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CHARACTERS ON INVERSE SEMIGROUPS

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The main purpose of this paper is to extend the results of ŠT. SCHWARZ [5] concerning characters of finite abelian semigroups to an other class of semigroups.

ŠT. SCHWARZ has investigated the structure of $G^\wedge$, the semigroup of characters for a finite abelian semigroup G [5]. In section 1, we investigate the structure of $G^\wedge$ where G is an infinite abelian inverse semigroup. In particular, we prove two theorems which are related to Theorem 7, page 246 and Theorem 8, page 246 of [5]. In section 2, we prove an extension theorem for abelian inverse semigroups which is related to a theorem of K. A. ROSS [4]. A separation theorem is a consequence of this theorem.

Inverse semigroups have been investigated by G. B. PRESTON [3].

1. THE STRUCTURE OF $G^\wedge$

1.1 Definition. An inverse semigroup is a semigroup S satisfying the following conditions:

a) To each $a \in S$ there corresponds at least one $e \in S$ for which $ea = a$ and such that the equation $ax = e$ has a solution $x \in S$.

b) If e and f are any two idempotents of S, then $ef = fe$.

It is shown in [3] that these conditions imply that to each $a \in S$ correspond unique idempotents e and f, called the left and right units of a respectively, and a unique inverse element a^{-1} such that $aa^{-1} = e$, $a^{-1}a = f$, and $fa^{-1} = a^{-1} = a^{-1}e$. The left and right units of a^{-1} are f and e and the inverse of a^{-1} is a. The inverse of ab is $b^{-1}a^{-1}$. If S is abelian, $e = f$.

It is shown in [2] that an equivalent definition is

1.2 Definition. An inverse semigroup is a semigroup S in which $a \in S$ implies there exists a unique $x \in S$ such that $axa = a$ and $xax = x$. Clearly, $x = a^{-1}$.

1.3 Definition. If is S a semigroup, χ is a character of S if and only if χ is a complex function on S such that $a, b \in S$ implies $\chi(ab) = \chi(a)\,\chi(b)$. If S has an identity 1, $\chi(1) \neq 0$.[1])

[1]) We wish to thank Prof. A. H. CLIFFORD for helpful suggestions in relation to this paper.

1.4 Definition. If G is an inverse semigroup, we will denote the set of characters of G by $G^\wedge$. We will define multiplication in $G^\wedge$ as follows. If $\chi_1, \chi_2 \in G^\wedge$, $\chi_1\chi_2(x) = \chi_1(x)\,\chi_2(x)$.

1.5 Lemma. *An abelian inverse semigroup G is a semilattice of groups* [5],.

Proof: Let G_e be the maximal sugroup of G containing e where e is an idempotent of G. G_e is the group of units of eGe. If $e \neq f$, $G_e \cap G_f = \phi$. If $x \in G$, then there exists $e \in G$ such that $x = exe$ and $x^{-1} \in G$ such that $x^{-1}x = xx^{-1} = e$ and $x^{-1} = ex^{-1}e$. Hence, $x \in G_e$. Thus, $G = \bigcup_{e \in E} G_e$ where E is the set of idempotents of G. If $a \in G_e$ and $b \in G_f$, $ef(ab)\,ef = ab$,

$$(ab)\,(ab)^{-1} = (ab)^{-1}(ab) = ef \quad \text{and} \quad (ab)^{-1} = (ef)\,(ab)^{-1}(ef)\,.$$

1.6 Lemma. *$G^\wedge$ is a semilattice of groups.*

Proof: If $\chi \in G^\wedge$, define $\chi^{-1}(x) = \dfrac{1}{\chi(x)}$ if $\chi(x) \neq 0$ and $\chi^{-1}(x) = 0$ if $\chi(x) = 0$. Define the unit χ_e of χ as follows: $\chi_e(x) = 1$ if $\chi(x) \neq 0$ and $\chi_e(x) = 0$ if $\chi(x) = 0$. Then $\chi\chi^{-1} = \chi^{-1}\chi = \chi_e$, $\chi\chi_e = \chi_e\chi = \chi$. Thus, $G^\wedge$ is an abelian inverse semigroup. Therefore, $G^\wedge$ is a semilattice of groups by lemma 1.5,

1.7 Remark. Let $(G, .)$ be an abelian inverse semigroup without an identity. Now let $(G^e, \circ)$ be defined as follows:

$G^e = G \cup e$, $a \circ e = e \circ a = a$ for all $a \in G^e$, $a \circ b = a \,.\, b$ for all $a, b \in G$. Then $(G^e, \circ)$ is an abelian inverse semigroup with an identity. From definition 1.3 it is clear that $G^{e\wedge}$ is isomorphic to $G^\wedge$. Hence in the theorems investigating the structure of $G^\wedge$ it will be only necessary to consider the case where G has an identity.

