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AN ESTIMATION FOR THE FIRST EXPONENTIAL FORMULA IN THE THEORY OF SEMI-GROUPS OF LINEAR OPERATIONS

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In this paper Hille's theorem concerning the "*first exponential formula*" in the theory of semi-groups of linear operations has been sharpened without imposing any other conditions than the original ones. Moreover a pair of convergence theorems similar to Butzer's have also been given.

1. Introduction. The main result of Chapter 9 of E. HILLE's comprehensive treatise [1] is known as the "*first exponential formula*" which is contained in Theorem 9.3.4 of the book. The principal object of this note is to reformulate the exponential formula in a more sharp form.

As in the chapter 9 of [1], denote by $\mathfrak{S} = \{T(\xi)\}$, $(\xi > 0)$, a one-parameter semi-group of linear operations on a complex Banach space X to itself so that $T(\xi_1 + \xi_2)\, x = T(\xi_1)[T(\xi_2)\, x]$ for all $\xi_1, \xi_2 > 0$ and all $x \in X$. Besides, a boundedness condition of the form $\|T(\xi)\| \leqq M < +\infty$ is assumed for $0 < \alpha \leqq \leqq \xi \leqq \max(\alpha + 1, 2\alpha)$, where M is in general depending upon $T(\xi)$ itself.

Denote $A_\eta = \dfrac{1}{\eta}[T(\eta) - I]$, of which the strong limit $A = \lim\limits_{\eta \to 0} A_\eta$ (whenever it exists) is known as the infinitesmal generator of $\mathfrak{S}$. Moreover, $\mu(\delta, x)$ is used to denote the rectified modulus of continuity of $T(\xi)\, x$ in a certain given interval $[\alpha, \beta] \subset (0, \infty)$, viz. $\mu(\delta, x) = \sup \|T(\xi_1)\, x - T(\xi_2)\, x\|$, the "sup" being taken over all ξ_1, ξ_2 with $\alpha \leqq \xi_1, \xi_2 \leqq \beta$ and $|\xi_1 - \xi_2| \leqq \delta$. Similarly we denote $\mu(\delta) = \sup \|T(\xi_1) - T(\xi_2)\|$ in case $T(\xi)$ is uniformly continuous for $\xi > 0$ and not merely strongly continuous.

Hille's theorem concerning the first exponential formula can be now sharpened to the following form (cf. loc. cit., p. 187):

Theorem 1. *If $T(\xi)$ is strongly continuous for $\xi > 0$, then for every $x \in X$ and every $\xi (0 < \alpha \leqq \xi \leqq \beta)$ and for $\eta > 0$ being small we have*

(1) $$\|\exp[(\xi - \alpha)\, A_\eta]\, T(\alpha)\, x - T(\xi)\, x\| \leqq \mu(\eta^{\frac{1}{3}}, x) + K \,.\, \eta^{\frac{1}{3}} \,.\, \|x\|\,,$$

where $K = K(\beta, M)$ is a positive constant independent of η. Moreover, if $T(\xi)$ is uniformly continuous for $\xi > 0$, then

$$\|\exp[(\xi - \alpha) A_\eta] T(\alpha) - T(\xi)\| \leqq \mu(\eta^{\frac{1}{3}}) + K \cdot \eta^{\frac{1}{3}} . \tag{2}$$

Particular mention should be made to the work of Butzer [2] in one of his recent papers, in which a quite sharp estimation for the left-hand side of (1) has been given under certain types of Lipschitz condition together with the uniform boundedness condition $\|T(\xi)\| \leqq M < + \infty$ $(0 < \xi < \infty)$ for $T(\xi)$. However, as will be seen from our proof of Theorem 1, it seems not quite easy to improve our estimates of (1) and (2) without imposing any further conditions upon $\{T(\xi)\}$.

2. A special proposition. As may be observed, Theorem 1 can be proved in a manner completely parallel to that of proving the following special proposition:

Let $f(s)$ be a continuous function defined on $0 \leqq s < \infty$ and satisfying the condition $|f(s)| \leqq M^{1+s}$ $(s \leqq 0)$ with $M \geqq 1$. For each fixed $s \geqq 0$, define

$$E_f(s) = e^{-st} \sum_{k=0}^{\infty} \frac{(st)^k}{k!} f\left(\frac{k}{t}\right), \quad (t > 0) . \tag{3}$$

Then for all sufficiently large t we have

$$|E_f(s) - f(s)| \leqq \omega_f^* \left(\left(\frac{1}{t}\right)^{\frac{1}{3}}\right) + K \cdot \left(\frac{1}{t}\right)^{\frac{1}{3}} , \tag{4}$$

where K is a positive constant independent of t, and $\omega_f^*(\delta) = \max |f(s_1) - f(s_2)|$ $(|s_1 - s_2| \leqq \delta)$ stands for the modulus of continuity of $f(u)$ as restricted to a certain neighborhood of $u = s$.

