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_Czechoslovak Mathematical Journal_, Vol. 11 (1961), No. 1, 115–133

Persistent URL: [http://dml.cz/dmlcz/100446](http://dml.cz/dmlcz/100446)

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APPLICATIONS OF COMPLETE FAMILIES OF CONTINUOUS FUNCTIONS TO THE THEORY OF Q-SPACES

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(Received December 9, 1959)

In the present paper the concept of a complete family of continuous functions is introduced and applied to the theory of $N(m)$-spaces (intersections of $m$ $N$-sets in their Stone-Čech extensions) and, in particular, $Q$-spaces. $N(m)$-spaces may be defined as the inverse images under continuous closed compact mappings to the topological product of $m$ real lines. The section 3 is devoted to the problem, under what conditions on the mapping is the image of a $N(m)$-space (in particular, of a $Q$-space) an $N(m)$-space (a $Q$-space, respectively).

In [2] the concept of a complete indexed family of open coverings of a space has been introduced. For convenience, we recall the definition. An indexed family of open coverings

\[
\{\mathcal{B}_a; a \in A\}
\]

is said to be complete if the following condition is satisfied:

If $\{F\}$ is a centered family of closed subsets of $P$ such that for each $a$ in $A$ there exists a $V_a$ in $\mathcal{B}_a$ containing some $F_a \in \{F\}$, then $\bigcap \{F\} \neq \emptyset$.

In [2] the following theorem was proved:

A completely regular space $P$ is an intersection of $m$ open sets in every compact extension of $P$ if and only if there exists a complete indexed family (1) of open coverings of $P$ such that the potency of $A$ is $m$.

In the present paper we investigate spaces possessing a complete family of open coverings (1) of a special sort. If $f$ is a continuous real-valued function on $P$ then the open cover consisting of sets

\[
\{x; |f(x)| < n\}, \quad n = 1, 2, \ldots
\]

will be denoted by $\mathcal{B}(f)$. We shall consider coverings of the form $\mathcal{B}(f)$ only. We shall prove that a completely regular space $P$ possesses a family of continuous functions $\mathcal{F}$ such that

\[
\{\mathcal{B}(f); f \in \mathcal{F}\}
\]
is complete (such a family $\mathcal{F}$ is said to be complete) if and only if there exists an indexed family $\{N_f; f \in \mathcal{F}\}$ of $\mathcal{N}$-sets in $\beta P$ such that

$$P = \bigcap\{N_f; f \in \mathcal{F}\}.$$  

If the potency of $\mathcal{F}$ is at most $m$, then such spaces will be called $\mathcal{N}(m)$-spaces. A space is a $\mathcal{Q}$-space (for Hewitt's definition of $\mathcal{Q}$-spaces see [3]) if and only if it is a $\mathcal{N}(m)$-space for some cardinal $m$.

If $f$ is a continuous function, then $f$ is bounded on a set $M$ if and only if there exists a set in $\mathcal{B}(f)$ containing $M$. Thus we obtain a definition of complete families of continuous functions which does not use coverings.

In section 1 we shall study complete families of continuous functions on an arbitrary space. For convenience we shall use a more general definition of a complete family. But for completely regular spaces both definitions are identical.

In section 2 we shall investigate complete families on completely regular spaces, more precisely, we shall study $\mathcal{N}(m)$-spaces (in particular, $\mathcal{Q}$-spaces) using the concept of a complete family of continuous functions.

The section 3 is devoted to the question:

Let $\Phi$ be a mapping from a $\mathcal{N}(m)$-space onto a space $Q$. Under what conditions on $\Phi$ may we assert that $Q$ is a $\mathcal{N}(m)$-space.

If $\mathfrak{B}$ is a family of sets, then the intersection of $\mathfrak{B}$ will be denoted by $\bigcap \mathfrak{B}$, that is

$$\bigcap \mathfrak{B} = \bigcap\{Z; Z \in \mathfrak{B}\}.$$  

For convenience we shall use the following convention: If $V$ is a property of sets, then a indexed family $\{M_a; a \in A\}$ is said to have the property $V$ if the set of all $M_a$ has the property $V$. If $V$ is a property of indexed families, then a set $\mathfrak{M}$ has the property $V$ if the indexed family $\{M; M \in \mathfrak{M}\}$ has the property $V$.

A topological space (in the sequel a space, merely) $P$ is said to be an extension of a space $R$ if $R$ is a dense subspace of $P$. An extension $P$ of $R$ is said to be Hausdorff, regular, completely regular, compact if $P$ is a Hausdorff, regular, completely regular, compact space, respectively. The Čech-Stone extension of a completely regular space $P$ will be denoted by $\beta P$. It is well-known that $\beta P$ is the compact extension of $P$ uniquely determined by the property:

every bounded real-valued continuous function on $P$ has a continuous extension over $\beta P$.

It is also well-known that if $K$ is a compact extension of $P$, then there exists one and only one continuous mapping $\Phi$ from $\beta P$ onto $K$ such that the restriction of $\Phi$ to $P$ is the identity mapping. This mapping will be called Čech-Stone mapping.

Function will always mean a real-valued function. A subset $M$ of a space is said to be a $Z$-set if there exists a continuous function $f$ on $P$ such that

$$M = Z(f) = \{x; f(x) = 0\}.$$  

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A subset $M$ of a space $P$ is said to be a $N$-set if $P - M$ is a $Z$-set. We shall use the notation

$$N(f) = \{x; f(x) \neq 0\}.$$

1. COMPLETE FAMILIES OF FUNCTIONS

1.1. Definition. Let $\mathcal{F}$ be a family of continuous functions on a space $P$. $\mathcal{F}$ is said to be complete if the following conditions is satisfied:

1.1.1. If $\mathcal{F}$ is centered family of $Z$-sets in $P$ and if for each $f$ in $\mathcal{F}$ there exists a $Z_f$ in $\mathcal{F}$ such that $f$ is bounded on $Z_f$, then $\bigcap \mathcal{F} \neq \emptyset$.

Note. We have at once that a family of continuous functions containing a complete family is a complete family.

1.2. Lemma. Let $\mathcal{F}$ be a maximal centered family of $Z$-sets in a space $P$ such that the intersection of every countable subfamily is non-void. For every continuous function $f$ on $P$ there exists a $Z$ in $\mathcal{F}$ on which $f$ is bounded.

Proof. Let $f$ be a continuous function on $P$. For every $n = 1, 2, \ldots$ denote by $Z_n$ the set

$$(2) \quad \{x; x \in P, |f(x)| \geq n\}.$$  

If for some $n$ the set $Z_n$ does not belong to $\mathcal{F}$, then there exists a $Z$ in $\mathcal{F}$ with $Z_n \cap Z = \emptyset$. Then $|f(x)| \leq n$ for $x$ in $Z$ and hence $f$ is bounded on $Z$. In the other case we have $Z_n \in \mathcal{F}$ for every $n = 1, 2, \ldots$ By our assumption we have

$$Z_0 = \bigcap_{n=1}^{\infty} Z_n \neq \emptyset.$$  

According to (2)

$$x \in Z_0 \Rightarrow |f(x)| \geq n$$

for every $n$, which is impossible since $f$ is finite-valued.

