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APPLICATIONS OF COMPLETE
FAMILIES OF CONTINUOUS FUNCTIONS TO THE THEORY
OF Q-SPACES

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In the present paper the concept of a complete family of continuous functions is introduced and applied to the theory of $N(m)$-spaces (intersections of $m$ $N$-sets in their Stone-Čech extensions) and, in particular, $Q$-spaces. $N(m)$-spaces may be defined as the inverse images under continuous closed compact mappings to the topological product of $m$ real lines. The section 3 is devoted to the problem, under what conditions on the mapping is the image of a $N(m)$-space (in particular, of a $Q$-space) an $N(m)$-space (a $Q$-space, respectively).

In [2] the concept of a complete indexed family of open coverings of a space has been introduced. For convenience, we recall the definition. An indexed family of open coverings

\[ \{\mathfrak{M}_a; a \in A\} \]

is said to be complete if the following condition is satisfied:

If \{\mathcal{F}\} is a centered family of closed subsets of $P$ such that for each $a$ in $A$ there exists a $V_a$ in $\mathfrak{M}_a$ containing some $F_a \in \{\mathcal{F}\}$, then $\bigcap \{\mathcal{F}\} \neq \emptyset$.

In [2] the following theorem was proved:

A completely regular space $P$ is an intersection of $m$ open sets in every compact extension of $P$ if and only if there exists a complete indexed family (1) of open coverings of $P$ such that the potency of $A$ is $m$.

In the present paper we investigate spaces possessing a complete family of open coverings (1) of a special sort. If $f$ is a continuous real-valued function on $P$ then the open cover consisting of sets

\[ \{x; |f(x)| < n\}, \quad n = 1, 2, \ldots \]

will be denoted by $\mathfrak{N}(f)$. We shall consider coverings of the form $\mathfrak{N}(f)$ only. We shall prove that a completely regular space $P$ possesses a family of continuous functions $\mathfrak{F}$ such that

\[ \{\mathfrak{N}(f); f \in \mathfrak{F}\} \]
is complete (such a family $\mathcal{F}$ is said to be complete) if and only if there exists a indexed family $\{N_f; f \in \mathcal{F}\}$ of $N$-sets in $\beta P$ such that

$$P = \bigcap \{N_f; f \in \mathcal{F}\}.$$  

If the potency of $\mathcal{F}$ is at most $m$, then such spaces will be called $N(m)$-spaces. A space is a $Q$-space (for Hewitt's definition of $Q$-spaces see [3]) if and only if it is a $N(m)$-space for some cardinal $m$.

If $f$ is a continuous function, then $f$ is bounded on a set $M$ if and only if there exists a set in $\mathfrak{B}(f)$ containing $M$. Thus we obtain a definition of complete families of continuous functions which does not use coverings.

In section 1 we shall study complete families of continuous functions on an arbitrary space. For convenience we shall use a more general definition of a complete family. But for completely regular spaces both definitions are identical.

In section 2 we shall investigate complete families on completely regular spaces, more precisely, we shall study $N(m)$-spaces (in particular, $Q$-spaces) using the concept of a complete family of continuous functions.

The section 3 is devoted to the question:

Let $\Phi$ be a mapping from a $N(m)$-space onto a space $Q$. Under what conditions on $\Phi$ may we assert that $Q$ is a $N(m)$-space.

If $\mathcal{F}$ is a family of sets, then the intersection of $\mathcal{F}$ will be denoted by $\bigcap \mathcal{F}$, that is

$$\bigcap \mathcal{F} = \bigcap \{Z; Z \in \mathcal{F}\}.$$  

For convenience we shall use the following convention: If $V$ is a property of sets, then a indexed family $\{M_a; a \in A\}$ is said to have the property $V$ if the set of all $M_a$ has the property $V$. If $V$ is a property of indexed families, then a set $\mathfrak{M}$ has the property $V$ if the indexed family $\{M; M \in \mathfrak{M}\}$ has the property $V$.

A topological space (in the sequel a space, merely) $P$ is said to be an extension of a space $R$ if $R$ is a dense subspace of $P$. An extension $P$ of $R$ is said to be Hausdorff, regular, completely regular, compact if $P$ is a Hausdorff, regular, completely regular, compact space, respectively. The Čech-Stone extension of a completely regular space $P$ will be denoted by $\beta P$. It is well-known that $\beta P$ is the compact extension of $P$ uniquely determined by the property:

every bounded real-valued continuous function on $P$ has a continuous extension over $\beta P$.

It is also well-known that if $K$ is a compact extension of $P$, then there exists one and only one continuous mapping $\Phi$ from $\beta P$ onto $K$ such that the restriction of $\Phi$ to $P$ is the identity mapping. This mapping will be called Čech-Stone mapping.

Function will always mean a real-valued function. A subset $M$ of a space is said to be a $Z$-set if there exists a continuous function $f$ on $P$ such that

$$M = Z(f) = \{x; f(x) = 0\}.$$
A subset $M$ of a space $P$ is said to be a $N$-set if $P - M$ is a $Z$-set. We shall use the notation

$$N(f) = \{x; f(x) \neq 0\}.$$ 

1. COMPLETE FAMILIES OF FUNCTIONS

1.1. Definition. Let $\mathfrak{F}$ be a family of continuous functions on a space $P$. $\mathfrak{F}$ is said to be complete if the following conditions is satisfied:

1.1.1. If $\mathfrak{F}$ is centered family of $Z$-sets in $P$ and if for each $f$ in $\mathfrak{F}$ there exists a $Z_f$ in $\mathfrak{F}$ such that $f$ is bounded on $Z_f$, then $\bigcap \mathfrak{F} \neq \emptyset$.

Note. We have at once that a family of continuous functions containing a complete family is a complete family.

1.2. Lemma. Let $\mathfrak{F}$ be a maximal centered family of $Z$-sets in a space $P$ such that the intersection of every countable subfamily is non-void. For every continuous function $f$ on $P$ there exists a $Z$ in $\mathfrak{F}$ on which $f$ is bounded.

Proof. Let $f$ be a continuous function on $P$. For every $n = 1, 2, \ldots$ denote by $Z_n$ the set

$$(2) \quad \{x; x \in P, |f(x)| \geq n\}.$$ 

If for some $n$ the set $Z_n$ does not belong to $\mathfrak{F}$, then there exists a $Z$ in $\mathfrak{F}$ with $Z_n \cap Z = \emptyset$. Then $|f(x)| \leq n$ for $x$ in $Z$ and hence $f$ is bounded on $Z$. In the other case we have $Z_n \in \mathfrak{F}$ for every $n = 1, 2, \ldots$. By our assumption we have

$$Z_0 = \bigcap_{n=1}^{\infty} Z_n \neq \emptyset.$$ 

According to (2)

$$x \in Z_0 \Rightarrow |f(x)| \geq n$$ 

for every $n$, which is impossible since $f$ is finite-valued.

As an immediate consequence of 1.2 we have

1.3. Theorem. If there exists a complete family of continuous functions on a space $P$, then the following condition is satisfied:

1.3.1. If $\mathfrak{F}$ is a maximal centered family of $Z$-sets in $P$ such that the intersection of every its countable subfamily is non-void, then $\bigcap \mathfrak{F} \neq \emptyset$.