Actually (3) is a well-known singular series, of which the convergence property has already been investigated by several authors (see, for instance, G. Mirakyan [3], G. Szegö [4] and O. Szasz [5]). Here we are going to establish (4) under the much wider condition $|f(s)| \leqq M^{1+s}$.

Obviously it suffices to prove, for $s > 0$ the inequality:

$$S(t) \equiv e^{-st} \sum_{k=0}^{\infty} \left| f\left(\frac{k}{t}\right) - f(s) \right| \frac{(st)^k}{k} \leqq \omega_f^* \left(\left(\frac{1}{t}\right)^{\frac{1}{3}}\right) + K \cdot \left(\frac{1}{t}\right)^{\frac{1}{3}} .$$

Let us split the summation $S(t)$ as a sum of two parts

$$S(t) = e^{-st}(\Sigma' + \Sigma'') ,$$

the summations Σ' and Σ'' being extended over all k $(k = 0, 1, 2, \ldots)$ subject to the following conditions respectively

$$\Sigma' : |k - st| \leqq t^{2/3} , \quad \Sigma'' : |k - st| > t^{2/3} .$$

Accordingly, for the sake of comparison, we introduce a summation $\sum_k'''$ subject to the condition

$$\Sigma''': |k - st \,.\, M^{1/t}| > \tfrac{1}{2} t^{2/3} \,.$$

Notice that the inequality $M^{1/t} \leqq 1 + \frac{1}{t}\,(M-1)$ holds for $t \geqq 1$. Thus $st \leqq \leqq stM^{1/t} \leqq st + s(M-1)$; and we see that for all large t the summation Σ'' is included in Σ''', i. e. $\Sigma'' < \Sigma'''$.

It is also known that the following simple inequality (Lemma 9.3.2 of [1])

$$e^{-\omega} \sum_k{}^{*} \frac{\omega^k}{k!} < N^{-2} \,.\, \omega \tag{5}$$

holds for every $\omega > 0$, where Σ^* extends over all those values of k for which $|k - \omega| > N$. Thus by repeated application of (5) it is seen that for all large t we have the following estimates (in which A is freely used to denote a positive constant, not depending on t and not necessarily the same at each occurrence):

$$\begin{aligned} e^{-st} \sum{}'' &\leqq e^{-st} \sum{}'' \left(\left| f\left(\frac{k}{t}\right) \right| + |f(s)| \right) \frac{(st)^k}{k!} \leqq \\ &\leqq e^{-st} \sum{}'' \frac{1}{k!} M^{1+\frac{k}{t}} (st)^k + A e^{-st} \sum{}'' \frac{1}{k!} (st)^k \leqq \\ &\leqq M \,.\, e^{-st} \sum{}'' \frac{1}{k!} (stM^{1/t})^k + A \,.\, t^{-4/3} \,.\, (st) \leqq \\ &\leqq M \,.\, e^{-st} \sum{}''' \frac{1}{k!} (stM^{1/t})^k + A \,.\, \left(\frac{1}{t}\right)^{1/3} \leqq \\ &\leqq M \,.\, e^{-st} e^{stM^{1/t}} \left(\frac{1}{2}\, t^{2/3}\right)^{-2} (stM^{1/t}) + A \,.\, \left(\frac{1}{t}\right)^{1/3} \leqq \\ &\leqq M \,.\, e^{s(M-1)} \,.\, A \,.\, \left(\frac{1}{t}\right)^{1/3} + A \,.\, \left(\frac{1}{t}\right)^{1/3} \leqq A \,.\, \left(\frac{1}{t}\right)^{1/3} . \end{aligned}$$

On the other hand we easily find, for large t,

$$e^{-st} \sum{}' \leqq \omega_f^* \left(\left(\frac{1}{t}\right)^{\frac{1}{3}} \right) .$$

Hence the inequality (4) is proved.