As an immediate consequence of 1.2 we have

1.3. Theorem. If there exists a complete family of continuous functions on a space $P$, then the following condition is satisfied:

1.3.1. If $\mathcal{F}$ is a maximal centered family of $Z$-sets in $P$ such that the intersection of every its countable subfamily is non-void, then $\bigcap \mathcal{F} \neq \emptyset$.

1.4. Lemma. Let $\mathcal{F}$ be a maximal centered family of $Z$-sets in $P$. If the intersection of some countable subfamily of $\mathcal{F}$ is empty, then there exists a continuous function $f$ on $P$ which is bounded on no $Z$ in $\mathcal{F}$.

Proof. Let $\{Z_n\}$ be a sequence in $\mathcal{F}$ such that

$$(3) \quad \bigcap_{n=1}^{\infty} Z_n = \emptyset.$$
Choose continuous functions $f_n$ on $P$ such that $Z_n = Z(f_n)$ and $0 \leq f_n \leq 1$.

Consider the continuous function $g = 1/f$ where $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. Clearly:

$$x \in \bigcap_{i=1}^{n} Z_i \Rightarrow f(x) \leq \frac{1}{2^n},$$

and hence, $g(x) \geq 2^n$ for each $x$ in $\bigcap_{i=1}^{n} Z_i$. It follows immediately that $f$ is bounded on no $Z$ in $\mathfrak{Z}$.

As a corollary of 1.4 we have:

1.5. Theorem. If a space $P$ satisfies the condition 1.3.1, then the family of all continuous functions is complete.

1.6. Definition. A space is said to be quasi-compact if the intersection of every centered family of $Z$-sets is non-void. A subspace $R$ of $P$ is said to be relatively quasi-compact in $P$ if the following condition is satisfied:

1.6.1. If $\mathfrak{Z}$ is a family of $Z$-sets in $P$ and if $\mathfrak{Z} \cap R$ is a centered family, then $\bigcap \mathfrak{Z} \cap R \neq \emptyset$.

Note. Evidently every $Z$-set of a quasicompact space $P$ is relatively quasicompact in $P$. Moreover, every intersection of $Z$-sets of a quasi-compact space $P$ is relatively quasi-compact in $P$. For further information see [1], 200—202.

1.7. Theorem. Let $\mathfrak{F}$ be a family of continuous functions on a space $P$. $\mathfrak{F}$ is complete if and only if the following two conditions 1.7.1 and 1.7.2 are satisfied:

1.7.1. If $F$ is intersection of $Z$-sets in $P$ and if every $f \in \mathfrak{F}$ is bounded on $F$, then $F$ is relatively quasi-compact in $P$.

1.7.2. If $\{Z_f; f \in \mathfrak{F}\}$ is a centered indexed family of $Z$-sets and if $f$ is bounded on $Z_f$, then

$$\bigcap \{Z_f; f \in \mathfrak{F}\} \neq \emptyset .$$

Proof. The necessity of conditions 1.7.1 and 1.7.2 is quite obvious. To prove the sufficiency, suppose that $\mathfrak{Z}$ is a centered family of $Z$-sets in $P$ and that for each $f$ in $\mathfrak{F}$ there exists a $Z_f$ in $\mathfrak{Z}$ on which $f$ is bounded. By 1.7.2 the set

$$F = \bigcap \{Z_f; f \in \mathfrak{F}\}$$

is non-void. By 1.7.1 the set $F$ is relatively quasi-compact in $P$. Consequently, to prove $\bigcap \mathfrak{Z} \neq \emptyset$ it is sufficient to show that $\mathfrak{Z} \cap F$ is a centered family. But if both $Z_1$ and $Z_2$ belong to $\mathfrak{Z}$, there again by 1.7.2 the set

$$\bigcap \{Z_1 \cap Z_2 \cap Z_f; f \in \mathfrak{F}\} = F \cap Z_1 \cap Z_2$$

is non-void. The proof is complete.

We shall need the following
1.8. Lemma. If $\mathcal{Z}$ is a maximal centered family of $Z$-sets in $P$ and if the $Z$-sets $Z_1, \ldots, Z_k$ cover some $Z \in \mathcal{Z}$, then some $Z_i$ belongs to $\mathcal{Z}$.

Proof. Suppose on the contrary that no $Z_i$ belongs to $\mathcal{Z}$. According to the maximality of $\mathcal{Z}$ there exist $Z'_i \in \mathcal{Z}$, $i = 1, 2, \ldots, k$, such that $Z_i \cap Z'_i = \emptyset$. Then $Z \cap \bigcap_{i=1}^{k} Z'_i$ belongs to $\mathcal{Z}$, which is impossible since

$$Z \cap \bigcap_{i=1}^{k} Z'_i \subseteq Z - \bigcup_{i=1}^{k} Z_i = \emptyset.$$ 

This contradiction completes the proof.

Now we proceed to characterize complete families in terms of mappings of a special sort.

1.9. Definition. A mapping from $P$ to $Q$ is said to be quasi-compact if the inverse image of every point of $Q$ is relatively quasicompact in $P$. A mapping from $P$ to $Q$ is said to be a $Z$-mapping if the image of every $Z$-set of $P$ is closed in $Q$.

1.10. Theorem. Let $\mathcal{F}$ be a family of continuous functions on a space $P$. Consider the space

$$E^\mathcal{F} = \bigoplus_{f \in \mathcal{F}} X\{E_f; f \in \mathcal{F}\},$$

where the $E_f$ are real lines; and also the continuous mapping $\Phi: P \to E^\mathcal{F}$ defined as follows:

$$\Phi(x) = \{f(x); f \in \mathcal{F}\}.$$ 

The family $\mathcal{F}$ is complete if and only if $\Phi$ is a quasi-compact $Z$-mapping of $P$ to $E^\mathcal{F}$.

Proof. First let us suppose that $\mathcal{F}$ is a complete family. To prove quasi-compactness of $\Phi$ we shall show that