1.4. Lemma. Let $\mathfrak{F}$ be a maximal centered family of $Z$-sets in $P$. If the intersection of some countable subfamily of $\mathfrak{F}$ is empty, then there exists a continuous function $f$ on $P$ which is bounded on no $Z$ in $\mathfrak{F}$.

Proof. Let $\{Z_n\}$ be a sequence in $\mathfrak{F}$ such that

$$\bigcap_{n=1}^{\infty} Z_n = \emptyset.$$ 

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Choose continuous functions $f_n$ on $P$ such that $Z_n = Z(f_n)$ and $0 \leq f_n \leq 1$.

Consider the continuous function $g = 1/f$ where $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. Clearly:

$$x \in \bigcap_{i=1}^{n} Z_i \Rightarrow f(x) \leq \frac{1}{2^n},$$

and hence, $g(x) \geq 2^n$ for each $x$ in $\bigcap_{i=1}^{n} Z_i$. It follows immediately that $f$ is bounded on no $Z$ in $\mathcal{Z}$.

As a corollary of 1.4 we have:

1.5. **Theorem.** If a space $P$ satisfies the condition 1.3.1, then the family of all continuous function is complete.

1.6. **Definition.** A space is said to be quasi-compact if the intersection of every centered family of $Z$-sets is non-void. A subspace $R$ of $P$ is said to be relatively quasi-compact in $P$ if the following condition is satisfied:

1.6.1. If $\mathcal{Z}$ is a family of $Z$-sets in $P$ and if $\mathcal{Z} \cap R$ is a centered family, then $\bigcap \mathcal{Z} \cap R \neq \emptyset$.

**Note.** Evidently every $Z$-set of a quasi-compact space $P$ is relatively quasi-compact in $P$. Moreover, every intersection of $Z$-sets of a quasi-compact space $P$ is relatively quasi-compact in $P$. For further information see [1], 200—202.

1.7. **Theorem.** Let $\mathcal{F}$ be a family of continuous functions on a space $P$. $\mathcal{F}$ is complete if and only if the following two conditions 1.7.1 and 1.7.2 are satisfied:

1.7.1. If $F$ is intersection of $Z$-sets in $P$ and if every $f \in \mathcal{F}$ is bounded on $F$, then $F$ is relatively quasi-compact in $P$.

1.7.2. If $\{Z_f; f \in \mathcal{F}\}$ is a centered indexed family of $Z$-sets and if $f$ is bounded on $Z_f$, then

$$\bigcap \{Z_f; f \in \mathcal{F}\} \neq \emptyset.$$

**Proof.** The necessity of conditions 1.7.1 and 1.7.2 is quite obvious. To prove the sufficiency, suppose that $\mathcal{Z}$ is a centered family of $Z$-sets in $P$ and that for each $f$ in $\mathcal{F}$ there exists a $Z_f$ in $\mathcal{Z}$ on which $f$ is bounded. By 1.7.2 the set

$$F = \bigcap \{Z_f; f \in \mathcal{F}\}$$

is non-void. By 1.7.1 the set $F$ is relatively quasi-compact in $P$. Consequently, to prove $\bigcap \mathcal{Z} \neq \emptyset$ it is sufficient to show that $\mathcal{Z} \cap F$ is a centered family. But if both $Z_1$ and $Z_2$ belong to $\mathcal{Z}$, there again by 1.7.2 the set

$$\bigcap \{Z_1 \cap Z_2 \cap Z_f; f \in \mathcal{F}\} = F \cap Z_1 \cap Z_2$$

is non-void. The proof is complete.

We shall need the following
1.8. Lemma. If $\mathcal{Z}$ is a maximal centered family of Z-sets in $P$ and if the Z-sets $Z_1, \ldots, Z_k$ cover some $Z \in \mathcal{Z}$, then some $Z_i$ belongs to $\mathcal{Z}$.

Proof. Suppose on the contrary that no $Z_i$ belongs to $\mathcal{Z}$. According to the maximality of $\mathcal{Z}$ there exist $Z'_i \in \mathcal{Z}$, $i = 1, 2, \ldots, k$, such that $Z_i \cap Z'_i = \emptyset$. Then $Z \cap \bigcap_{i=1}^{k} Z'_i$ belongs to $\mathcal{Z}$, which is impossible since

$$Z \cap \bigcap_{i=1}^{k} Z'_i \subset Z - \bigcup_{i=1}^{k} Z_i = \emptyset.$$ 

This contradiction completes the proof.

Now we proceed to characterize complete families in terms of mappings of a special sort.

1.9. Definition. A mapping from $P$ to $Q$ is said to be quasi-compact if the inverse image of every point of $Q$ is relatively quasicompact in $P$. A mapping from $P$ to $Q$ is said to be a Z-mapping if the image of every Z-set of $P$ is closed in $Q$.

1.10. Theorem. Let $\mathcal{F}$ be a family of continuous functions on a space $P$. Consider the space

$$E^\mathcal{F} = \bigtimes \{E_f; f \in \mathcal{F}\},$$

where the $E_i$ are real lines; and also the continuous mapping $\Phi: P \to E^\mathcal{F}$ defined as follows:

$$\Phi(x) = \{f(x); f \in \mathcal{F}\}.$$ 

The family $\mathcal{F}$ is complete if and only if $\Phi$ is a quasi-compact Z-mapping of $P$ to $E^\mathcal{F}$.

Proof. First let us suppose that $\mathcal{F}$ is a complete family. To prove quasi-compactness of $\Phi$ we shall show that

1.10.1. The inverse image of every compact subspace $K$ of $E^\mathcal{F}$ is relatively quasi-compact in $P$.

It is easy to see that every function $f$ from $\mathcal{F}$ is bounded on $\Phi^{-1}[K]$. Indeed, we have we have $f(x) = \pi_f(\Phi(x))$ and $f[\Phi^{-1}[K]] = \pi_f[K]$ where $\pi_f$, denotes the projections of $E^\mathcal{F}$ onto $E_f$. Since $\pi_f$ is a continuous function and $K$ is a compact space, $\pi_f[K]$ is a compact subspace of $E_f$, and consequently, $\pi_f[K]$ is a bounded subspace of $E_f$ (in the usual metric). $K$ is a compact subspace of the completely regular space $E^\mathcal{F}$ and therefore $K$ is an intersection of Z-sets in $E^\mathcal{F}$. Since $\Phi$ is a continuous mapping, it follows at once that $f^{-1}[K]$ is an intersection of Z-sets in $P$. By Theorem 1.7 the subspace $f^{-1}[K]$ of $P$ is relatively quasi-compact in $P$. Thus 1.10.1 holds and $\Phi$ is a quasi-compact mapping. It remains to prove that $\Phi$ is a Z-mapping. Let $Z_0$ be a Z-set in $P$. Suppose on the contrary that $\Phi[Z_0] = \emptyset$.
$F$ is not closed in $E^0$. Then we may choose $y = \{y_f; f \in \mathcal{F}\}$ in $F - F$. Consider the family

$$\mathcal{J} = \{Z_n, f; f \in \mathcal{F}, \ n = 1, 2, \ldots\} \cup (Z_0)$$

of $Z$-sets in $P$, where

$$Z_{n,f} = \{x; x \in P, |f(x) - y_f| \leq \frac{1}{n}\}.$$ 

The point $y$ being an accumulation point of $F$, the family $\mathcal{J}$ is centered. Moreover, each $f$ in $\mathcal{F}$ is bounded on $Z_{n,f}$. It follows that $\bigcap \mathcal{J} \neq \emptyset$. But this is impossible since

$$\bigcap \mathcal{J} = \Phi^{-1}[y] \cap Z_0$$

and by our assumption $y$ does not belong to $F = \Phi[Z_0]$, that is, $\bigcap \mathcal{J} = \emptyset$. This contradiction completes the proof of necessity.

