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AN EXAMPLE CONCERNING COUNTABLY COMPACT SPACES

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In the present note a Hausdorff topological space $P$ is constructed such that $P$ is not countably compact and every open point-finite covering of $P$ contains a finite subcovering.

A topological space is said to be countably compact if it contains no infinite discrete closed subset. In [2] and [3] the following theorem is proved:

**Theorem 1.** Every point-finite open covering of a countably compact space contains a finite subcovering. If a space $P$ is regular and if every point-finite open covering of $P$ contains a finite subcovering, then $P$ is countably compact.

The following simple example is known of a $T_1$-space $P$ such that $P$ is not countably compact and every point-finite open covering of $P$ is finite. For an uncountable set $P$ a topology is defined such that closed sets are precisely $P$, $\Phi$ and all countable sets. In [2] an example is given of a Hausdorff space $R$ possessing the following two properties:

(a) $R$ contains a discrete closed subset of potency $2^{2^{\aleph_0}}$,

(b) If $\{U\}$ is an open covering of $R$ and if the family $\{\bar{U}\}$ is point-finite, then the covering $\{U\}$ contains a finite subcovering.

In the present note an example is given solving a problem raised in [2] and [3].

**Example.** There exists a Hausdorff space $P$ such that

(1) $P$ contains an infinite closed discrete subset (i.e., $P$ is not countably compact).

(2) Every point-finite open covering of $P$ contains a finite subcovering.

**Construction.** Let $J$ be the interval $\{x; 0 \leq x < 1\}$ of real numbers. Denoting by $T$ the set of all countable ordinals, we order the set $S = T \times J$ lexicographically, i.e. $(\gamma, x) > (\delta, y)$ if and only if either $\gamma > \delta$ or $\gamma = \delta$ and $x > y$. The ordered set $S$ with the order topology is a topological space which will also be denoted by $S$. Let $N$ be the set of all positive integers. We define a topology for the set $P = S \cup N$ in the following manner. The space $S$ is an
open subspace of $P$. It is sufficient to define local bases at points of $N$. Let $K_n (n \in N)$ be the set of all real numbers of the form $i/2^n$, where $i$ is odd integer and $1 \leq i \leq 2^n$. For $n \in N$, $\gamma \in T$ and an open subset $G$ of the open interval $(0, 1)$ such that $K_n \subset G$, we define

$$U_n (G, \gamma) = (n) \cup \bigcup \{ (\delta) \times G ; \delta \in T, \ \delta > \gamma \}.$$ 

The family $\{U_n (G, \gamma)\}_{G, \gamma}$ is by definition a local base at the point $n$. It is easy to show that the topological space $P$ is a Hausdorff space.

Now we shall prove that the space $P$ satisfies the conditions (1) and (2). The subspace $N$ is discrete and closed in $P$ and therefore the condition (1) is satisfied. First we state two lemmas.

**Lemma 1.** Let $\{G_n\}$ be a sequence of open subsets of the open interval $(0, 1)$ such that for each $n \in N$ there is a $k \in N$ with $K_k \subset G_n$. Then the sequence $\{G_n\}$ is not point-finite, i. e., there exists a $x \in (0, 1)$ such that the set

$$\{n ; n \in N, \ x \in G_n\}$$

is infinite.

**Lemma 2.** The space $S$ is countably compact.

The proof of lemma 1 is quite elementary and may be left to the reader. Now we shall prove lemma 2. Assume that $R$ is an infinite subset of $S$. Putting

$$J(\gamma) = \{x ; x \in S, \ (\gamma, 0) \leq x \leq (\gamma + 1, 0)\}$$

for $\gamma \in T$, we see that $J(\gamma)$ and the closed interval $[0, 1]$ of real numbers are homeomorphic to each other. It follows that if for some $\gamma \in T$ the set $R \cap J(\gamma)$ is infinite, then $R$ has an accumulation point in $J(\gamma)$. If for each $\gamma \in T$ the set $R \cap J(\gamma)$ is finite, then there exists an infinite set $T' \subset T$ such that

$$\gamma \in T' \Rightarrow J(\gamma) \cap R \neq \Phi.$$ 

It is well-known that the space $T$ is countably compact. It follows that there exists an accumulation point $\delta$ of the set $T'$ in the space $T$. Clearly the point $(\delta, 0)$ is an accumulation point of the set $R$. This completes the proof.

**Proof of condition (2).** Let $\mathcal{A}$ be an open point-finite covering of the space $P$. By lemma 2, the space $S$ is countably compact and therefore according to theorem 1, there is a finite family $\mathcal{A}' \subset \mathcal{A}$ such that

$$S \subset \bigcup \{A ; A \in \mathcal{A}'\}.$$ 

Consequently, to prove the condition (2), it is sufficient to show that the set

$$\mathcal{A}_1 = \{A ; A \in \mathcal{A}, \ A \cap N \neq \Phi\}$$

is finite. Suppose the contrary, that $\mathcal{A}_1$ is infinite. Arranging a countably infinite subset of $\mathcal{A}_1$ in a sequence $\{A_n ; n \in N\}$ we choose a $k_n \in A_n \cap N$. The sets $A_n$ are open and therefore we can choose points $\gamma_n \in T$ and sets $G_n$ open in the open interval $(0, 1)$ such that $G_n \supset K_{k_n}$ and

$$U_n = U_{k_n} (G_n, \gamma_n) \subset A_n.$$
There exists a $\gamma \in T$ such that $\gamma_n < \gamma$ for $n \in \mathbb{N}$. From lemma 1 we conclude that the sequence $\{U_n \cap J(\gamma)\}$ is not point-finite. It follows that the sequence $\{A_n\}$ is not point-finite and consequently, the family $\mathcal{A}$ is not point-finite. But this is a contradiction. The proof of the condition (2) is complete.

**Literature**


**Резюме**

ПРИМЕР, КАСАЮЩИЙСЯ СЧЕТНО-КОМПАКТНЫХ ПРОСТРАНСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolik), Прага

В статье построено пространство Хаусдорфа $P$, имеющее следующие свойства:

(1) $P$ не является счетно компактным, т. е., $P$ содержит бесконечное замкнутое дискретное множество.

(2) Всякое точечно-конечное открытое покрытие пространства $P$ содержит конечное покрытие.