1.10.1. The inverse image of every compact subspace $K$ of $E^\mathcal{F}$ is relatively quasi-compact in $P$.

It is easy to see that every function $f$ from $\mathcal{F}$ is bounded on $\Phi^{-1}[K]$. Indeed, we have $f(x) = \pi_f(\Phi(x))$ and $f[\Phi^{-1}[K]] = \pi_f[K]$ where $\pi_f$ denotes the projections of $E^\mathcal{F}$ onto $E_f$. Since $\pi_f$ is a continuous function and $K$ is a compact space, $\pi_f[K]$ is a compact subspace of $E_f$, and consequently, $\pi_f[K]$ is a bounded subspace of $E_f$ (in the usual metric). $K$ is a compact subspace of the completely regular space $E^\mathcal{F}$ and therefore $K$ is an intersection of $Z$-sets in $E^\mathcal{F}$. Since $\Phi$ is a continuous mapping, it follows at once that $f^{-1}[K]$ is an intersection of $Z$-sets in $P$. By Theorem 1.7 the subspace $f^{-1}[K]$ of $P$ is relatively quasi-compact in $P$. Thus 1.10.1 holds and $\Phi$ is a quasi-compact mapping. It remains to prove that $\Phi$ is a $Z$-mapping. Let $Z_0$ be a $Z$-set in $P$. Suppose on the contrary that $\Phi[Z_0] = \emptyset$.
= F is not closed in $E^n$. Then we may choose $y = \{y_i; f \in \mathcal{F}\}$ in $F - F$. Consider the family

$$\mathcal{Z} = \{Z_{n,f}; f \in \mathcal{F}, n = 1, 2, \ldots\} \cup \{Z_0\}$$

of Z-sets in $P$, where

$$Z_{n,f} = \left\{x; x \in P, |f(x) - y_f| \leq \frac{1}{n}\right\}.$$ 

The point $y$ being an accumulation point of $F$, the family $\mathcal{Z}$ is centered. Moreover, each $f$ in $\mathcal{F}$ is bounded on $Z_{n,f}$. It follows that $\bigcap \mathcal{Z} \neq \emptyset$. But this is impossible since

$$\bigcap \mathcal{Z} = \Phi^{-1}[y] \cap Z_0$$

and by our assumption $y$ does not belong to $F = \Phi[Z_0]$, that is, $\bigcap \mathcal{Z} = \emptyset$. This contradiction completes the proof of necessity.

To prove sufficiency let us suppose that $\Phi$ is a quasi-compact Z-mapping. Let $\mathcal{Z}$ be a maximal centered family of Z-sets in $P$ and suppose that for each $f$ in $\mathcal{F}$ there exists a $Z_f$ in $\mathcal{Z}$ such that $f$ is bounded on $Z_f$. From quasi-compactness of $\Phi$ it follows at once that it is sufficient to prove the existence of a point $y = \{y_i; f \in \mathcal{F}\}$ in $E^n$ such that $\mathcal{Z} \cap \Phi^{-1}[y]$ is a centered family. We proceed to construct such a point $y$.

Choose $f$ in $\mathcal{F}$. By our assumption $f$ is bounded on $Z_f$. Hence, there exists a bounded interval $I_f$ of $E_f$ such that

$$f[Z_f] \subset I_f.$$ 

Let $K_1, \ldots, K_k$ be a finite cover of $I_f$ by closed intervals of length less than $\frac{1}{n}$. Since $f^{-1}[K_i]$ are Z-sets in $P$ and

$$\bigcup_{i=1}^{k} f^{-1}[K_i] \supset Z_f \in \mathcal{Z},$$

it follows at once from lemma 1.8 that for some $i = 1, \ldots, k$, $f^{-1}[K_i]$ belongs to $\mathcal{Z}$.

Thus, for every $n = 1, 2, \ldots$ and for each $f$ in $\mathcal{F}$ there exists a closed interval $K_{n,f}$ in $E_f$ of length less than $\frac{1}{n}$ such that

$$Z_{n,f} = f^{-1}[K_{n,f}] \in \mathcal{Z}.$$ 

Evidently for every $f$ in $\mathcal{F}$, $\{K_{n,f}; n = 1, 2, \ldots\}$ is a centered family of compact sets. It follows that

$$\bigcap_{n=1}^{\infty} K_{n,f} \neq \emptyset.$$ 

This intersection contains only one point, namely $y_f$, since the lengths of $K_{n,f}$ converge to zero with $n \to \infty$. The point $\{y_i; f \in \mathcal{F}\}$ will be denoted by $y$. Since
\( \Phi[P] \) is a closed subspace of \( E^\theta \), \( y \) belongs to \( \Phi[P] \). It remains to prove that \( Z \cap \Phi^{-1}[y] \) is a centered family. It is of course sufficient to show that 

\[
Z \in 3 \Rightarrow Z \cap \Phi^{-1}[y] \neq \emptyset.
\]

Let us suppose on the contrary that some \( Z \in 3 \) does not meet \( \Phi^{-1}[y] \). The mapping \( \Phi \) is a \( Z \)-mapping and hence \( F = \Phi[Z] \) is a closed subspace of \( E^\theta \). By our assumption \( y \) does not belong to \( F \). In consequence, there exists a neighborhood \( U \) of \( y \) which does not meet \( F \). Since the lengths of \( K_n \) converge to zero with \( n \to 0 \), there exist \( K_i = K_{n,i} \) (\( i = 1, \ldots, k \)) such that

\[
\bigcap_{i=1}^k f_i^{-1}[K_i] \cap Z = \emptyset.
\]

But this is a contradiction, since \( f_i^{-1}[K_i] \) belong to \( 3 \). The proof is complete.

As a corollary of 1.10 and 1.10.1 we have

**1.11. Theorem.** If \( \Phi \) is a quasi-compact \( Z \)-mapping from \( P \) to the topological product \( R \) of a family of real lines, then the inverse image of every compact subspace of \( R \) is a relatively quasi-compact subspace of \( P \).

2. **Q-SPACES AND \( N(m) \)-SPACES**

In this section we shall study complete families of continuous functions on a completely regular space.

**2.1. Definition.** Let \( m \) be a cardinal number. A space \( P \) is said to be an \( N(m) \)-space provided that \( P \) is completely regular and there exists a complete family \( \mathcal{F} \) of continuous functions on \( P \) such that the potency of \( \mathcal{F} \) is \( \leq m \). A space is said to be an exact \( N(m) \)-space provided that it is an \( N(m) \)-space but not an \( N(n) \)-space for any cardinal \( n < m \). A space is a \( Q \)-space if it is an \( N(m) \)-space for some cardinal \( m \).

Thus a completely regular space is a \( Q \)-space if and only if the set of all continuous functions is complete.

**2.2. Definition.** A mapping \( \Phi \) of \( P \) to \( Q \) is said to be compact if the inverse images of points of \( Q \) are compact spaces. \( \Phi \) is closed if the image of every closed subset of \( P \) is closed in \( Q \).

**2.3. Lemma.** A relatively quasi-compact subspace \( R \) of a completely regular space \( P \) is a compact space. A quasi-compact mapping from a completely regular space to a space is a compact mapping. A quasi-compact \( Z \)-mapping from a completely regular space to a space is a compact closed mapping.

**Proof.** Let \( R \) be relatively quasi-compact in a completely regular space \( P \). Let \( \{F\} \) be a centered family of closed subsets of \( R \). Let \( 3 \) be the family of all \( Z \)-sets in \( P \) such that for some \( F \) in \( \{F\} \) the inclusion \( F^P \subset Z \) holds. Since \( P \) is
a completely regular space, we have \( \cap \mathfrak{Z} = \cap \{F\} \). By quasicompactness of \( R \) we have
\[
R \cap \cap \mathfrak{Z} = \emptyset .
\]
Combining the above two relations we obtain \( \cap \{F\} = \emptyset \).

The second statement of the lemma is an immediate consequence of the first.