To prove sufficiency let us suppose that $\Phi$ is a quasi-compact $Z$-mapping. Let $\mathcal{J}$ be a maximal centered family of $Z$-sets in $P$ and suppose that for each $f$ in $\mathcal{F}$ there exists a $Z_f$ in $\mathcal{J}$ such that $f$ is bounded on $Z_f$. From quasi-compactness of $\Phi$ it follows at once that it is sufficient to prove the existence of a point $y = \{y_f; f \in \mathcal{F}\}$ in $E^0$ such that $\mathcal{J} \cap \Phi^{-1}[y]$ is a centered family. We proceed to construct such a point $y$.

Choose $f$ in $\mathcal{F}$. By our assumption $f$ is bounded on $Z_f$. Hence, there exists a bounded interval $I_f$ of $E_f$ such that

$$f[Z_f] \subset I_f.$$ 

Let $K_1, \ldots, K_k$ be a finite cover of $I_f$ by closed intervals of length less than $\frac{1}{n}$. Since $f^{-1}[K_i]$ are $Z$-sets in $P$ and

$$\bigcup_{i=1}^{k} f^{-1}[K_i] \supset Z_f \in \mathcal{J},$$

it follows at once from lemma 1.8 that for some $i = 1, \ldots, k$, $f^{-1}[K_i]$ belongs to $\mathcal{J}$.

Thus, for every $n = 1, 2, \ldots$ and for each $f$ in $\mathcal{F}$ there exists a closed interval $K_{n,f}$ in $E_f$ of length less than $\frac{1}{n}$ such that

$$Z_{n,f} = f^{-1}[K_{n,f}] \in \mathcal{J}.$$ 

Evidently for every $f$ in $\mathcal{F}$, $\{K_{n,f}; \ n = 1, 2, \ldots\}$ is a centered family of compact sets. It follows that

$$\bigcap_{n=1}^{\infty} K_{n,f} \neq \emptyset.$$ 

This intersection contains only one point, namely $y_f$, since the lengths of $K_{n,f}$ converge to zero with $n \to \infty$. The point $\{y_f; f \in \mathcal{F}\}$ will be denoted by $y$. Since
\( \Phi[P] \) is a closed subspace of \( E^\mathfrak{A} \), \( y \) belongs to \( \Phi[P] \). It remains to prove that \( \mathfrak{A} \cap \Phi^{-1}[y] \) is a centered family. It is of course sufficient to show that

\[
Z \in \mathfrak{A} \Rightarrow Z \cap \Phi^{-1}[y] = \emptyset.
\]

Let us suppose on the contrary that some \( Z \) in \( \mathfrak{A} \) does not meet \( \Phi^{-1}[y] \). The mapping \( \Phi \) is a \( Z \)-mapping and hence \( F = \Phi[Z] \) is a closed subspace of \( E^\mathfrak{A} \). By our assumption \( y \) does not belong to \( F \). In consequence, there exists a neighborhood \( U \) of \( y \) which does not meet \( F \). Since the lengths of \( K_{n,i} \) converge to zero with \( n \to 0 \), there exist \( K_i = K_{n_i,i} \) \( (i = 1, \ldots, k) \) such that

\[
\bigcap_{i=1}^{k} f_i^{-1}[K_i] \cap Z = \emptyset.
\]

But this is a contradiction, since \( f_i^{-1}[K_i] \) belong to \( \mathfrak{A} \). The proof is complete.

As a corollary of 1.10 and 1.10.1 we have

1.11. Theorem. If \( \Phi \) is a quasi-compact \( Z \)-mapping from \( P \) to the topological product \( R \) of a family of real lines, then the inverse image of every compact subspace of \( R \) is a relatively quasi-compact subspace of \( P \).

2. \( Q \)-SPACES AND \( N(m) \)-SPACES

In this section we shall study complete families of continuous functions on a completely regular space.

2.1. Definition. Let \( m \) be a cardinal number. A space \( P \) is said to be an \( N(m) \)-space provided that \( P \) is completely regular and there exists a complete family \( \mathfrak{A} \) of continuous functions on \( P \) such that the potency of \( \mathfrak{A} \) is \( \leq m \). A space is said to be an exact \( N(m) \)-space provided that it is an \( N(m) \)-space but not an \( N(n) \)-space for any cardinal \( n < m \). A space is a \( Q \)-space if it is an \( N(m) \)-space for some cardinal \( m \).

Thus a completely regular space is a \( Q \)-space if and only if the set of all continuous functions is complete.

2.2. Definition. A mapping \( \Phi \) of \( P \) to \( Q \) is said to be compact if the inverse images of points of \( Q \) are compact spaces. \( \Phi \) is closed if the image of every closed subset of \( P \) is closed in \( Q \).

2.3. Lemma. A relatively quasi-compact subspace \( R \) of a completely regular space \( P \) is a compact space. A quasi-compact mapping from a completely regular space to a space is a compact mapping. A quasi-compact \( Z \)-mapping from a completely regular space to a space is a compact closed mapping.

Proof. Let \( R \) be relatively quasi-compact in a completely regular space \( P \). Let \( \{F\} \) be a centered family of closed subsets of \( R \). Let \( \mathfrak{A} \) be the family of all \( Z \)-sets in \( P \) such that for some \( F \) in \( \{F\} \) the inclusion \( F^P \subset Z \) holds. Since \( P \) is
a completely regular space, we have $\bigcap \mathcal{Z} = \bigcap \{F\}$. By quasicompactness of $R$ we have

$$R \cap \bigcap \mathcal{X} = \emptyset.$$ Combining the above two relations we obtain $\bigcap \{F\} = \emptyset$.

The second statement of the lemma is an immediate consequence of the first.