1.8 Example. Let I be the non-negative integers under the multiplication:

$$a \circ b = 0 \text{ if } a \neq b\,, \quad a \circ a = a\,.$$

Clearly I is an abelian inverse semigroup and $I^\wedge$ consists of the following characters $\chi(a) = 0$ for all $a \in I$; $\chi(a) = 1$ for all $a \in I$; $[\chi_a \mid a \neq 0 \in I,\ \chi_a(a) = 1$ and $\chi_a(b) = 0$ for $a \neq b]$.

$I^{e\wedge}$ consists of the following characters: $\chi(a) = 1$ for all $a \in I^e$; $[\chi_a \mid a \neq 0 \in I^e$, $\chi_a(a) = 1$, $\chi_a(b) = 0$ for $b \neq a$, $b \neq e$ and $\chi_a(e) = 1]$.

1.9 Example. Let $G = I^+$, the positive integers under the multiplication $a \circ b = \min(a, b)$. Then $G^\wedge = I^+ \cup 0$ where $a \circ b = \max(a, b)$, $a, b \in I^+$ and $a \circ 0 = 0 \circ a = 0$ for all $a \in G$. Clearly $G^\wedge$ is isomorphic to $G^{e\wedge}$.

1.10 Theorem. *Let G be abelian inverse semigroup with an identity such that every non-void subset of E_G has a minimal element. Then E_G and $E_{G^\wedge}$ are anti-isomorphic as semi-lattices.*[2])

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[2]) E_G and $E_{G^\wedge}$ denote the set of idempotents of G and $G^\wedge$ respectively.

Proof: Let $e \in E_G$ and let χ_e be defined as follows: $\chi_e(x) = 1$ if and only if $e \leqq f$ where $x \in G_f$. $\chi_e(x) = 0$ if and only if $f \ngeqq e$ where $x \in G_f$.

We wish to show first that the mapping $e \to \chi_e$ is one to one of E_G onto $E_{G^\wedge}$. Clearly $\chi_e \in E_{G^\wedge}$. If $\chi_e = \chi_f$, $\chi_e(f) = \chi_f(f) = 1$. Thus $f \geqq e$. In addition $\chi_f(e) = \chi_e(e) = 1$ and $e \geqq f$. Hence, $e = f$. If $\chi \in E_{G^\wedge}$, let $H = \{f \in E_G \mid \chi(f) = 1\}$. Since $\chi(1) \neq 0$, $H \neq \phi$. Let e be the minimal element of H. Now, $\chi(e) = 1$. If $e \leqq f$, $ef = e$ and $\chi(f) = 1$. If $f \ngeqq e$, then $ef = h \neq e$. Thus $he = h$ and $h \leqq e$. Therefore, $h < e$ and $\chi(h) = 0$. Hence, $\chi(f) = 0$. Therefore, $\chi(x) = 1$ if and only if $e \leqq f$ where $x \in G_f$ and $\chi(x) = 0$ if and only if $f \ngeqq e$ where $x \in G_f$. Hence $\chi = \chi_e$. Next, suppose $e \leqq f$, i. e. $ef = e$. If $x \in G_h$ and $f \leqq h$, then $\chi_e\chi_f(x) = \chi_f(x)$. If $x \in G_h$ and $h \ngeqq f$, then $\chi_e\chi_f(x) = \chi_f(x)$. If $\chi_e\chi_f = \chi_f$, $\chi_e(f) = 1$ and $e \leqq f$. Hence $e \leqq f$ if and only if $\chi_f \leqq \chi_e$, i. e. the mapping $e \to \chi_e$ is a semi-lattice anti-isomorphism of E_G onto $E_{G^\wedge}$.

1.11 Example. We give an example to show that "minimal" cannot be replaced by "maximal" in theorem 1.10. Let G be positive integers under the multiplication $x \circ y = \max(x, y)$. Then, $G^\wedge$ consists of the following characters, $\chi(x) = 1$, $x \leqq n$, $\chi(x) = 0$, $x > n$ for $n = 1, 2, 3, \dots$ and $\chi(x) = 1$ for all $x \in G$. Suppose that there exists an anti-isomorphism: $\Phi : i \to \chi_i$ of G $(= E_G)$ onto $G^\wedge$ $(= E_{G^\wedge})$. Let the χ such that $\chi(x) = 1$ for all $x \in G$ (the identity character) be denoted by χ_k. Then $\chi_s\chi_k = \chi_s$ for all $s \in G$, i. e. $\chi_s \leqq \chi_k$ for all $s \in G$. Choose $t \in G$ such that $t < k$. Then $\chi_t > \chi_k$ and we have a contradiction.

