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RELATIONS OF COMPLETENESS

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A problem of G. CHOQUET [1] is solved. The concept of the relation of completeness is introduced. A (completely regular) space P is a G_δ-space or topologically complete in the sense of E. ČECH (*i.e.* P is a G_δ-subset of the Stone-Čech compactification of P) if and only if there exists a relation of completeness on the space P. Analogously, the spaces containing a topologically complete space as a dense subspace are characterized internally.

1. All spaces are assumed to be completely regular. For convenience in this section we shall recall the definitions and some theorems from [2] that are connected with our subject.

Definition 1. A space P is said to be topologically complete in the sense of E. Čech (in the terminology of [2] a G_δ-space), or merely topologically complete, if P is a G_δ-subset of the Stone-Čech compactification $\beta(P)$ of P. A space is said to be almost topologically complete in the sense of E. Čech (in the terminology of [2], an almost G_δ-space) or merely an almost topologically complete space, if P contains a topologically complete space as a dense subspace.

If a topologically complete space P is a dense subspace of a space R, then P is G_δ in R. If a space P is a G_δ-subset of a topologically complete space, then P is a topologically complete space.

Using complete sequences of open coverings (almost coverings, respectively), an internal characterization (*i.e.* without references to larger spaces) of topologically complete (almost topologically complete) spaces is given in [3]. First let us recall that a family $\mathfrak{M}$ of subsets of a space P is said to be an almost covering (of P) if the union of $\mathfrak{M}$ is a dense subset of P.

Definition 2. A sequence $\{\mathfrak{A}_n\}$ of open coverings (almost coverings, respectively) is said to be complete if, whenever a family $\mathfrak{A}$ of open subsets has the finite intersection property and $\mathfrak{A} \cap \mathfrak{A}_n \neq \emptyset$ for all $n = 1, 2, \ldots$, then $\bigcap \{\bar{A}; A \in \mathfrak{A}\} \neq \emptyset$.

Theorem 1. *A necessary and sufficient condition that P be a topologically complete (an almost topologically complete) space is that there exist a complete sequence of open coverings (almost coverings, respectively) of the space P.*[1])

[1]) For proof see [3], Theorems 2.8 and 4.5.

It is evident that if $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings (almost coverings) and if $\mathfrak{B}_n$ is an open refinement of $\mathfrak{A}_n$, $n = 1, 2, \ldots$, then $\{\mathfrak{B}_n\}$ is also a complete sequence. It may be proved that whenever $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings (almost coverings) then $\{\mathfrak{B}_n\}$ is a complete one, where $\mathfrak{B}_n$ consist of unions of all finite subfamilies of $\mathfrak{A}_n$.[2]) Thus we have proved the following.

Theorem 2. *If P is a topologically complete space (an almost topologically complete space) then there exists a complete sequence $\{\mathfrak{A}_n\}$ of open coverings (almost coverings, respectively) such that*

(i) $\mathfrak{A}_n \supset \mathfrak{A}_{n+1}$, $n = 1, 2, \ldots$,

(ii) *If A is open and $A \subset B \in \mathfrak{A}_n$, then $A \in \mathfrak{A}_n$,*

(iii) *Every $\mathfrak{A}_n$ is (finitely) additive, i.e. if both A and B belong to $\mathfrak{A}_n$, then $A \cup B$ belongs to $\mathfrak{A}_n$.*

Finally we shall need the following (see [3], theorem 2.14):

Theorem 3. *A sequence $\{\mathfrak{A}_n\}$ of open coverings of space P is complete if and only if the following two conditions are satisfied:*

(j) *If $M \subset \bigcap_{n=1}^{\infty} A_n$ where $A_n \in \mathfrak{A}_n$, then $\overline{M}$ is a compact subspace of P.*

(jj) *If $\{F_n\}$ is a sequence of closed subsets such that $F_n \supset F_{n+1} \neq \emptyset$, $(n = 1, 2, \ldots)$ and for some $A_n \in \mathfrak{A}_n$ we have $F_n \subset A_n$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

2. Now we are prepared to formulate the definition of the relation of completeness. For convenience we shall use the term "relation" in the following special manner:

Definition 3. A relation r on a space P is a binary relation defined for open subsets of P and such that

(1) $r(A, B) \Rightarrow A \supset B$.