To prove the third statement let us suppose that $\Phi$ is a quasi-compact $Z$-mapping from a completely regular space $P$ to $Q$. Then $\Phi$ is a compact mapping and it remains to show that $\Phi$ is a closed mapping. Let $F$ be a closed subspace of $P$. Denote by $\mathcal{Z}$ the family of all $Z$-sets in $P$ containing $F$. Since $P$ is completely regular, we have $\bigcap \mathcal{Z} = F$. Put $F_0 = \Phi[F]$. It is sufficient to prove

$$F_1 = \bigcap \{\Phi[Z]; Z \in \mathcal{Z}\} = F_0.$$

The inclusion $F_1 \supset F_0$ is trivial. For the other one, suppose that there exists a point $y$ in $F_1 - F_0$. We see at once that

$$(4) \quad \mathcal{Z} \cap \Phi^{-1}[y]$$

is a centered family of closed subsets of the compact space $\Phi^{-1}[y]$. Thus we may choose a point $x$ in the intersection of the family (4). But this is impossible since

$$x \in \bigcap \mathcal{Z} = F, \quad \Phi(x) = y \notin \Phi[F].$$

From 1.7 and 2.3 we have

2.4. Theorem. Suppose that $\mathcal{F}$ is a family of continuous functions on a completely regular space $P$. $\mathcal{F}$ is complete if and only if the following two conditions are satisfied:

2.4.1. If $K$ is closed in $P$ and if each $f$ from $\mathcal{F}$ is bounded on $K$, then $K$ is a compact space.

2.4.2. If $\{Z_f; f \in \mathcal{F}\}$ is a centered indexed family of $Z$-sets in $P$ such that $f$ is bounded on $Z_f$, then

$$\bigcap \{Z_f; f \in \mathcal{F}\} = \emptyset.$$ Definition. $m$ being a cardinal number, denote by $E^m$ the topological product of $m$ real lines.

As an immediate consequence of 1.10 and 2.2 we have

2.5. Theorem. Let $\mathcal{F}$ be a family of continuous functions on a completely regular space $P$. Define $E^\mathcal{F}$ and $\Phi$ as in 1.10. $\mathcal{F}$ is complete if and only if $\Phi$ is a closed compact mapping.

2.6. Theorem. A space $P$ is an $N(m)$-space if and only if $P$ is completely regular and there exists a continuous closed compact mapping $\Phi$ from $P$ to $E^m$.

Proof. First suppose that $P$ is an $N(m)$-space. Hence $P$ is completely regular and there exists a complete family $\mathcal{F}$ of continuous function on $P$ such that the
potency of \( F \) at most \( m \). Without loss of generality we may assume that the potency of \( F \) is \( m \). Define \( E^F \) and \( \Phi \) as in 1.10. By 2.5 \( \Phi \) is closed and compact.

Evidently \( \Phi \) is continuous and \( E^F = E^m \).

Conversely, let \( \Phi \) be a continuous closed compact mapping from a completely regular space \( P \) to

\[
E^m = \{ E_a; a \in A \}
\]

where the potency of the index set \( A \) is \( m \) and the \( E_a \) are real lines. For each \( a \) in \( A \) denote by \( \pi_a \) the projection of \( E^m \) onto \( E_a \). Denote by \( f_a \) the function \( \pi_a(\Phi) \). Every \( f_a \) is continuous as the superposition of two continuous mappings. Evidently for each \( x \) in \( P \),

\[
\Phi(x) = \{ f_a(x); a \in A \}.
\]

Applying 2.5 we obtain that the family of all \( f_a \) is complete.

If \( \Phi \) is a closed compact mapping from \( P \) to \( Q \) and if \( F \) is a closed subset of \( P \), then the restriction of \( \Phi \) to \( F \) is a closed compact mapping. From this fact and from 2.5 and 2.6 we have at once

2.7. Theorem. If \( F \) is a complete family of continuous functions on a completely regular space \( P \) and if \( F \) is a closed subspace of \( P \), then the family of the restrictions (to \( F \)) of all \( f \in F \) is a complete family on \( F \). Closed subspaces of \( N(m) \)-spaces are \( N(m) \)-spaces.

Now we proceed to characterize \( N(m) \)-spaces as intersections of \( m \) \( N \)-sets in their \( Čech-Stone \) extensions.

2.8. Proposition. Let \( \mathcal{F} \) be a family of continuous functions on a completely regular space \( P \) such that \( f \geq 1 \) for each \( f \) in \( \mathcal{F} \). For each \( f \) in \( \mathcal{F} \) denote by \( f^* \) the continuous extension of \( 1/f \) over \( \beta P \) (\( 1/f \) is bounded).

Then \( \mathcal{F} \) is complete if and only if

2.8.1.

\[
\bigcap \{ N(f^*); f \in \mathcal{F} \} = P.
\]

Proof. First let us suppose that 2.8.1 holds. Let \( \mathcal{Z} \) be a centered system of \( Z \)-sets in \( P \) such that for each \( f \) in \( \mathcal{F} \) there exists a \( Z_f \) in \( \mathcal{Z} \) on which \( f \) is bounded. \( \beta P \) being a compact space, the set

\[
F_0 = \bigcap \{ \bar{Z}^{\beta P}; Z \in \mathcal{Z} \}
\]

is non-void. It is sufficient to show that \( F \subseteq P \). According to 2.8.1 it is sufficient to show that

(5) \( \bar{Z}_f^{\beta P} \subseteq N(f^*) \)

for each \( f \) in \( \mathcal{F} \), \( f \) is bounded on \( Z_f \),

\[
x \in Z_f \Rightarrow |f(x)| \leq M
\]

say, and hence \( (f^*) \) is continuous

\[
x \in \bar{Z}_f^{\beta P} \Rightarrow |f^*(x)| \geq M^{-1}
\]

which implies (5).

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To prove necessity, let us suppose that there exists a point $x$ in 
\[ \bigcap \{N(f^*); f \in \mathcal{F}\} = \emptyset. \]
Let $\mathcal{Z}$ be the family of all $Z$-sets in $\beta P$ containing $x$ in their interior. Evidently 
\[ \bigcap \{Z; Z \in \mathcal{Z}\} = (x) \subset \beta P \setminus P. \]

Thus $\mathcal{Z} \cap P$ is a centered family of $Z$-sets in $P$ with empty intersection. To prove that $\mathcal{Z}$ is not complete, it is sufficient to show that for each $f$ in $\mathcal{F}$ there exists a $Z_f$ in $\mathcal{Z}$ such that $f$ is bounded on $Z_f$. Fix $f \in \mathcal{F}$. Since $f^*(x) = 0$, there exist a $Z_f$ in $\mathcal{Z}$ and an $\varepsilon > 0$ with 
\[ y \in Z_f \Rightarrow |f^*(y)| \geq \varepsilon. \]
If follows that $y \in Z \cap P \Rightarrow |f(y)| \leq 1/\varepsilon$. The proof is complete.

As an immediate consequence of 2.8 we have:

**2.9. Theorem.** A completely regular space $P$ is an $N(m)$-space if and only if there exists a set $\mathfrak{N}$ of $N$-sets in $\beta P$ such that the potency of $\mathfrak{N}$ is at most $m$ and 
\[ \bigcap \{N; N \in \mathfrak{N}\} = P. \]

Now we shall proceed to give the usual characterisation of $N(m)$-spaces. First we prove the following crucial property of continuous closed compact mappings.

**2.10. Theorem.** Let $\Phi$ be a continuous closed compact mapping from a regular space $P$ to a space $Q$. There exists no proper regular extension $R$ of $P$ on which $\Phi$ may be continuously extended.

