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RELATIVE INVERTIBILITY IN SEMIGROUPS

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The purpose of this paper is the study of two functions $u(x)$ and $v(x)$ defined on any Γ-compact semigroup.

A semigroup is a non-void Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition. In what follows S will always denote a semigroup and E will denote the set of its idempotents,

$$E = \{x \mid x \in S \text{ and } x^2 = x\} .$$

A subset G of S is a *subgroup* if G is non-void and if $xG = G = Gx$ for each $x \in G$, and a subset T of S is a *subsemigroup* of S if $T^2 \subset T$ and T is non-void. It is known (e. g. [1]) that any subgroup of S is contained in a maximal subgroup and that no two maximal subgroups of S intersect. Let H be the union of all the maximal subgroups of S so that H is non-void if and only if E is non-void. Indeed, if we let H_e denote the maximal subgroup containing $e \in E$, then

$$H = \cup\{H_e \mid e \in E\} .$$

Construction I. For $x \in H$ let $u(x)$ denote the unit of the maximal subgroup containing x and let $v(x)$ denote the inverse of x in this subgroup. In this way two functions $u : H \to E$ and $v : H \to H$ are defined (generally discontinuous) such that $u(u(x)) = u(x)$, $v(v(x)) = x$ and $x\, v(x) = u(x) = v(x)\, x$.

If $x \in S$ let

$$\Gamma_n(x) = \{x^m \mid m \geqq n\}^*$$

(the * denoting closure), write $\Gamma(x)$ for $\Gamma_1(x)$ and let

$$N(x) = \cap\{\Gamma_n(x) \mid n \geqq 1\} .$$

If $\Gamma(x)$ is compact then it is a commutative subsemigroup of S, $N(x)$ is a compact subgroup which is the minimal ideal and the maximal subgroup of the semigroup $\Gamma(x)$ and $\Gamma(x)$ contains exactly one idempotent, the unit of $N(x)$. Here an ideal of a semigroup M is such a non-void subset I of M that $MI \subset I \supset IM$. For the above result see for example [3] or [4].

We say that S is *Γ-compact* if $\Gamma(x)$ is compact for each $x \in S$.

Construction II. Let S be Γ-compact and for $x \in S$ let $u(x)$ be the unit of $N(x)$ and let $v(x)$ be the inverse of $x\,u(x) = u(x)\,x$ in the group $N(x)$. In this way two functions $u : S \to E$ and $v : S \to H$ are defined (generally discontinuous) such that $u(u(x)) = u(x)$, $x\,v(x) = u(x) = v(x)\,x$ (notice that $u(x)\,v(x) = v(x) = v(x)\,u(x)$) and $u(u(x)) = x\,v(x)$, as is easily verified from the definitions.

Some years ago the question arose as to the equivalence of Constructions I and II. It is readily seen that the functions are the same (where they are defined) if S is compact and later R. J. Koch observed that this remark remains true if S is locally compact. We shall prove here that the constructions are the same for S Γ-compact, this being essential for second construction.

Suppose then that S is Γ-compact, that the functions u and v are given by Construction II, and that $a \in H$. Then a is a member of some maximal subgroup of S and we let b be the inverse of a in that subgroup and denote by G the smallest subgroup containing a so that also $b \in G$. If we let

$$O(x) = \{x^n \mid n \geqq 1\}$$

then $O(x)^* = \Gamma(x)$ and

$$G = O(a) \cup O(b) \cup \{e\} \quad (e \text{ the unit of } G)$$

so that

$$G^* = \Gamma(a) \cup \Gamma(b) \cup \{e\}$$

and hence G^* is compact. Now it readily follows from the continuity of multiplication and the compactness of G^* (S is Hausdorff) that G^* is a subgroup. We have $u(a) \in G^* \cap E$ and thus $u(a) = e$, which is to say that $u(a)$ is the unit of the maximal subgroup H_e of S which contains a. From $a \in H_e$ we have $a\,u(a) = a$ and we see that $v(a)$ is the inverse of a in H_e. This completes the proof.

It ensues from the above reasoning that if S is Γ-compact then each element of H is contained in a compact subgroup.

Using a result due to Št. Schwarz [4] or an unpublished result of A. L. Shields it can be shown that (S being Γ-compact) the functions u and v are endomorphisms if S is commutative, cf. [2].

The following observation may be of interest: Suppose that S is discrete and that $O(x)$ (see the above proof) is finite for each $x \in S$. (Otherwise, S is periodic.) We infer then from the preceeding remark that any element of S which is contained in a subgroup is also contained in a finite subgroup. Of course this is not difficult to prove directly.

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Резюме

ОТНОСИТЕЛЬНАЯ ОБРАТИМОСТЬ В ПОЛУГРУППАХ

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Полугруппа S называется Γ-компактной, если для всякого $x \in S$ замыкание последовательности $\{x, x^2, x^3, \ldots\}$ компактно.

Целью этой заметки является доказательство одной теоремы, известной для компактных полугрупп, в случае Γ-компактных полугрупп.