3. Proof of Theorem 1. For proving the general theorem, it requires only to notice that

$$\begin{aligned} &\|\exp\,[(\xi - \alpha)\, A_\eta]\, T(\alpha)\, x - T(\xi)\, x\| \leqq \\ &\leqq e^{-st} \sum_{k=0}^{\infty} \frac{1}{k!} (st)^k \left\| \left[T\left(\alpha + \frac{k}{t}\right) - T(\alpha + s) \right] x \right\| , \end{aligned} \tag{6}$$

where $s = \xi - \alpha$, $t = 1/\eta$. Moreover, we may restrict ourselves to the typical case $M \geqq 1$, so that $\left\| T\left(\alpha + \frac{k}{t}\right) x \right\| \leqq M^{1+\frac{k}{t}} . \|x\|$. The rigt-hand side of (6) can be thus treated in exactly the same way as in the estimation of $S(t)$, and so we have the inequalities (1) and (2).

Clearly the theorem can also be extended to cover the case $\alpha = 0$, if $T(\xi)$ is assumed to be strongly continuous for $\xi \geqq 0$, in which $T(0)$ is defined as the strong limit $T(0) = \lim_{\eta \to 0} T(\eta)$.

It seems somewhat interesting to determine whether the estimate on the right-hand side of (1) can be improved to the form $\mu(\eta^{\Theta}, x) + K . \eta^{\Theta} . \|x\|$ with $\Theta > \frac{1}{3}$. In fact, this has not yet been decided in this work, though we may observe that the estimate of (4) seems improvable by using our device of proof.

4. An application. It is easy to deduce from Theorem 1 (or the special proposition) the following consequence:

If $f(s)$ is a continuous function satisfying the condition $|f(s)| \leqq M^{1+s}$ $(0 \leqq s < \infty)$ with $M \geqq 1$, then for any given interval $\alpha \leqq s \leqq \beta$ $(0 \leqq \alpha < \beta < < \infty)$ there is a sequence of polynomials of the form

$$P_t(s) = \left(\sum_{h=0}^{m} \frac{(-st)^h}{h!}\right)\left(\sum_{k=0}^{n} \frac{(st)^k}{k!} f\left(\frac{k}{t}\right)\right)$$

with $m = [10\beta t]$, $n = [(\beta + 1)\, t]$ $(t = 1, 2, 3, \ldots)$ such that, for t being large,

$$|f(s) - P_t(s)| < \omega_f\left(\left(\frac{1}{t}\right)^{\frac{1}{3}}\right) + c . \left(\frac{1}{t}\right)^{\frac{1}{3}}, \tag{7}$$

where $C = c(\beta, M)$ is a positive constant independent of t, and $\omega_f(\delta)$ denotes the modulus of continuity of $f(s)$ for $\alpha \leqq s \leqq \beta$.

In fact we easily find that (cf. the estimation of $e^{-st}\Sigma''$ in § 2)

$$\left| e^{-st} \sum_{k=n+1}^{\infty} \frac{(st)^k}{k!} f\left(\frac{k}{t}\right)\right| \leqq A . \left(\frac{1}{t}\right)^{\frac{1}{3}}, \tag{8}$$

and moreover, we have (with $0 < \Theta = \Theta(m) < 1$)

(9)

$$\left|\sum_{h=m+1}^{\infty} \frac{(-st)^h}{h!}\right| \leqq \frac{(st)^{m+1}}{(m+1)!} e^{-\Theta st} \leqq \frac{(\beta t)^{m+1}}{(m+1)!} \leqq \left(\frac{e\beta t}{m+1}\right)^{m+1} \leqq \left(\frac{e}{10}\right)^{10\beta t} < e^{-10\beta t}.$$

The inequality (7) may therefore be inferred at once from (8), (9) and (4).

5. Convergence theorems similar to Butzer's. As a simple constructive proof for the Weierstrass polynomial approximation theorem, it has been shown by the author [6] that the following limit relation

$$\lim_{n\to\infty} \sum_{k=0}^{n} \Psi_{k,n}(s) \cdot f\left(\frac{k}{n}\right) = f(s)$$

holds uniformly for any continuous function $f(s)$ defined on an interval $\varepsilon \leqq s \leqq \leqq 1 - \varepsilon$ with any small $\varepsilon > 0$, where

$$\Psi_{k,n}(s) = \frac{1}{\sqrt{n\pi}} \left[1 - \left(\frac{k}{n} - s\right)^2\right]^n . \tag{10}$$

Thus by making use of a theorem of BUTZER [2] we get at once the following

Theorem 2. *If $T(\xi)$ is strongly continuous for $\xi > 0$, then for every $x \in X$ the limit relation*

$$\lim_{n\to\infty} \left\| \left\{ \sum_{k=0}^{n} \Psi_{k,n}(s)\, T\left(\frac{k}{n}\right) \right\} x - T(s)\, x \right\| = 0 \tag{11}$$

holds uniformly for $\varepsilon \leqq s \leqq 1 - \varepsilon$ with any small $\varepsilon > 0$.