**Proof.** Let us suppose, on the contrary, that there exists a proper regular extension $R$ of $P$ and a continuous mapping $\Phi^*$ from $R$ to $Q$ such that $\Phi$ is the restriction of $\Phi^*$. Choose $x$ in $R \setminus P$. Since $\Phi[P]$ is a closed subset of $Q$ ($\Phi$ is closed) and since by continuity of $\Phi^*$
\[ \Phi^*[R] \subset \Phi[P] \]
we have at once that $\Phi^*[R] = \Phi[P]$. Hence, there exists a $y$ in $\Phi[P]$ such that $\Phi^*(x) = y$. Denote by $K$ the inverse image under $\Phi$ of $y$ (that is, the set $\Phi[y]$). $\Phi$ is a compact mapping, and consequently, $K$ is a compact space. It follows that 
\[ x \text{ non } \epsilon \overline{K}^R = K. \]
Since $R$ is a regular space, we may choose a closed (in $R$) neighborhood $F$ of $x$ with $F \cap K = \emptyset$. Consider the set $F \cap P$. Since $F$ being a closed mapping, $\Phi[F \cap P]$ is a closed subset of $Q$. Since $F \cap K = \emptyset$, it follows that 
\[ y \text{ non } \epsilon \Phi[F \cap P] = \Phi^*[F \cap P]. \]
But this is impossible, since $\Phi^*$ is continuous, $x \in \overline{F} \cap \overline{P}$ and $\Phi^*(x) = y$. This contradiction establishes the theorem.
2.11. Definition. Suppose that $P$ and $Q$ are completely regular spaces. A continuous mapping $\Phi$ from $P$ to $Q$ is said to be non-extensible if for any proper completely regular extension $R$ of $P$ and any continuous mapping $\Phi^*$ from $R$ to $Q$, the restriction of $\Phi^*$ to $P$ is different from $\Phi$.

Combining 2.6 and 2.10 we obtain at once:

2.12. Proposition. If $P$ is an $N(m)$-space then there exists a continuous non-extensible mapping from $P$ to $E^m$.

In the converse direction we shall prove:

2.13. Proposition. Let us suppose that there exists a continuous non-extensible mapping $\Phi$ from a completely regular space $P$ to $E^m$. Then $P$ is an $N(m)$-space.

Proof. Introduce the same notation as in the proof of 2.6:

$$E^m = X\{E_a; a \in A\}, \quad \Phi(x) = \{f_a(x); a \in A\}.$$  

It is sufficient to show that $\{f_a; a \in A\}$ is a complete family. Suppose, on the contrary, that $\{f_a; a \in A\}$ is not complete. Thus, there exists a maximal centered family $\mathcal{Z}$ of $Z$-sets in $P$ such that

$$\bigcap\{Z; Z \in \mathcal{Z}\} = \emptyset$$

and for each $a$ in $A$ there is a $Z_a$ in $\mathcal{Z}$ such that $f_a$ is bounded on $Z_a$. $\beta P$ being compact and $\mathcal{Z}$ being a maximal centred family of $Z$-sets, the intersection of the family $\{Z^{\beta P}; Z \in \mathcal{Z}\}$ contains exactly one point, namely $x$. Since $\Phi$ is a non-extensible continuous mapping, there exists a $f_a$ which is non-extensible over $P \cup \{x\}$, and clearly, since every bounded continuous function of $P$ is extensible over $\beta P$, there must be

$$\lim_{z \to x, z \in P} f_a(z) = \pm \infty.$$  

But, $x$ is contained in the closure of every $Z$ in $\mathcal{Z}$, and consequently, we have

$$Z \in \beta \Rightarrow \lim_{z \to x, z \in Z} f_a(z) = \pm \infty.$$  

Particularly, $f_a$ is not bounded on $Z_a$. This contradiction establishes the Theorem.

Combining 2.12 a 2.13 we obtain:

2.14. Theorem. A completely regular space $P$ is an $N(m)$-space if and only if there exists a continuous non-extensible mapping from $P$ to $E^m$.

We shall need the following proposition (see [4] and [3]).

2.15. A space is a $Q$-space if and only if it is homeomorphic with some closed subspace of $E^m$ for some $m$.

Proof. Let $\mathcal{F}$ be the set of all continuous functions on $P$. Define $E^\mathcal{F}$ and $\Phi$ as in 1.10 It is well known that $\Phi$ is a homeomorphic mapping if and only if $P$ is
a completely regular space. Now the statement follows from the note preceding 1.2 and 2.5.

Now we are prepared to prove the following theorem.

2.16. Theorem. Let \( \Phi \) be a continuous mapping from a completely regular space \( P \) to a \( Q \)-space \( Q \). The following two conditions on \( \Phi \) are equivalent:

2.15.1. \( \Phi \) is closed and compact.

2.15.2. \( \Phi \) is non-extensible.

Proof. By 2.10 the assertion 2.15.1 implies 2.15.2. Conversely, suppose that \( \Phi \) is non-extensible. By 2.15 there exists a homeomorphic mapping \( \Psi \) of \( Q \) onto a closed subspace of \( E^m \) for some cardinal \( m \). We see at once that the superposition \( \Psi(\Phi) \) of \( \Psi \) and \( \Phi \) is a non-extensible mapping from \( P \) to \( E^m \). By 2.13 \( \Psi(\Phi) \) is a closed compact mapping, and consequently, \( \Psi \) being homeomorphic, \( \Phi \) is a closed compact mapping.

2.17. Theorem. Let \( P, Q \) and \( R \) be completely regular spaces. If \( \Phi \) is a continuous closed compact mapping from \( P \) to \( Q \) and if \( \Psi \) is a continuous closed compact mapping from \( Q \) to \( R \), then the superposition of \( \Psi \) and \( \Phi \) is a continuous closed compact mapping.

2.18. If \( \Phi \) is a continuous closed compact mapping of a space \( P \) onto a compact space \( Q \), then \( P \) is a compact space. (It may be noticed that if \( \Phi \) is a continuous mapping from \( P \) onto a compact space \( Q \), then \( P \) is compact if and only if \( \Phi \) is closed and compact.)

The proof of 2.18 is quite routine and may be left to the reader.

As an immediate consequence of 2.17 we have:

2.19. Theorem. A completely regular space \( P \) is an \( N(m) \)-space if and only if there exists a continuous closed compact mapping from \( P \) to an \( N(m) \)-space.

2.20. Theorem. Let \( \{ P_a; a \in A \} \) be an indexed family such that \( P_a \) is an \( N(m_a) \)-space. Then the topological product \( P = \prod_{a \in A} P_a \) is an \( N(m) \)-space, where \( m = \sum m_a \).

To prove 2.20 it is sufficient to show that:

2.21. Theorem. Let \( \{ P_a; a \in A \} \) and \( \{ Q_a; a \in A \} \) be indexed families of completely regular spaces. For each \( a \in A \) let \( \Phi_a \) be a continuous closed compact mapping from \( P_a \) to \( Q_a \).