To prove the third statement let us suppose that \( \Phi \) is a quasi-compact \( Z \)-mapping from a completely regular space \( P \) to \( Q \). Then \( \Phi \) is a compact mapping and it remains to show that \( \Phi \) is a closed mapping. Let \( F \) be a closed subspace of \( P \). Denote by \( \mathfrak{Z} \) the family of all \( Z \)-sets in \( P \) containing \( F \). Since \( P \) is completely regular, we have \( \cap \mathfrak{Z} = F \). Put \( F_0 = \Phi[F] \). It is sufficient to prove
\[
F_1 \overset{\text{def}}{=} \cap \{\Phi[Z]; Z \in \mathfrak{Z}\} = F_0 .
\]
The inclusion \( F_1 \subset F_0 \) is trivial. For the other one, suppose that there exists a point \( y \) in \( F_1 - F_0 \). We see at once that
\[
(4) \quad \mathfrak{Z} \cap \Phi^{-1}[y]
\]
is a centered family of closed subsets of the compact space \( \Phi^{-1}[y] \). Thus we may choose a point \( x \) in the intersection of the family \( (4) \). But this is impossible since
\[
x \in \cap \mathfrak{Z} = F , \quad \Phi(x) = y \in \Phi[F] .
\]

From 1.7 and 2.3 we have

2.4. Theorem. Suppose that \( \mathfrak{F} \) is a family of continuous functions on a completely regular space \( P \). \( \mathfrak{F} \) is complete if and only if the following two conditions are satisfied:

2.4.1. If \( K \) is closed in \( P \) and if each \( f \) from \( \mathfrak{F} \) is bounded on \( K \), then \( K \) is a compact space.

2.4.2. If \( \{Z_f; f \in \mathfrak{F}\} \) is a centered indexed family of \( Z \)-sets in \( P \) such that \( f \) is bounded on \( Z_f \), then
\[
\cap \{Z_f; f \in \mathfrak{F}\} = \emptyset .
\]

Definition. \( m \) being a cardinal number, denote by \( E^m \) the topological product of \( m \) real lines.

As an immediate consequence of 1.10 and 2.2 we have

2.5. Theorem. Let \( \mathfrak{F} \) be a family of continuous functions on a completely regular space \( P \). Define \( E^\mathfrak{F} \) and \( \Phi \) as in 1.10.
\( \mathfrak{F} \) is complete if and only if \( \Phi \) is a closed compact mapping.

2.6. Theorem. A space \( P \) is an \( N(m) \)-space if and only if \( P \) is completely regular and there exists a continuous closed compact mapping \( \Phi \) from \( P \) to \( E^m \).

Proof. First suppose that \( P \) is an \( N(m) \)-space. Hence \( P \) is completely regular and there exists a complete family \( \mathfrak{F} \) of continuous function on \( P \) such that the
potency of $ \mathfrak{F} $ at most $ m $. Without loss of generality we may assume that the potency of $ \mathfrak{F} $ is $ m $. Define $ E^{\mathfrak{F}} $ and $ \Phi $ as in 1.10. By 2.5 $ \Phi $ is closed and compact.

Evidently $ \Phi $ is continuous and $ E^{\mathfrak{F}} = E^m $.

Conversely, let $ \Phi $ be a continuous closed compact mapping from a completely regular space $ P $ to

$$ E^m = \{ E_a; a \in A \} $$

where the potency of the index set $ A $ is $ m $ and the $ E_a $ are real lines. For each $ a $ in $ A $ denote by $ \pi_a $ the projection of $ E^m $ onto $ E_a $. Denote by $ f_a $ the function $ \pi_a(\Phi) $. Every $ f_a $ is continuous as the superposition of two continuous mappings. Evidently for each $ x $ in $ P $,

$$ \Phi(x) = \{ f_a(x); a \in A \}. $$

Applying 2.5 we obtain that the family of all $ f_a $ is complete.

If $ \Phi $ is a closed compact mapping from $ P $ to $ Q $ and if $ F $ is a closed subset of $ P $, then the restriction of $ \Phi $ to $ F $ is a closed compact mapping. From this fact and from 2.5 and 2.6 we have at once

2.7. Theorem. If $ \mathfrak{F} $ is a complete family of continuous functions on a completely regular space $ P $ and if $ F $ is a closed subspace of $ P $, then the family of the restrictions (to $ F $) of all $ f \in \mathfrak{F} $ is a complete family on $ F $. Closed subspaces of $ N(m) $-spaces are $ N(m) $-spaces.

Now we proceed to characterize $ N(m) $-spaces as intersections of $ m $ $ N $-sets in their Čech-Stone extensions.

2.8. Proposition. Let $ \mathfrak{F} $ be a family of continuous functions on a completely regular space $ P $ such that $ f \not\geq 1 $ for each $ f $ in $ \mathfrak{F} $. For each $ f $ in $ \mathfrak{F} $ denote by $ f^* $ the continuous extension of $ 1/f $ over $ \beta P $ ($ 1/f $ is bounded).

Then $ \mathfrak{F} $ is complete if and only if

2.8.1.

$$ \bigcap \{ N(f^*); f \in \mathfrak{F} \} = P. $$

Proof. First let us suppose that 2.8.1 holds. Let $ \mathfrak{Z} $ be a centered system of $ Z $-sets in $ P $ such that for each $ f $ in $ \mathfrak{F} $ there exists a $ Z_f $ in $ \mathfrak{Z} $ on which $ f $ is bounded. $ \beta P $ being a compact space, the set

$$ F_0 = \bigcap \{ \bar{Z}_f; f \in \mathfrak{Z} \} $$

is non-void. It is sufficient to show that $ F \subset P $. According to 2.8.1 it is sufficient to show that

(5) $$ \bar{Z}_f \subset N(f^*) $$

for each $ f $ in $ \mathfrak{F} $. $ f $ is bounded on $ Z_f $,

$$ x \in Z_f \Rightarrow |f(x)| \leq M $$

say, and hence ( $ f^* $ is continuous)

$$ x \in \bar{Z}_f \Rightarrow |f^*(x)| \geq M^{-1} $$

which implies (5).
To prove necessity, let us suppose that there exists a point $x$ in $$ \bigcap \{N(f^*); f \in \mathfrak{F}\} = P. $$ Let $\mathfrak{F}$ be the family of all $Z$-sets in $\beta P$ containing $x$ in their interior. Evidently $$ \bigcap \{Z; Z \in \mathfrak{F}\} = (x) \subset \beta P - P. $$

Thus $\mathfrak{F} \cap P$ is a centered family of $Z$-sets in $P$ with empty intersection. To prove that $\mathfrak{F}$ is not complete, it is sufficient to show that for each $f$ in $\mathfrak{F}$ there exists a $Z_f$ in $\mathfrak{F}$ such that $f$ is bounded on $Z_f$. Fix $f \in \mathfrak{F}$ Since $f^*(x) = 0$, there exist a $Z_f$ in $\mathfrak{F}$ and an $\varepsilon > 0$ with $$ y \in Z_f \Rightarrow |f^*(y)| \geq \varepsilon. $$ If follows that $y \in Z \cap P \Rightarrow |f(y)| \leq 1/\varepsilon$. The proof is complete.