(2) If $r(A, B)$ and both C and D are open, $C \supset A$, $D \subset B$, then $r(C, D)$.

Definition 4. A relation of almost completeness on a space P is a relation r on P satisfying the following two conditions:

(3) If a family $\mathfrak{A}$ of open sets has the finite intersection property and if for every positive integer n there exist $A_1, \ldots, A_{n+1} \in \mathfrak{A}$ such that $r(A_i, A_{i+1})$, $i = 1, \ldots, n$, then $\bigcap \{\overline{A}; A \in \mathfrak{A}\} \neq \emptyset$.

(4) If A is a non-void open set then there exists a B with $r(A, B)$.

Definition 5. A relation of completeness on a space P is a relation r on P satisfying (3) and

(5) For every open set A the family $\{B; r(A, B)\}$ is a base for open subsets of A.

[2]) For proof see [3], Theorem 2.14.

Note 1. Evidently (5) implies (4). Thus every relation of completeness is a relation of almost completeness. A space P is compact if and only if the inclusion relation $\supset$ is a relation of completeness. A space is locally compact if and only if the following relation is a relation of completeness: $r(A, B)$ if and only if $A \supset B$ and the closure of B is a compact space. A metric space (P, φ) is complete if and only if the following rel tion is a relation of completeness: $r(A, B)$ if and only if $A \supset B$ and the diameter of B is finite and less than the half that of A.

First we shall consider the connection between relations of completeness and complete sequences of open coverings.

Theorem 4. *Let P be a space. There exists a complete sequence of open coverings (almost coverings) if and only if there exists a relation of completeness (almost completeness, respectively).*

Proof. First let us suppose that $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings (almost coverings) of the space P. Without loss of generality we may assume that conditions (i) and (ii) are satisfied. Now if A is an open set which does not belong to $\mathfrak{A}_1$, put $n(A) = 0$. In the other case put

$$n(A) = \sup \{n;\ A \in \mathfrak{A}_n\} .$$

Thus $n(A)$ is either an integer $0, 1, 2, \ldots$ or ∞. Let us define a relation r on the space P such that $r(A, B)$ if and only if either $n(A) < n(B)$ or $n(A) = n(B) = \infty$ and $B \subset A$.

Evidently the axioms (1) and (2) are fulfilled. Now we shall prove (3). Let us suppose that a family $\mathfrak{A}$ of open sets has the finite intersection property and for every positive integer n there exist $A_1, \ldots, A_{n+1} \in \mathfrak{A}$ with $r(A_1, A_{1+1})$, $i = 1, 2, \ldots, u$. To prove $\bigcap \{\bar{A};\ A \in \mathfrak{A}\} \neq \emptyset$, it is sufficient to show $\mathfrak{A} \cap \mathfrak{A}_n \neq \emptyset$ for all $n = 1, 2, \ldots$. But according to the definition of r, if $r(A_i, A_{i+1})$ for $i = 1, 2, \ldots n$, then A_{n+1} belongs to $\mathfrak{A}_{n+1}$. The proof of (3) is complete. It remains to prove that if $\mathfrak{A}_n$ are coverings (almost coverings) then the axiom (5) (the axiom (4), respectively) is fulfilled. But this is evident and may be left to the reader.