Consider the product spaces \( P = \prod_{a \in A} P_a \) and \( Q = \prod_{a \in A} Q_a \) and the mapping \( \Phi = \{ \Phi_a; a \in A \} \) defined as follows:

\[ \Phi(x) = \{ \Phi_a(x_a); a \in A \} . \]

The mapping \( \Phi \) is continuous, closed and compact.
Proof. The proof of continuity is quite routine and may be left to the reader. Denote by \( \pi_a \) the projection of \( Q \) onto \( Q_a \). Let \( y \) be an element of \( Q \). Clearly
\[
\Phi^{-1}[y] = \bigcup \{ \Phi_a^{-1}[\pi_a(y)]; a \in A \}.
\]
The spaces \( \Phi_a^{-1}[\pi_a(y)] \) being compact, the space \( \Phi^{-1}[y] \) is compact by Tychonoff's theorem. Thus \( \Phi \) is a compact mapping. It remains to prove that \( \Phi \) is a closed mapping. First, let \( F \) be a closed subset of \( P \) of the form
\[
(6) \quad \bigcup \{ F_a; a \in A \}
\]
where \( F_a \) is a closed subset of \( P_a \). Clearly
\[
\Phi[F] = \bigcup \{ \Phi_a[F_a]; a \in A \}.
\]
\( \Phi_a \) being closed, the set \( \Phi_a[F_a] \) is closed in \( Q_a \), and consequently, \( \Phi[F] \) is closed in \( Q \). Now, let \( F \) be an arbitrary closed subset of \( P \). Let \( \mathcal{M} \) be the family of all closed subsets of \( P \) of the form \( (6) \), and containing \( F \). \( \Phi[\mathcal{M}] \) being closed in \( Q \), the set
\[
F_0 = \bigcap \{ \Phi[M]; M \in \mathcal{M} \}
\]
is closed in \( Q \), and consequently, it is sufficient to show that \( F_0 = \Phi[F] \). Clearly \( F_0 \supset \Phi[F] \). Suppose that there exists a \( y \) in \( F_0 - \Phi[F] \). Thus \( \Phi^{-1}[y] = K \) is a compact subspace of \( P \) disjoint with \( F \). Since
\[
\bigcap \{ M; M \in \mathcal{M} \} = F,
\]
there exists a \( M \) in \( \mathcal{M} \) with \( M \cap K = \emptyset \). \( \Phi[M] \) being closed, we have at once that \( y \) non \( \in \Phi[M] \supset F_0 \). This contradiction completes the proof of 2.21.

Now we give another proof of 2.20 using 2.9 and Stone-Čech theorem (and also Tychonoff's theorem). By 2.9, for each \( a \) in \( A \) there exists a family \( \mathcal{N}_a \) of \( N \)-sets in \( \beta P_a \) such that the potency of \( \mathcal{N}_a \) is at most \( m_a \) and
\[
\bigcap \{ N; N \in \mathcal{N}_a \} = P_a.
\]
Consider the space \( K = \bigcup \{ \beta P_a; a \in A \} \). Denote by \( \pi_a \) the projection of \( K \) onto \( \beta P_a \). Let \( \mathcal{N}_a^* \) be the family of all \( \pi_a^{-1}[N] \) where \( N \in \mathcal{N}_a \). Denote by \( \mathcal{N}^* \) the union of the indexed family \( \{ \mathcal{N}_a^*; a \in A \} \). Evidently \( \mathcal{N}^* \) is a family of \( N \)-sets in \( K \) and
\[
\bigcap \{ N; N \in \mathcal{N}^* \} = P.
\]
Since the potency of \( \mathcal{N}^* \) is at most \( m \), the space \( P \) is the intersection of \( m \) \( N \)-sets in \( K \). Let \( \Phi \) be the Čech-Stone mapping from \( \beta P \) onto \( K \). Evidently
\[
\bigcap \{ \Phi^{-1}[N]; N \in \mathcal{N}^* \} = P
\]
and \( \Phi^{-1}[N] \) are \( N \)-sets in \( \beta P \). By 2.9, the space \( P \) is an \( N(m) \)-space. The second proof of 2.20 is complete.

In conclusion we give a summary of definitions of \( N(m) \)-spaces:

2.22. Theorem. The following condition on a completely regular space \( P \) are equivalent:
There exists a complete family \( \mathcal{F} \) of continuous functions on \( P \) such that the potency of \( \mathcal{F} \) is \( \leq m \).

There exists a continuous closed compact mapping from \( P \) to \( E^m \).

There exists a continuous non-extensible mapping from \( P \) to \( E^m \).

There exists a continuous closed compact mapping from \( P \) to an \( N(m) \)-space.

There exists a continuous non-extensible mapping from \( P \) to an \( N(m) \)-space.

\( P \) is an intersection of \( m \) \( N \)-sets in some compactification of \( P \).

\( P \) is an intersection of \( m \) \( N \)-sets in \( \beta P \).

3. IMAGES OF Q-SPACES

All spaces are assumed to be completely regular. Let \( \Phi \) be a continuous mapping from a Q-space \( P \) onto \( Q \). Under what conditions on \( \Phi \) may we assert that \( Q \) is a Q-space?

We recall that a subspace \( P' \) of \( P \) is said to be relatively pseudocompact in \( P \), if for every sequence \( \{Z_n\} \) of Z-sets in \( P \) such that \( \{Z_n \cap P'\} \) is centered, the intersection \( P' \cap \bigcap_{n=1}^{\infty} Z_n \) is non-void. Equivalently, \( P' \) is relatively pseudocompact if and only if every continuous function on \( P \) is bounded on \( P' \).

3.1. Theorem. Let \( \Phi \) be a continuous mapping from \( P \) onto \( Q \) such that

3.1.1. The images of Z-sets are Z-sets, that is, if \( Z \) is a Z-set in \( P \), then \( \Phi[Z] \) is a Z-set in \( Q \).

3.1.2. The inverses of points under \( \Phi \) are relatively pseudocompact, that is, for each \( y \) in \( Q \) the subspace \( \Phi^{-1}[y] \) of \( P \) is relatively pseudocompact in \( P \).

Then if \( P \) is a Q-space, \( Q \) is also a Q-space.

Proof. Let us suppose that \( \mathcal{Z} \) is a maximal centered family of Z-sets in \( Q \) such that the intersection of every countable subfamily of \( \mathcal{Z} \) is non-void. Let \( \mathcal{Z}' \) be the family of all \( \Phi^{-1}[Z] \) where \( Z \in \mathcal{Z} \). Evidently, \( \mathcal{Z}' \) is a centered family of Z-sets in \( P \). Let \( \mathcal{Z}'' \) be a maximal centered family of Z-sets in \( P \) containing \( \mathcal{Z}' \). We shall prove that the intersection of every countable subfamily of \( \mathcal{Z}'' \) is non-void. Indeed, let \( \{Z'_n\} \) be a sequence of Z-sets in \( \mathcal{Z}'' \). By 3.1.1 the sets \( Z_n = \Phi[Z'_n] \) are Z-sets in \( Q \), and clearly \( Z_n \in \mathcal{Z} \). Choose a point \( y \) in \( \bigcap_{n=1}^{\infty} Z_n \). By 3.1.2 we have

\[
\Phi^{-1}[y] \cap \bigcap_{n=1}^{\infty} Z'_n = \emptyset.
\]

\( P \) being a Q-space, the set \( \bigcap\{Z'; Z' \in \mathcal{Z}''\} \) is non-void. Choose a point \( x \) in this intersection. Evidently

\[
\Phi(y) \in \bigcap\{Z; Z \in \mathcal{Z}\}.
\]

The theorem is proved.
We proceed to quotient mappings:

3.2. **The image of a Q-space under an open continuous mapping may fail to be a Q-space.**

**Proof.** Let us suppose that \( Q \) is not a Q-space and let

\[
Q = \bigcup \{K_a; a \in A\},
\]

where \( K_a \) are compact subspaces of \( Q \). Finally, suppose the indexed set \( A \) endow with the discrete topology is a Q-space. Under these assumptions we shall construct a Q-space \( P \) and a continuous open mapping \( \Phi \) from \( P \) onto \( Q \).