We also note that the replacement of "maximal" for "minimal" and isomorphism for anti-isomorphism in theorem 1.10 is not valid. For let G and $G^\wedge$ be as above. Suppose $\Phi : i \to \chi_i$ is an isomorphism between G and $G^\wedge$. Let χ_k be the character such that $\chi(1) = 1$ and $\chi(n) = 0$ for $n > 1$. $\chi_k\chi_s = \chi_k$, i. e. $\chi_k \leqq \chi_s$ for all $s \in G$. Choose $t < k$. Then $\chi_t < \chi_k$.

1.12 Corollary. *If G is an abelian inverse semigroup with an identity and every non-void subset of E_G has a minimal element, then $G^\wedge_{\chi_e}$ is isomorphic to the character group of G_e where $e \to \chi_e$ is the anti-isomorphism of E_G onto $E_{G^\wedge}$ referred to in Theorem 1.10. If G_e is finite, then G_e is isomorphic to $G^\wedge_{\chi_e}$.*

Proof. Let $\chi \in G^\wedge_{\chi_e}$ and denote by $\bar{\chi}$ the restriction of χ to G_e. Let $C(G_e)$ denote the character group of G_e. Clearly, $\bar{\chi} \in C(G_e)$. We wish to show the mapping $\chi \overset{\Theta}{\to} \bar{\chi}$ is an isomorphism of $G^\wedge_{\chi_e}$ onto $C(G_e)$. If $\chi_0 \in C(G_e)$, we define

$$\chi(x) = \chi_0(xe) \text{ if and only if } x \in G_f \text{ and } e \leqq f,$$
$$\chi(x) = 0 \quad\quad \text{if and only if } x \in G_f \text{ and } f \ngeqq e\,.$$

Clearly, $\chi \in G^\wedge_{\chi_e}$ and $\chi(x) = \chi_0(x)$ for all $x \in G_e$. Thus, $\chi_0 = \bar{\chi}$ and Θ is onto. If $\bar{\chi}_1 = \bar{\chi}_2$, then $\bar{\chi}_1(x) = \bar{\chi}_2(x)$ for all $x \in G_e$. If $x \in G_f$ and $e \leqq f$, then $\chi_1(ex) = \chi_2(ex)$. Hence, $\chi_1(x) = \chi_2(x)$. If $x \in G_f$ and $f \ngeqq e$, then $\chi_1(x) = \chi_2(x) = 0$. Hence, Θ is one to one. Thus, Θ is an isomorphism. If G_e is finite, then G_e is isomorphic to $C(G_e)$ [6] and hence is isomorphic to $G^\wedge_{\chi_e}$.

Let $G^{\wedge\wedge}$ denote the semigroup of characters of $G^{\wedge}$. Clearly $G^{\wedge\wedge}$ is a semi-lattice of groups.

1.13 Corollary. *Let G be an abelian inverse semigroup with an identity. Suppose every non-void subset of E_G has a maximal element and a minimal element. Then E_G and $E_{G^{\wedge\wedge}}$ are isomorphic as semi-lattices and as semigroups.*

Proof. Let $e \overset{\Phi}{\to} \chi_e$ denote the semi-lattice anti-isomorphism of E_G onto $E_{G^\wedge}$ of theorem 1.10. There exists a semi-lattice anti-isomorphism $\Phi' : (\chi_e \to \Phi'\chi_e)$ of $E_{G^\wedge}$ onto $E_{G^{\wedge\wedge}}$ since every non-void subset of $E_{G^\wedge}$ has a minimal element. Hence $\Phi'\Phi(e \to \to \Phi'\chi_e)$ is a semi-lattice isomorphism of E_G onto $E_{G^{\wedge\wedge}}$. Hence $\Phi'\Phi$ is a semigroup isomorphism.

1.14 Example. An example to show that it is not enough to just assume the maximal condition in corollary 1.13. Let G be positive integers under the following multiplication: $x \circ y = \max(x, y)$. Then $G^{\wedge}$ = positive integers $\cup$ e under the following multiplication: $x \circ y = \min(x, y)$ if and only if $x \neq e$, $y \neq e$ and $x \circ e = e \circ x = x$ for all $x \in G^{\wedge}$. Then $G^{\wedge\wedge}$ has a zero, namely the character $\chi(x) = 0$ for $x \neq e$ and $\chi(e) = 1$ while G has no zero.

1.15 Example. An example of an abelian inverse semigroup G such that E_G is an infinite set in which every non-void subset has a maximal element and a minimal element is given by example 1.8.