Conversely, let r be a relation of completeness (almost completeness), respectively on the space P. Let $\mathfrak{A}_1$ be the family of all non-void open subsets of P. By induction, put

$$\mathfrak{A}_{n+1} = \{A;\ r(B, A) \quad \text{for some} \quad B \in \mathfrak{A}_n\} .$$

We shall prove that $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings (almost coverings). Let us suppose that a family of open subsets of P has the finite intersection property and $\mathfrak{A}_n \cap \mathfrak{A} \neq \emptyset$ for all $n = 1, 2, \ldots$. Without loss of generality we may assume

$$B \quad \text{open}, \quad B > A \in \mathfrak{A} \Rightarrow B \in \mathfrak{A} .$$

It follows that if $A_{n+1} \in \mathfrak{A}_{n+1} \cap \mathfrak{A}$, then there exist $A_1, \ldots, A_n \in \mathfrak{A}$ such that $r(A_i, A_{i+1})$, $i = 1, \ldots, n$. In consequence, by (3) we have $\bigcap \{\bar{A};\ A \in \mathfrak{A}\} \neq \emptyset$. It remains to prove that $\mathfrak{A}_n$ are coverings or almost coverings provided that the condition (5) or (4), respectively, is fulfilled by r. But this is evident.

Note 2. The sequence $\{\mathfrak{A}_n\}$ from the second part (*i.e.* the "if" part) of the proof of the preceding theorem satisfies the conditions (i) and (ii) of theorem 2.

As a consequence of the preceding theorem and theorem 1 we have at once

Theorem 5. *A necessary and sufficient condition that P be a topologically complete space (almost topologically complete space) is that there exists a relation of completeness (almost completeness, respectively) on the space P.*

In the following section we shall prove the preceding theorem 5 directly (*i.e.* without reference to theorems 1 and 4). We shall also prove a characterization of complete sequences in theorem 3.

3. Proposition 1. *Let us suppose that P is a dense subspace of a space K and that there exists a relation r of completeness (almost completeness) on the space P. Then P is a G_δ-subset of P (P contains a dense G_δ-subset of K, respectively).*

Proof. We shall prove the assertion concerning the relation of completeness only. For every open subset A of P let A' be the union of all open $U \subset K$ with $U \cap P = A$. Thus we have $A' \cap P = A$. Let U_n be the union of all A' for which there exist sets $A_1, \ldots, A_n$ open in P such that $r(A_n, A)$ and

$$r(A_i, A_{i+1}) \quad (i = 1, \ldots, n - 1).$$

Put

$$G = \bigcap_{n=1}^{\infty} U_n .$$

Clearly $G \supset P$. To prove the converse inclusion, let us suppose that there exists a point x in $G - P$. Let $\mathfrak{B}$ be the family of all open neighborhoods of the point x and let $\mathfrak{A}$ be the family of all $A = B \cap P$ where $B \in \mathfrak{B}$. Since P is dense in K, the family $\mathfrak{A}$ has the finite intersection property. Clearly the assumption of (3) is satisfied, and hence

$$\bigcap \{\bar{A}^P;\ A \in \mathfrak{A}\} \neq \emptyset .$$

Choosing a point y in this intersection, we have $y \neq x$. But this is impossible since

$$\bigcap \{\bar{B}^K;\ B \in \mathfrak{B}\} = (x) .$$

Proposition 2. *Let us suppose that P is dense and G_δ in a space K and that there exists a relation of completeness (almost completeness) on K. Then there exists a relation of completeness (almost completeness, respectively) on P.*

Proof. Again we shall prove the assertion concerning the relation of completeness only. Let r be a relation of completeness on K and let

$$P = \bigcap \{U_n;\ n = 1, 2, \ldots\} ,$$

where U_n are open subsets of K and $U_n \supset U_{n+1}$. Let us define a relation r_1 of P as follows:

For every open $A \subset P$ put $u(A) = 0$ if the closure of A in K is not contained in U_1. In the other case put

$$n(A) = \sup \{n;\ \bar{A}^K \subset U_n\}\,.$$

Thus $n(A)$ is 0, 1, … or ∞. Now we shall define $r(A, B)$ if and only if $r(A', B')$ where A' is the interior of the closure of A in K and either $n(B) = n(A) = \infty$ or $n(A) < n(B)$.