Consider the product space \( R = Q \times A \) and the subspace

\[
P = \bigcup \{K_a \times a; a \in A\}
\]

of \( R \). By 2.19 the space \( P \) is a Q-space. Indeed, the mapping \( x \in K_a \times a \rightarrow a \) is a closed compact continuous mapping of \( P \) onto \( A \). Denote by \( \Phi \) the projection map of \( P \) onto \( Q \), i.e., \( \Phi \) is the restriction of the projection of \( R \) onto \( Q \).

It is easy to show that the mapping \( \Phi \) is open and continuous. The proofs of existence of \( Q, K_a \) and \( A \) may be left to the reader.

Modifying the construction in 3.2 (to consider the disjoint union) we obtain at once:

3.3. **The image of a Q-space under a compact open continuous mapping may fail to be a Q-space.**

**Note.** If the topological product \( P \times Q \) is a Q-space \((N(m)\)-space\), then both \( P \) and \( Q \) are Q-spaces \((N(m)\)-spaces, respectively\).

We shall need the following assertion:

3.4. **Lemma.** Let \( \Phi \) be an open, closed and continuous mapping from \( P \) onto \( Q \). Let \( f \) be a continuous function on \( P \). For each \( y \) in \( Q \) put

\[
F(y) = \sup \{f(x); \Phi(x) = y\}.
\]

If \( F(y) \) is a real number for each \( y \) in \( Q \), then \( F \) is a continuous function on \( Q \).

**Proof.** Let \( y_0 \) be an element of \( Q \) and let \( \epsilon \) be a positive real number. For each \( x \) in \( X = \Phi^{-1}[y_0] \) choose an open neighborhood \( U(x) \) of \( x \) on which \( f \) varies less than \( \epsilon \). Denote by \( U \) the union of all \( U(x), x \in X \). Put

\[
V = \bigcup \{\Phi^{-1}[y]; \Phi^{-1}[y] \subset U\}.
\]

\( \Phi \) being closed, \( V \) is an open subset of \( P \). Evidently

\[
y \in \Phi[V] \Rightarrow F(y) \leq F(y_0) + \epsilon.
\]

\( \Phi \) being an open mapping, \( \Phi[V] \) is an open neighborhood of \( y_0 \). Thus \( F \) is an upper semi-continuous function. It remains to prove that \( F \) is lower semi-
continuous. Choose a point \( x_0 \) in \( X \) such that \( f(x_0) > F(y_0) - \frac{\varepsilon}{2} \). Choose an open neighborhood \( W \) of \( x_0 \) such that \[ x \in W \Rightarrow f(x) > f(x_0) - \frac{\varepsilon}{2}. \]

\( \Phi \) being open, \( W' = \Phi[W] \) is an open neighborhood of \( y_0 \). We have at once

\[ y \in W' \Rightarrow F(y) > F(y_0) - \varepsilon. \]

This establishes lower semi-continuity of \( F \) and completes the proof of 3.4.

**3.5. Proposition.** Let us suppose that \( \Phi \) is a closed, open and continuous mapping from \( P \) onto \( Q \). Let \( \mathcal{F} \) be a complete family of continuous non-negative functions on \( P \). Suppose that for each \( f \) in \( \mathcal{F} \) the function \( F \) defined as in 3.4 is real-valued, that is, \( F \) is finite. Denote by \( \mathcal{F}' \) the family of all \( F \) where \( f \in \mathcal{F} \).

Then \( \mathcal{F}' \) is a complete family of continuous functions on \( Q \).

**Proof.** By 3.4 the functions \( F \in \mathcal{F}' \) are continuous. To prove completeness of \( \mathcal{F}' \), let \( \mathcal{Z} \) be a centered family of closed subsets of \( Q \) such that for each \( F \) in \( \mathcal{F}' \) there exists a \( Z_F \) in \( \mathcal{Z} \) with \( F \) bounded on \( Z_F \). Denote by \( \mathcal{Z}' \) the family of all \( \Phi^{-1}[Z] \) where \( Z \in \mathcal{Z} \). Evidently, \( \mathcal{Z}' \) is a centered family of closed subsets of \( P \). Moreover, for each \( f \) in \( \mathcal{F} \) there exists a \( Z_f \) in \( \mathcal{Z}' \) such that \( f \) is bounded on \( Z_f \).

Indeed, if \( F \) is a function corresponding to \( f \), we may put \( Z_f = \Phi^{-1}[Z_F]. \) \( \mathcal{F} \) being a complete family, we have

\[ F_0 = \bigcap \{Z; Z \in \mathcal{Z}'\} \neq \emptyset. \]

Clearly \( \Phi[F_0] \subset \bigcap \{Z; Z \in \mathcal{Z}\} \). Thus \( \mathcal{F}' \) is complete.

As an immediate consequence of 3.5 we have:

**3.6. Theorem.** Let \( \Phi \) be a closed, open and continuous mapping from \( P \) onto \( Q \). Suppose that the tranches of \( \Phi \) (that is, the sets of the form \( \Phi^{-1}[y], y \in Q \)) are relatively pseudocompact spaces. If \( P \) is an \( N(m) \)-space, then \( Q \) is an \( N(m) \)-space. In particular, if \( P \) is a \( Q \)-space, then \( Q \) is a \( Q \)-space.

4. \( N(1) \)-SPACES AND \( N(\aleph_0) \)-SPACES

**4.1. Theorem.** Let \( m \geq 1 \). A discrete space \( M \) is an \( N(m) \)-space if and only if it is homeomorphic with some closed subspace (discrete, of course) of \( E^m \).

Proof. The theorem is obvious for finite \( m \). Suppose \( m \geq \aleph_0 \). We shall use 2.22, condition 2.22.2. To prove necessity let us suppose that \( \Phi \) is a continuous, closed and compact mapping from \( M \) to \( E^m \). The tranches of \( \Phi \) being compact and discrete, they are finite. Thus \( M \) and \( \Phi[M] \) has the same potency. The image under closed mappings of a discrete space is a discrete space. The discrete
spaces $M$ and $\Phi[M]$ have the same potency, and consequently, they are homeomorphic. The sufficiency is obvious.