Moreover, in order to approximate any bounded continuous function $f(s)$ defined on the infinite interval $(0, \infty)$, the author has introduced polynomials of the form

$$\frac{1}{\sqrt{n\pi}} \sum_{k=0}^{n} f\left(\frac{k}{n^{3/4}}\right) \left[1 - \left(\frac{k}{n} - \frac{s}{n^{1/4}}\right)^2\right]^n , \quad (0 < s < \infty) .$$

Thus if we define

$$\Phi_{k.n}(s) = \frac{1}{\sqrt{n\pi}} \left[1 - \left(\frac{k}{n} - \frac{s}{n^{1/4}}\right)^2\right]^n , \quad (n = 1, 2, 3, \ldots) \tag{12}$$

then by the same method as used in proving Theorem 2 we may also obtain the following result:

Theorem 3. *If $T(\xi)$ is strongly continuous for $\xi > 0$, and if $\|T(\xi)\| \leqq M < \infty$ $(\xi > 0)$, then for every $x \in X$, the limit relation*

$$\lim_{n\to\infty} \|\{ \sum_{k=0}^{n} \Phi_{k,n}(s)\, T(k/n^{\frac{3}{4}})\}\, x - T(s)\, x\| = 0 \tag{13}$$

holds uniformly on any interval $\alpha \leqq s \leqq \beta$ with $0 < \alpha < \beta < \infty$.

The whole proof of this result is just the same as that of Theorem 3 of [6], and may therefore be omitted here.

Literature

[1] *E. Hille*: Functional analysis and semi-groups. Amer. Math. Soc. Coll. Publ., XXXI. (1948), Chap. 9.

[2] *P. L. Butzer*: Halbgruppen von linearen Operatoren und eine Anwendung in der Approximationstheorie. Jour. reine angew. Math., Band *197* (1957), 112—120.

[3] *G. Mirakyan*: Approximation des fonctions continues au moyen de polynômes de la forme $e^{-nx}\sum_{k=0}^{m}C_{k,n}x^k$. C. R. (Doklady) Acad. Sci. URSS., 31 (1941), 201—205.

[4] *G. Szegö*: Lösung der Aufgaben, 105, 106. Jahresber. D. M. V., *43*, 10—11, 15—16.

[5] *O. Szász*: A generalization of Bernstein polynomials etc. Jour. Res. Nat. Bureau Standards, *45* (1950), 239—244.

[6] *L. C. Hsu*: A new type of polynomials approximating a continuous or integrable function. Studia Mathematica *18* (1959), 43—48.

Резюме

ОЦЕНКА ДЛЯ „ПОКАЗАТЕЛЬНОЙ ФОРМУЛЫ“ ХИЛЛЕ В ТЕОРИИ ПОЛУГРУПП ЛИНЕЙНЫХ ОПЕРАТОРОВ

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Пусть $\{T(\xi)\}$, $\xi > 0$ — множество линейных операторов, отображающих пространство Банаха X на себя и удовлетворяющих следующим требованиям:

(1) $T(\xi_1 + \xi_2)\, x = T(\xi_2)[T(\xi_1)\, x]$, каждое $x \in X,\ \xi_1 > 0,\ \xi_2 > 0$;

(2) $\lim\limits_{\eta\to\xi} \|T(\eta)\, x - T(\xi)\, x\| = 0$, каждое $x \in X,\ \xi > 0$;

(3) $\|T(\xi)\| \leqq M = M(T) < +\infty$ для $0 < \alpha \leqq \xi \leqq \max(\alpha + 1, 2\alpha)$.

Условие (2) означает, что $T(\xi)\, x$ в качетсве функции ξ сильно непрерывен для $\xi > 0$.

Мы доказали следующую теорему (ср. Хилле [1], стр. 187):

Теорема 1. *Если* $\{T(\xi)\}$ — *семейство операторов, удовлетворяющих условиям* (1), (2) *и* (3), *то*

$$\|\exp[\xi - \alpha)\, A_\eta]\, T(\alpha)\, x - T(\xi)\, x\| \leqq \mu(\eta^{\frac{1}{3}}, x) + K \,.\, \eta^{\frac{1}{3}}\, \|x\|$$

для $0 < \alpha \leqq \xi \leqq \beta$ *и для малого числа* $\eta > 0$;

$$A_\eta = \frac{1}{\eta}[T(\eta) - I]\,, \quad K = K(\beta, M) > 0\,.$$

Кроме того, в работе доказаны еще две теоремы о сходимости (теорема 2 и теорема 3).