It is easy to see that r_1 is a relation of completeness on the space P. Indeed, (5) is evident and if $\mathfrak{A}$ satisfies the assumption of (3) with respect to r_1 then the family of all A', $A \in \mathfrak{A}$, satisfies the assumptions of (3) with respect to r. Thus

$$F = \bigcap \{\bar{A}'^K,\ A \in \mathfrak{A}\} \neq \emptyset\,.$$

But from the definition of r_1 we have that $F \subset U_n$ for every n and hence $F \subset P$. The proof is complete.

Proposition 3. *Let P be a closed subspace of a space K. If there exists a relation of completeness on K, then there exists one on P also.*

Proof. Let r be a relation of completeness on the space K. For every pair of open subsets A and B of P pur $r_1(A, B)$ if and only if there exist open subsets A' and B' of K such that $r(A', B')$ and $A' \cap P = A$, $B' \cap P = B$. Evidently r_1 satisfies (1), (2) and (5). To prove the condition (3), it is sufficient to prove the following.

Proposition 4. *If r is a relation of completeness on a space P, then the following condition (3′) is satisfied:*

(3′) *If a family $\mathfrak{M}$ of subsets of P has the finite intersection property and if for every positive integer n there exist $A_1, \ldots, A_{n+1} \in \mathfrak{M}$ with $r(A_i, A_{i+1})$, $(i = 1, \ldots, n)$ then $\bigcap \{\bar{M};\ M \in \mathfrak{M}\} \neq \emptyset$.*

Proof. Let $\mathfrak{A}$ be the family of all open subsets A of P containing a set $M \in \mathfrak{M}$. Evidently $\mathfrak{A}$ has the finite intersection property and the assumptions of (3) are satisfied. Thus we have

$$F = \bigcap \{\bar{A};\ A \in \mathfrak{A}\} \neq \emptyset\,.$$

The space P being regular, every closed subset K ot P is the intersection of closures of all open sets containing K. Thus for every M in $\mathfrak{M}$ we have $F \subset \bar{M}$ and consequently $F \subset \bigcap \{\bar{M};\ M \in \mathfrak{M}\}$ which completes the proof of proposition 4 and also that of proposition 3.

Note. For almost relations the analogue of proposition 3 does not hold.

As an immediate consequence of the preceding propositions 1–4 and theorem 4 we have the following theorem:

Theorem 6. *The following conditions on a space P are equivalent:*

(1) *P is G_δ in the Čech-Stone compactification of P.*

(2) *P is G_δ in some compactification of P.*

(3) *There exists a relation of completeness of the space P.*

(4) *There exists a complete sequence of open coverings of the space P.*

Theorem 7. *The following conditions on a space P are equivalent:*

(1) *There exists a dense G_δ-subset R of the Stone-Čech compactification of P such that $R \subset P$.*

(2) *There exists a dense G_δ-subset S of some compactification of P with $S \subset P$.*

(3) *There exists a relation of almost completeness of P.*

(4) *There exists a complete sequence of open almost coverings of the space P.*

Finally, we shall prove the following analogue of theorem 3:

Theorem 8. *Let r be a relation on a space P such that condition (5) is satisfied. Then r is a relation of completeness if and only if the following two conditions are satisfied:*

(k) *Let M be a subset of P such that for every positive integer n there exist $A_1, \ldots, A_{n+1}$ such that $M \subset A_{n+1}$ and $r(A_i, A_{i+1})$, $i = 1, 2, \ldots, n$. Then the closure of M is a compact subspace of P.*

(kk) *Let $\{F_n\}$ be a sequence of non-void closed subsets of P such that $F_n > F_{n+1}$ and that for every positive integer n there exist $A_1, \ldots, A_{n+1}$ such that $A_{n+1} \supset F_{n+1}$ and $r(A_i, A_{i+1})$, $i = 1, \ldots, n$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof. First let us suppose that r is a relation of completeness. The condition (kk) follows at once from (3). To prove (k) it is sufficient to notice that if $\mathfrak{M}$ is a family of subsets of M with the finite intersection property, then the assumptions of proposition 4 are satisfied and hence $\bigcap \{\overline{N};\ N \in \mathfrak{M}\} \neq \emptyset$, which proves the compactness of $\overline{M}$ and completes the first part of the proof.