As a corollary of 4.1 and of the fact that $E^\omega$ is a metrizable and separable space we have

4.2. Theorem. The following conditions on a discrete space are equivalent:

- 4.2.1. $M$ is a $N(1)$-space.
- 4.2.2. $M$ is a $N(\aleph_0)$-space.
- 4.2.3. The potency of $M$ is at most $\aleph_0$.

4.3. Theorem. The following conditions on a space $P$ are equivalent

- 4.3.1. $P$ is a $N(1)$-space.
- 4.3.2. There exists a continuous function $f$ on $P$ such that every closed subspace $K$ of $P$ is compact if and only if the function $f$ is bounded on $K$.
- 4.3.3. There exists a sequence $\{K_n\}$ of compact subspaces of $P$ such that $K_n \subseteq \text{int } K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = P$.
- 4.3.4. $P$ is locally compact and $\sigma$-compact.

Proof. By 2.4 the conditions 4.3.1 and 4.3.2 are equivalent. If $f$ is the function from 3.1.2 and if we put

$$K_n = \{x; |f(x)| \leq n\},$$

we obtain a sequence $\{K_n\}$ satisfying 3.1.3. Thus 3.1.2 implies 3.1.3. Suppose 3.1.3. Choose continuous functions $f_n$, $n = 3, 4, \ldots$, such that

$$f_n(x) = \begin{cases} n & \text{for } x \notin K_{n-1}, \\ 0 & \text{for } x \in K_{n-1} \end{cases}$$

and $0 \leq f_n(x) \leq n$ for every $x$. Put $f = \sum_{n=3}^{\infty} f_n$. Evidently, $f$ is bounded on a set $M$ if and only if the set $M$ is contained in some finite union $\bigcup_{i=1}^{n} K_i$. It follows at once that $f$ satisfies 3.1.2. Thus 3.1.3 implies 3.1.2. The proof of equivalence of 3.1.3 and 3.1.4 is quite routine and may be left to the reader.

4.4. Theorem. A metrizable space is an $N(1)$-space if and only if it is separable and locally compact.

Proof. First suppose that $P$ is a metrizable $N(1)$-space. Evidently $P$ is locally compact. By 4.2 and 2.7 the space $P$ contains no uncountable discrete closed subset. Thus $P$ is separable. Conversely, $P$ being a separable and metrizable space, $P$ has a compact metrizable extension $K$. $P$ being locally compact, $P$ is open in $K$. An open subset of a metrizable space is an $N$-set. Thus $P$ is an $N$-set of a compact space $K$. It follows that $P$ is an $N(1)$-space.
Recall that a space $P$ is said to be a $G_\delta$-space if it is a $G_\delta$-set in every extension. It is well-known that a metrizable space is a $G_\delta$-space if and only if there exists a metric $\varphi$ for $P$ such that $(P, \varphi)$ is a complete metric space (for further information see [2]). By [2], theorem 2.8, a completely regular space $P$ is a $G_\delta$-space if and only if it is a $G_\delta$-subset of some compact space. Thus every $N(\mathfrak{N}_0)$-space is a $G_\delta$-space.

4.5. **Theorem.** The following two conditions on a metrizable space $P$ are equivalent:

4.5.1. $P$ is an $N(\mathfrak{N}_0)$-space.

4.5.2. $P$ is a separable $G_\delta$-space.

**Proof.** As we note above, an $N(\mathfrak{N}_0)$-space is a $G_\delta$-space. Thus, to prove that 4.5.1 implies 4.5.2, it is sufficient to show that every metrizable $N(\mathfrak{N}_0)$-space is separable. By 4.2 and 2.7 the space $P$ contains no uncountable discrete closed subspace. It follows that $P$ is separable. Conversely, suppose 4.5.2. $P$ being separable, there exists a compact metrizable extension $K$ of $P$. $P$ being a $G_\delta$-space, $P$ is a $G_\delta$-subset of $K$, and consequently, $P$ is an $N(\mathfrak{N}_0)$-set in $K$. Thus $P$ is an $N(\mathfrak{N}_0)$-space.

**Bibliography**


**Резюме**

ПРИЛОЖЕНИЯ ПОЛНЫХ СЕМЕЙСТВ ФУНКЦИЙ В ТЕОРИИ ФУНКЦИОНАЛЬНО ЗАМКНУТЫХ ПРОСТРАНСТВ

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Подмножество $N$ пространства $P$ называют $N$-множеством, если существует непрерывная функция $f$ на $P$ так, что $N = N(f) = \{x; x \in P, f(x) \neq 0\}$.

Если $N$ является $N$-множеством, то $P - N$ называют $Z$-множеством. Пространство $P$ называется функционально замкнутым (или $Q$-пространством, см. [3]) если выполнено следующие условия: всякая максимальная
счетно центрированная система $Z$-множеств (т. е. максимальная система $Z$-множеств такая, что любая счетная подсистема имеет непустое пересечение) имеет непустое пересечение. В статье дается определение $Q$-пространств при помощи полных семейств непрерывных функций. Семейство $\mathcal{F}$ непрерывных функций называется полным, если выполнено следующее условие:

Если $\mathcal{Z}$ — такая центрированная система $Z$-множеств, что всякая функция из $\mathcal{F}$ ограничена на некотором множестве из $\mathcal{Z}$, то пересечение системы $\mathcal{Z}$ не пусто.

Оказывается, что вполне регулярное пространство является $Q$-пространством тогда и только тогда, если семейство всех непрерывных функций полно.

В статье определены т. наз. $N(m)$-пространства ($m$ — некоторое кардинальное число). $P$ называется $N(m)$-пространством если существует полное семейство $\{f_\alpha; \alpha \in A\}$ непрерывных функций на $P$ такое, что мощность множества $A$ равна $m$. Итак, вполне регулярное пространство является $Q$-пространством тогда и только тогда, если оно является $N(m)$-пространством для некоторого $m$. Пусть $\Phi$ — отображение пространства $P$ в пространство $Q$; $\Phi$ называется бикомпактным, если прообразы точек бикомпактны, замкнутыми, если образы замкнутых множеств замкнуты; наконец, непрерывное $\Phi$ называется нерасширимым, если, какого бы ни было пространство $R$, $R > P$, $R = \bar{P}$, $R \neq P$, отображение $\Phi$ нельзя расширить до непрерывного отображения пространства $R$ в $Q$. Доказана следующая

**Теорема.** Следующие свойства вполне регулярного пространства $P$ эквивалентны ($m$ — кардинальное число):

(1) $P$ является $N(m)$-пространством.

(2) Существует бикомпактное замкнутое и непрерывное отображение пространства $P$ в топологическое произведение $m$ прямых.

(3) Существует непрерывное нерасширимое отображение пространства $P$ в топологическое произведение $m$ прямых.

(4) $P$ является пересечением $m$ $N$-множеств в некотором своем бикомпактном расширении.

(5) $P$ является пересечением $m$ $N$-множеств с своим чеховским бикомпактным расширением.

В последней части рассматривается вопрос, при каких условиях непрерывный образ $N(m)$-пространства является $N(m)$-пространством. Указывается, что достаточно предполагать, что отображение замкнуто, открыто, и полные прообразы точек относительно псевдокомпактны.