As an immediate consequence of 2.8 we have:

2.9. **Theorem.** A completely regular space $P$ is an $N(m)$-space if and only if there exists a set $\mathfrak{R}$ of $N$-sets in $\beta P$ such that the potency of $\mathfrak{R}$ is at most $m$ and $$ \bigcap \{N; N \in \mathfrak{R}\} = P. $$

Now we shall proceed to give the usual characterisation of $N(m)$-spaces. First we prove the following crucial property of continuous closed compact mappings.

2.10. **Theorem.** Let $\Phi$ be a continuous closed compact mapping from a regular space $P$ to a space $Q$. There exists no proper regular extension $R$ of $P$ on which $\Phi$ may be continuously extended.

**Proof.** Let us suppose, on the contrary, that there exists a proper regular extension $R$ of $P$ and a continuous mapping $\Phi^*$ from $R$ to $Q$ such that $\Phi$ is the restriction of $\Phi^*$. Choose $x$ in $R - P$. Since $\Phi[P]$ is a closed subset of $Q$ ($\Phi$ is closed) and since by continuity of $\Phi^*$ $$ \Phi^*[R] \subset \Phi[P] $$ we have at once that $\Phi^*[R] = \Phi[P]$. Hence, there exists a $y$ in $\Phi[P]$ such that $\Phi^*(x) = y$. Denote by $K$ the inverse image under $\Phi$ of $y$ (that is, the set $\Phi[y]$). $\Phi$ is a compact mapping, and consequently, $K$ is a compact space. It follows that $$ x \nonrel x K^R = K. $$

Since $R$ is a regular space, we may choose a closed (in $R$) neighborhood $F$ of $x$ with $F \cap K = 0$. Consider the set $F \cap P$. $\Phi$ being a closed mapping, $\Phi[F \cap P]$ is a closed subset of $Q$. Since $F \cap K = 0$, it follows that $$ y \nonrel y \Phi[F \cap P] = \Phi^*[F \cap P]. $$

But this is impossible, since $\Phi^*$ is continuous, $x \in \bar{F} \cap \bar{P}$ and $\Phi^*(x) = y$. This contradiction establishes the theorem.
2.11. Definition. Suppose that $P$ and $Q$ are completely regular spaces. A continuous mapping $\Phi$ from $P$ to $Q$ is said to be non-extensible if for any proper completely regular extension $R$ of $P$ and any continuous mapping $\Phi^*$ from $R$ to $Q$, the restriction of $\Phi^*$ to $P$ is different from $\Phi$.

Combining 2.6 and 2.10 we obtain at once:

2.12. Proposition. If $P$ is an $N(m)$-space then there exists a continuous non-extensible mapping from $P$ to $E^m$.

In the converse direction we shall prove:

2.13. Proposition. Let us suppose that there exists a continuous non-extensible mapping $\Phi$ from a completely regular space $P$ to $E^m$. Then $P$ is an $N(m)$-space.

Proof. Introduce the same notation as in the proof of 2.6:

$$E^m = \{E_a; a \in A\}, \quad \Phi(x) = \{f_a(x); a \in A\}.$$ 

It is sufficient to show that $\{f_a; a \in A\}$ is a complete family. Suppose, on the contrary, that $\{f_a; a \in A\}$ is not complete. Thus, there exists a maximal centered family $\mathcal{Z}$ of $Z$-sets in $P$ such that

$$\bigcap \{Z; Z \in \mathcal{Z}\} = \emptyset$$

and for each $a$ in $A$ there is a $Z_a$ in $\mathcal{Z}$ such that $f_a$ is bounded on $Z_a$. $\beta P$ being compact and $\mathcal{Z}$ being a maximal centered family of $Z$-sets, the intersection of the family $\{Z^\beta; Z \in \mathcal{Z}\}$ contains exactly one point, namely $x$. Since $\Phi$ is a non-extensible continuous mapping, there exists a $f_a$ which is non-extensible over $P \cup \{x\}$, and clearly, since every bounded continuous function of $P$ is extensible over $\beta P$, there must be

$$\lim_{z \to x^{z \in \beta P}} f_a(z) = \pm \infty.$$ 

But, $x$ is contained in the closure of every $Z$ in $\mathcal{Z}$, and consequently, we have

$$Z \in \beta \Rightarrow \lim_{z \to x^{z \in Z}} f_a(z) = \pm \infty.$$ 

Particularly, $f_a$ is not bounded on $Z_a$. This contradiction establishes the Theorem.

Combining 2.12 and 2.13 we obtain:

2.14. Theorem. A completely regular space $P$ is an $N(m)$-space if and only if there exists a continuous non-extensible mapping from $P$ to $E^m$.

We shall need the following proposition (see [4] and [3]).

2.15. A space is a $Q$-space if and only if $\Phi$ is a homeomorphic mapping if and only if $P$ is

Proof. Let $\mathfrak{F}$ be the set of all continuous functions on $P$. Define $E^0$ and $\Phi$ as in 1.10 It is well known that $\Phi$ is a homeomorphic mapping if and only if $P$ is
a completely regular space. Now the statement follows from the note preceding
1.2 and 2.5.

Now we are prepared to prove the following theorem.

2.16. Theorem. Let $\Phi$ be a continuous mapping from a completely regular
space $P$ to a $Q$-space $Q$. The following two conditions on $\Phi$ are equivalent:

2.15.1. $\Phi$ is closed and compact.

2.15.2. $\Phi$ is non-extensible.

Proof. By 2.10 the assertion 2.15.1 implies 2.15.2. Conversely, suppose that
$\Phi$ is non-extensible. By 2.15 there exists a homeomorphic mapping $\Psi$ of $Q$ onto
a closed subspace of $E^m$ for some cardinal $m$. We see at once that the super­
position $\Psi(\Phi)$ of $\Psi$ and $\Phi$ is a non-extensible mapping from $P$ to $E^m$. By 2.13
$\Psi(\Phi)$ is a closed compact mapping, and consequently, $\Psi$ being homeomorphic,
$\Phi$ is a closed compact mapping.

2.17. Theorem. Let $P$, $Q$ and $R$ be completely regular spaces. If $\Phi$ is a continuous
closed compact mapping from $P$ to $Q$ and if $\Psi$ is a continuous closed compact
mapping from $Q$ to $R$, then the superposition of $\Psi$ and $\Phi$ is a continuous closed
compact mapping.

2.18. If $\Phi$ is a continuous closed compact mapping of a space $P$ onto a compact
space $Q$, then $P$ is a compact space. (It may be noticed that if $\Phi$ is a continuous
mapping from $P$ onto a compact space $Q$, then $P$ is compact if and only if $\Phi$ is closed
and compact.)

The proof of 2.18 is quite routine and may be left to the reader.

As an immediate consequence of 2.17 we have:

2.19. Theorem. A completely regular space $P$ is an $N(m)$-space if and only if
there exists a continuous closed compact mapping from $P$ to an $N(m)$-space.