Conversely, suppose (k) and (kk) and let the family $\mathfrak{A}$ satisfy the assumptions of (3). Let $\mathfrak{M}$ be a maximal family of subsets of P with the finite intersection property and containing $\mathfrak{A}$. It is easy to construct by induction a sequence $\{F_n\}$ of closed subsets of P, satisfying the assumptions ot (kk). Put $F = \bigcap_{n=1}^{\infty} F_n$. By (kk) we have $F \neq \emptyset$, and by (k) the set F is compact. Now it remains to prove that the family of all $F \cap \overline{M}$, $M \in \mathfrak{M}$, has the finite intersection property. According to the maximality of $\mathfrak{M}$ it is sufficient to prove $\overline{M} \cap F \neq \emptyset$ for all $M \in \mathfrak{M}$. But this is evident, since

$$\overline{M} \cap F = \bigcap_{n=1}^{\infty} \overline{M} \cap F_n\,,$$

and the sequence $\{\overline{M} \cap F_n\}$ satisfies the condition (kk). The proof is complete.

References

[1] *G. Choquet:* Une classe reguliere d'espaces de Baire. C. R. Acad. Paris, *260* (1958), 218—220.

[2] *E. Čech:* On Bicompact Spaces. Ann. of Math. *38* (1937), 823—844.

[3] *Z. Frolík:* Generalizations of the G_δ-property of Complete Metric Spaces. Czechosl. Math. J. *10 (85)*, 1960, 359—379.

Резюме

ОТНОШЕНИЯ ПОЛНОТЫ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Пусть дано пространство P (рассматриваются только вполне регулярные топологические пространства). „Отношением на P" мы будем называть бинарное отношение r, определенное для открытых подмножеств P и такое, что

(1) если $r(A, B)$, то $A \supset B$;

(2) если $r(A, B)$, C и D открыты, $C \supset A$, $\emptyset \neq D \subset B$, то $r(C, D)$.

Отношение r на P называется *отношением почти-полноты,* если

(3) для любой центрированной системы $\mathfrak{A}$ открытых множеств такой, что при любом натуральном n найдутся $A_i \in \mathfrak{A}$, для которых $r(A_i, A_{i+1})$, $i = 1, \ldots, n$, пересечение всех $\bar{A}$, $A \in \mathfrak{A}$, непусто;

(4) для любого открытого $A \subset P$, $A \neq \emptyset$, существует B так, что $r(A, B)$.

Если же удовлетворяется условия (3) и

(5) для любого открытого A система $\{B; r(A, B)\}$ является открытой базой A, то r называется *отношением полноты.*

Основным результатом являются следующие теоремы (встречающееся в них понятие полной последовательности покрытий или „почти-покрытий" определено в работе [3]).

Теорема. *Следующие свойства пространства P эквивалентны:*

(1) *P является G_δ-множеством в своем чеховском расширении βP;*

(2) *P является G_δ-множеством в одном из своих компактных расширений;*

(3) *существует отношение полноты на P;*

(4) *существует полная последовательность открытых покрытий P.*

Теорема. *Следующие свойства пространства P эквивалентны:*

(1) *Существует $R \subset P$, являющееся плотным G_δ-подмножеством пространства βP;*

(2) *существует $R \subset P$, являющееся плотным G_δ-множеством в одном из компактных расширений пространства P;*

(3) *существует отношение почти-полноты на P;*

(4) *существует полная последовательность открытых почти-покрытий пространства P.*