2. EXTENSION THEOREM AND CONSEQUENCES

2.1 Lemma. *If χ is a bounded character on an inverse semigroup G, then $\chi(x) = 0$ or $\chi(x) = e^{i\Theta}$ for all $x \in G$.*

Proof. Clearly, $|\chi(x)| \leqq 1$. Let $a \in G$. Then there exists a unique $x \in G$ such that $axa = a$. Thus

$$|\chi(a)|\,|\chi(x)|\,|\chi(a)| = |\chi(a)|\,.$$

If $\chi(a) \neq 0$, $|\chi(a)|\,|\chi(x)| = 1$, i. e. $|\chi(a)| = 1$.

2.2 Theorem. *Let G be an abelian inverse semigroup and H be an inverse sub-semigroup of G. Suppose χ is a bounded character of H such that $\chi \not\equiv 0$ on H. Then χ may be extended to a bounded character $\chi^{\wedge}$ of G.*

Proof. Let $H_1 = \{x \in H \mid \chi(x) = 0\}$ and $H_2 = \{x \in H \mid \chi(x) \neq 0\}$. Clearly H_1 is a semigroup. If $x \in H_1$, $\chi(x) = 0$. Thus, if e is the unit of x, $\chi(e) = 0$. If x^{-1} is the inverse of x, $\chi(x^{-1}) = 0$. Thus, H_1 is an inverse semigroup. Similarly, H_2 is an inverse semigroup. By the single valuedness of χ, $H_1 \cap H_1 = \phi$. Clearly $H = H_1 \cup \cup H_2$. Let $a, b \in H$ and suppose $ax = b$. If $a, b \in H_1$, $a, b \in H_2$ or $a \in H_2$, $b \in H_1$, the result follows from Ross' Theorem since $|\chi(a)| = 0$ or 1 for all $a \in H$. Suppose $a \in H_1$ and $b \in H_2$. Now, $eaxb^{-1} = f$ where f denotes the unit of b and e denotes the unit of a. Now since $e \in H_1$, $\chi(e) = 0$. But, it follows easily from Lemma 2.1 that $|\chi(e)| \geqq |\chi(f)|$. Thus $\chi(f) = 0$. But this contradicts the fact $f \in H_2$.

2.3 Corollary. *Let G be an abelian inverse semigroup and let a and b be distinct elements of G. Then there exists a bounded character χ of G such that $\chi(a) \neq \chi(b)$.*

Proof. G is a semi-lattice of groups $\{G_e : e \in E\}$ where G_e is the maximal subgroup containing the idempotent e. Let a and b be distinct elements of G. We consider:

Case I: $a, b \in G_e$ for some idempotent e. By a result of A. WEIL [6] there exists a bounded character χ of G_e such that $\chi(a) \neq \chi(b)$. By theorem 2.2 χ may be extended to a bounded character of G.

Case II. $a \in G_e$, $b \in G_f$ with $ef = f$ and $e \neq f$. In this case $e \cup f$ is an inverse semigroup. Let $\chi(e) = 1$ and $\chi(f) = 0$. χ is a bounded character on $e \cup f$. Thus by Theorem 2,2 χ may be extended to a bounded character $\chi^\wedge$ of G such that $\chi^\wedge(a) \neq 0$, $\chi^\wedge(b) = 0$.

Case III. $a \in G_e$, $b \in G_f$, $ef \neq f$, $ef \neq e$, $e \neq f$. Clearly $e \cup f \cup ef$ is an inverse semigroup. Define $\chi(e) = 1$, $\chi(f) = 0$, $\chi(ef) = 0$. Thus χ is a bounded character on $e \cup f \cup ef$. Hence the conditions of Theorem 2,2 are satisfied and χ may be extended to a bounded character $\chi^\wedge$ of G such that $\chi^\wedge(a) \neq 0$ and $\chi^\wedge(b) = 0$.

This corollary is also a consequence of results of E. HEWITT and H. S. ZUCKERMANN [1].

PROBLEM. When are G and $G^{\wedge\wedge}$ (see section 1) isomorphic semigroups?

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Резюме

ХАРАКТЕРЫ ИЗВЕРЗНЫХ ПОЛУГРУПП

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Пусть G — абелева инверзная полугруппа, $G^\wedge$ — полугруппа характеров G. В отделе 1 настоящей работы доказывается несколько теорем, касающихся строения полугруппы $G^\wedge$. В отделе 2 доказывается теорема о продолжении характеров и теорема о существовании достаточного множества характеров.

На примерах показано, что предположения доказываемых теорем нельзя существенным образом ослабить.