2.20. Theorem. Let $\{P_a; a \in A\}$ be an indexed family such that $P_a$ is an $N(m_a)$-
space. Then the topological product $P = \times\{P_a; a \in A\}$ is an $N(m)$-space, where
$m = \Sigma\{m_a; a \in A\}$.

To prove 2.20 it is sufficient to show that:

2.21. Theorem. Let $\{P_a; a \in A\}$ and $\{Q_a; a \in A\}$ be indexed families of comple­
tely regular spaces. For each $a$ in $A$ let $\Phi_a$ be a continuous closed compact mapping
from $P_a$ to $Q_a$.

Consider the product spaces $P = \times\{P_a; a \in A\}$ and $Q = \times\{Q_a; a \in A\}$ and the
mapping $\Phi = \{\Phi_a; a \in A\}$ defined as follows:

$$\Phi(x) = \{\Phi_a(x_a); a \in A\}.$$  

The mapping $\Phi$ is continuous, closed and compact.
Proof. The proof of continuity is quite routine and may be left to the reader. Denote by \( \pi_a \) the projection of \( Q \) onto \( Q_a \). Let \( y \) be an element of \( Q \). Clearly
\[
\Phi^{-1}[y] = X\{\Phi_a^{-1}[\pi_a(y)]; a \in A\}.
\]
The spaces \( \Phi_a^{-1}[\pi_a(y)] \) being compact, the space \( \Phi^{-1}[y] \) is compact by Tychoff's theorem. Thus \( \Phi \) is a compact mapping. It remains to prove that \( \Phi \) is a closed mapping. First, let \( F \) be a closed subset of \( P \) of the form
\[
(6) \quad X\{F_a; a \in A\}
\]
where \( F_a \) is a closed subset of \( P_a \). Clearly
\[
\Phi[F] = X\{\Phi_a[F_a]; a \in A\}.
\]
\( \Phi_a \) being closed, the set \( \Phi_a[F_a] \) is closed in \( Q_a \), and consequently, \( \Phi[F] \) is closed in \( Q \). Now, let \( F \) be an arbitrary closed subset of \( P \). Let \( \mathcal{W} \) be the family of all closed subsets of \( P \) of the form (6), and containing \( F \). \( \Phi[M] \) being closed in \( Q \), the set
\[
F_0 = \bigcap \{\Phi[M]; M \in \mathcal{W}\}
\]
is closed in \( Q \), and consequently, it is sufficient to show that \( F_0 = \Phi[F] \). Clearly \( F_0 \supset \Phi[F] \). Suppose that there exists a \( y \) in \( F_0 \setminus \Phi[F] \). Thus \( \Phi^{-1}[y] = K \) is a compact subspace of \( P \) disjoint with \( F \). Since
\[
\bigcap \{M; M \in \mathcal{W}\} = F,
\]
there exists a \( M \) in \( \mathcal{W} \) with \( M \cap K = \emptyset \). \( \Phi[M] \) being closed, we have at once that \( y \) non \( \in \Phi[M] \supset F_0 \). This contradiction completes the proof of 2.21.

Now we give another proof of 2.20 using 2.9 and Stone-Cech theorem (and also Tychonoff's theorem). By 2.9, for each \( a \) in \( A \) there exists a family \( \mathcal{R}_a \) of \( \mathcal{N} \)-sets in \( \beta P_a \) such that the potency of \( \mathcal{R}_a \) is at most \( m_a \) and
\[
\bigcap \{N; N \in \mathcal{R}_a\} = P_a.
\]
Consider the space \( K = X\{\beta P_a; a \in A\} \). Denote by \( \pi_a \) the projection of \( K \) onto \( \beta P_a \). Let \( \mathcal{R}_a^* \) be the family of all \( \pi_a^{-1}[N] \) where \( N \in \mathcal{R}_a \). Denote by \( \mathcal{R}^* \) the union of the indexed family \( \{\mathcal{R}_a^*; a \in A\} \). Evidently \( \mathcal{R}^* \) is a family of \( \mathcal{N} \)-sets in \( K \) and
\[
\bigcap \{N; N \in \mathcal{R}^*\} = P.
\]
Since the potency of \( \mathcal{R}^* \) is at most \( m \), the space \( P \) is the intersection of \( m \) \( \mathcal{N} \)-sets in \( K \). Let \( \Phi \) be the \( \check{C} \)ech-Stone mapping from \( \beta P \) onto \( K \). Evidently
\[
\bigcap \{\Phi^{-1}[N]; N \in \mathcal{R}^*\} = P
\]
and \( \Phi^{-1}[N] \) are \( \mathcal{N} \)-sets in \( \beta P \). By 2.9, the space \( P \) is an \( N(m) \)-space. The second proof of 2.20 is complete.

In conclusion we give a summary of definitions of \( N(m) \)-spaces:

2.22. Theorem. The following condition on a completely regular space \( P \) are equivalent:
(1) There exists a complete family $\mathcal{F}$ of continuous functions on $P$ such that the potency of $\mathcal{F}$ is $\leq m$.

(2) There exists a continuous closed compact mapping from $P$ to $E^m$.

(3) There exists a continuous non-extensible mapping from $P$ to $E^m$.

(4) There exists a continuous closed compact mapping from $P$ to an $N(m)$-space.

(5) There exists a continuous non-extensible mapping from $P$ to an $N(m)$-space.

(6) $P$ is an intersection of $m$ $N$-sets in some compactification of $P$.

(7) $P$ is an intersection of $m$ $N$-sets in $\beta P$.

3. IMAGES OF Q-SPACES

All spaces are assumed to be completely regular. Let $\Phi$ be a continuous mapping from a $Q$-space $P$ onto $Q$. Under what conditions on $\Phi$ may we assert that $Q$ is a $Q$-space?

We recall that a subspace $P'$ of $P$ is said to be relatively pseudocompact in $P$, if for every sequence $\{Z_n\}$ of $Z$-sets in $P$ such that $\{Z_n \cap P'\}$ is centered, the intersection $P' \cap \bigcap_{n=1}^{\infty} Z_n$ is non-void. Equivalently, $P'$ is relatively pseudocompact if and only if every continuous function on $P$ is bounded on $P'$.

3.1. Theorem. Let $\Phi$ be a continuous mapping from $P$ onto $Q$ such that

3.1.1. The images of $Z$-sets are $Z$-sets, that is, if $Z$ is a $Z$-set in $P$, then $\Phi[Z]$ is a $Z$-set in $Q$.

3.1.2. The inverses of points under $\Phi$ are relatively pseudocompact, that is, for each $y$ in $Q$ the subspace $\Phi^{-1}[y]$ of $P$ is relatively pseudocompact in $P$.

Then if $P$ is a $Q$-space, $Q$ is also a $Q$-space.

