Suppose the contrary, that $\mathfrak{A}_1$ is infinite. Arranging a countably infinite subset of $\mathfrak{A}_1$ in a sequence $\{A_n; n \in N\}$ we choose a $k_n \in A_n \cap N$. The sets A_n are open and therefore we can choose points $\gamma_n \in T$ and sets G_n open in the open interval $(0, 1)$ such that $G_n \supset K_{k_n}$ and

$$U_n = U_{k_n}(G_n, \gamma_n) \subset A_n .$$

There exists a $\gamma \in T$ such that $\gamma_n < \gamma$ for $n \in N$. From lemma 1 we conclude that the sequence $\{U_n \cap J(\gamma)\}$ is not point-finite. It follows that the sequence $\{A_n\}$ is not point-finite and consequently, the family $\mathfrak{A}$ is not point-finite. But this is a contradiction. The proof of the condition (2) is complete.

Literature

[1] *R. Arens, J. Dugundji:* Remark on the concept of compactness. Portugaliae Math. *9* (1950), 141–143.

[2] *З. Фролик* (Z. Frolík): Обобщения компактности и свойства Линделефа (Summary: Generalisations of compact and Lindelöf spaces.) Czech. Math. Journal, *9* (1959), 172–217.

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Резюме

ПРИМЕР, КАСАЮЩИЙСЯ СЧЕТНО-КОМПАКТНЫХ ПРОСТРАНСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

В статье построено пространство Хаусдорфа P, имеющее следующие свойства:

(1) P не является счетно компактным, т. е., P содержит бесконечное замкнутое дискретное множество.

(2) Всякое точечно-конечное открытое покрытие пространства P содержит конечное покрытие.