Proof. Let us suppose that $\mathcal{Z}$ is a maximal centered family of $Z$-sets in $Q$ such that the intersection of every countable subfamily of $\mathcal{Z}$ is non-void. Let $\mathcal{Z}'$ be the family of all $\Phi^{-1}[Z]$ where $Z \in \mathcal{Z}$. Evidently, $\mathcal{Z}'$ is a centered family of $Z$-sets in $P$. Let $\mathcal{Z}''$ be a maximal centered family of $Z$-sets in $P$ containing $\mathcal{Z}'$. We shall prove that the intersection of every countable subfamily of $\mathcal{Z}''$ is non-void. Indeed, let $\{Z'_n\}$ be a sequence of $Z$-sets in $\mathcal{Z}''$. By 3.1.1 the sets $Z_n = \Phi[Z'_n]$ are $Z$-sets in $Q$, and clearly $Z_n \in \mathcal{Z}$. Choose a point $y$ in $\bigcap_{n=1}^{\infty} Z_n$. By 3.1.2 we have

$$\Phi^{-1}[y] \cap \bigcap_{n=1}^{\infty} Z_n' \neq \emptyset.$$ 

$P$ being a $Q$-space, the set $\bigcap\{Z'; Z' \in \mathcal{Z}\}$ is non-void. Choose a point $x$ in this intersection. Evidently

$$\Phi(y) \in \bigcap\{Z; Z \in \mathcal{Z}\}.$$ 

The theorem is proved.
We proceed to quotient mappings:

3.2. The image of a Q-space under an open continuous mapping may fail to be a Q-space.

Proof. Let us suppose that \( Q \) is not a Q-space and let

\[ Q = \bigcup \{ K_a; a \in A \}, \]

where \( K_a \) are compact subspaces of \( Q \). Finally, suppose the indexed set \( A \) endow the discrete topology is a Q-space. Under these assumptions we shall construct a Q-space \( P \) and a continuous open mapping \( \Phi \) from \( P \) onto \( Q \). Consider the product space \( R = Q \times A \) and the subspace

\[ P = \bigcup \{ K_a \times a; a \in A \} \]

of \( R \). By 2.19 the space \( P \) is a Q-space. Indeed, the mapping \( x \in K_a \times a \rightarrow a \) is a closed compact continuous mapping of \( P \) onto \( A \). Denote by \( \Phi \) the projection map of \( P \) onto \( Q \), i.e., \( \Phi \) is the restriction of the projection of \( R \) onto \( Q \). It is easy to show that the mapping \( \Phi \) is open and continuous. The proofs of existence of \( Q, K_a \) and \( A \) may be left to the reader.

Modifying the construction in 3.2 (to consider the disjoint union) we obtain at once:

3.3. The image of a Q-space under a compact open continuous mapping may fail to be a Q-space.

Note. If the topological product \( P \times Q \) is a Q-space (\( N(m) \)-space), then both \( P \) and \( Q \) are Q-spaces (\( N(m) \)-spaces, respectively).

We shall need the following assertion:

3.4. Lemma. Let \( \Phi \) be an open, closed and continuous mapping from \( P \) onto \( Q \). Let \( f \) be a continuous function on \( P \). For each \( y \) in \( Q \) put

\[ F(y) = \sup \{ f(x); \Phi(x) = y \}. \]

If \( F(y) \) is a real number for each \( y \) in \( Q \), then \( F \) is a continuous function on \( Q \).

Proof. Let \( y_0 \) be an element of \( Q \) and let \( \epsilon \) be a positive real number. For each \( x \) in \( X = \Phi^{-1}[y_0] \) choose an open neighborhood \( U(x) \) of \( x \) on which \( f \) varies less than \( \epsilon \). Denote by \( U \) the union of all \( U(x), x \in X \). Put

\[ V = \bigcup \{ \Phi^{-1}[y]; \Phi^{-1}[y] \subset U \}. \]

\( \Phi \) being closed, \( V \) is an open subset of \( P \). Evidently

\[ y \in \Phi[V] \Rightarrow F(y) \leq F(y_0) + \epsilon. \]

\( \Phi \) being an open mapping, \( \Phi[V] \) is an open neighborhood of \( y_0 \). Thus \( F \) is an upper semi-continuous function. It remains to prove that \( F \) is lower semi-
continuous. Choose a point \( x_0 \) in \( X \) such that \( f(x_0) > F(y_0) - \frac{\varepsilon}{2} \). Choose an open neighborhood \( W \) of \( x_0 \) such that
\[
x \in W \implies f(x) > f(x_0) - \frac{\varepsilon}{2}.
\]
\( \Phi \) being open, \( W' = \Phi[W] \) is an open neighborhood of \( y_0 \). We have at once
\[
y \in W' \implies F(y) > F(y_0) - \varepsilon.
\]
This establishes lower semi-continuity of \( F \) and completes the proof of 3.4.

3.5. Proposition. Let us suppose that \( \Phi \) is a closed open and continuous mapping from \( P \) onto \( Q \). Let \( \mathcal{F} \) be a complete family of continuous non-negative functions on \( P \). Suppose that for each \( f \) in \( \mathcal{F} \) the function \( F \) defined as in 3.4 is real-valued, that is, \( F \) is finite. Denote by \( \mathcal{F}' \) the family of all \( F \) where \( f \in \mathcal{F} \).

Then \( \mathcal{F}' \) is a complete family of continuous functions on \( Q \).

Proof. By 3.4 the functions \( F \in \mathcal{F}' \) are continuous. To prove completeness of \( \mathcal{F}' \), let \( \mathcal{Z} \) be a centered family of closed subsets of \( Q \) such that for each \( F \) in \( \mathcal{F}' \) there exists a \( Z \) in \( \mathcal{Z} \) with \( F \) bounded on \( Z \). Denote by \( \mathcal{Z}' \) the family of all \( \Phi^{-1}[Z] \) where \( Z \) is in \( \mathcal{Z} \). Evidently, \( \mathcal{Z}' \) is a centered family of closed subsets of \( P \).

Moreover, for each \( f \) in \( \mathcal{F} \) there exists a \( Z \) in \( \mathcal{Z}' \) such that \( f \) is bounded on \( Z \). Indeed, if \( F \) is a function corresponding to \( f \), we may put \( Z = \Phi^{-1}[Z_f] \). \( \mathcal{F} \) being a complete family, we have
\[
F = \bigcap\{Z; Z \in \mathcal{Z}'\} \neq \emptyset.
\]
Clearly \( \Phi[F] \subset \bigcap\{Z; Z \in \mathcal{Z}\} \). Thus \( \mathcal{F}' \) is complete.

As an immediate consequence of 3.5 we have:

3.6. Theorem. Let \( \Phi \) be a closed, open and continuous mapping from \( P \) onto \( Q \). Suppose that the tranches of \( \Phi \) (that is, the sets of the form \( \Phi^{-1}[y], y \in Q \)) are relatively pseudocompact spaces. If \( P \) is an \( N(m) \)-space, then \( Q \) is an \( N(m) \)-space. In particular, if \( P \) is a \( Q \)-space, then \( Q \) is a \( Q \)-space.

4. \( N(1) \)-SPACES AND \( N(\aleph_0) \)-SPACES

4.1. Theorem. Let \( m \geq 1 \). A discrete space \( M \) is an \( N(m) \)-space if and only if it is homeomorphic with some closed subspace (discrete, of course) of \( E^m \).

Proof. The theorem is obvious for finite \( m \). Suppose \( m \geq \aleph_0 \). We shall use 2.22, condition 2.22.2. To prove necessity let us suppose that \( \Phi \) is a continuous, closed and compact mapping from \( M \) to \( E^m \). The tranches of \( \Phi \) being compact and discrete, they are finite. Thus \( M \) and \( \Phi[M] \) has the same potency. The image under closed mappings of a discrete space is a discrete space. The discrete
spaces $M$ and $\Phi[M]$ have the same potency, and consequently, they are homeomorphic. The sufficiency is obvious.

As a corollary of 4.1 and of the fact that $E^\aleph_0$ is a metrizable and separable space we have

4.2. **Theorem.** The following conditions on a discrete space are equivalent:

4.2.1. $M$ is a $N(1)$-space.

4.2.2. $M$ is a $N(\kappa_0)$-space.

4.2.3. The potency of $M$ is at most $\kappa_0$.

4.3. **Theorem.** The following conditions on a space $P$ are equivalent

4.3.1. $P$ is a $N(1)$-space.

4.3.2. There exists a continuous function $f$ on $P$ such that every closed subspace $K$ of $P$ is compact if and only if the function $f$ is bounded on $K$.

4.3.3. There exists a sequence $\{K_n\}$ of compact subspaces of $P$ such that $K_n \subset \text{int } K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = P$.

4.3.4. $P$ is locally compact and $\sigma$-compact.

**Proof.** By 2.4 the conditions 4.3.1 and 4.3.2 are equivalent. If $f$ is the function from 3.1.2 and if we put

$$K_n = \{x; |f(x)| \leq n\},$$

we obtain a sequence $\{K_n\}$ satisfying 3.1.3. Thus 3.1.2 implies 3.1.3. Suppose 3.1.3. Choose continuous functions $f_n$, $n = 3, 4, \ldots$, such that

$$f_n(x) = \begin{cases} n & \text{for } x \notin K_{n-1}, \\ 0 & \text{for } x \in K_{n-1} \end{cases}$$

and $0 \leq f_n(x) \leq n$ for every $x$. Put $f = \sum_{n=3}^{\infty} f_n$. Evidently, $f$ is bounded on a set $M$ if and only if the set $M$ is contained in some finite union $\bigcup_{i=1}^{n} K_i$. It follows at once that $f$ satisfies 3.1.2. Thus 3.1.3 implies 3.1.2. The proof of equivalence of 3.1.3 and 3.1.4 is quite routine and may be left to the reader.

4.4. **Theorem.** A metrizable space is an $N(1)$-space if and only if it is separable and locally compact.

**Proof.** First suppose that $P$ is a metrizable $N(1)$-space. Evidently $P$ is locally compact. By 4.2 and 2.7 the space $P$ contains no uncountable discrete closed subset. Thus $P$ is separable. Conversely, $P$ being a separable and metrizable space, $P$ has a compact metrizable extension $K$. $P$ being locally compact, $P$ is open in $K$. An open subset of a metrizable space is an $N$-set. Thus $P$ is an $N$-set of a compact space $K$. It follows that $P$ is an $N(1)$-space.
Recall that a space $P$ is said to be a $G_δ$-space if it is a $G_δ$-set in every extension. It is well-known that a metrizable space is a $G_δ$-space if and only if there exists a metric $φ$ for $P$ such that $(P, φ)$ is a complete metric space (for further informations see [2]). By [2], theorem 2.8, a completely regular space $P$ is a $G_δ$-space if and only if it is a $G_δ$-subset of some compact space. Thus every $N(σ_0)$-space is a $G_δ$-space.

4.5. **Theorem.** The following two conditions on a metrizable space $P$ are equivalent:

4.5.1. $P$ is an $N(σ_0)$-space.

4.5.2. $P$ is a separable $G_δ$-space.

**Proof.** As we note above, an $N(σ_0)$-space is a $G_δ$-space. Thus, to prove that 4.5.1 implies 4.5.2, it is sufficient to show that every metrizable $N(σ_0)$-space is separable. By 4.2 and 2.7 the space $P$ contains no uncountable discrete closed subspace. It follows that $P$ is separable. Conversely, suppose 4.5.2. $P$ being separable, there exists a compact metrizable extension $K$ of $P$. $P$ being a $G_δ$-space, $P$ is a $G_δ$-subset of $K$, and consequently, $P$ is an $N(σ_0)$-set in $K$. Thus $P$ is an $N(σ_0)$-space.

**Bibliography**


счетно центрированная система Z-множеств (т. е. максимальная система Z-множеств такая, что любая счетная подсистема имеет непустое пересечение) имеет непустое пересечение. В статье дается определение Q-пространств при помощи полных семейств непрерывных функций. Семейство \( \mathfrak{F} \) непрерывных функций называется полным, если выполнено следующее условие:

Если \( \mathfrak{Z} \) — такая центрированная система Z-множеств, что всякая функция из \( \mathfrak{F} \) ограничена на некотором множестве из \( \mathfrak{Z} \), то пересечение системы \( \mathfrak{Z} \) не пусто.

Оказывается, что вполне регулярное пространство является Q-пространством тогда и только тогда, если семейство всех непрерывных функций полно.

В статье определены т. наз. \( N(m) \)-пространства (м — некоторое кардинальное число). \( P \) называется \( N(m) \)-пространством если существует полное семейство \( \{ f_\alpha; \, \alpha \in A \} \) непрерывных функций на \( P \) такое, что мощность множества \( A \) равна \( m \). Итак, вполне регулярное пространство является Q-пространством тогда и только тогда, если оно является \( N(m) \)-пространством для некоторого \( m \). Пусть \( \Phi \) — отображение пространства \( P \) в пространство \( Q \); \( \Phi \) называется бикомпактным, если прообразы точек бикомпактны, замкнутыми, если образы замкнутых множеств замкнуты; наконец, непрерывное \( \Phi \) называется нерасширимым, если, какого бы ни было пространство \( R, R \supset P, R = \bar{P}, R \neq P \), отображение \( \Phi \) нельзя расширить до непрерывного отображения пространства \( R \) в \( Q \). Доказана следующая

Теорема. Следующие свойства вполне регулярного пространства \( P \) эквивалентны (м — кардинальное число):

1. \( P \) является \( N(m) \)-пространством.
2. Существует бикомпактное замкнутое и непрерывное отображение пространства \( P \) в топологическое произведение \( m \) прямых.
3. Существует непрерывное нерасширимое отображение пространства \( P \) в топологическое произведение \( m \) прямых.
4. \( P \) является пересечением \( m \) N-множеств в некотором своем бикомпактном расширении.
5. \( P \) является пересечением \( m \) N-множеств с своим чеховским бикомпактным расширении.

В последней части рассматривается вопрос, при каких условиях непрерывный образ \( N(m) \)-пространства является \( N(m) \)-пространством. Указывается, что достаточно предполагать, что отображение замкнуто, открыто, и полные прообразы точек относительно псевдокомпактны.