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Relations of completeness


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RELATIONS OF COMPLETENESS

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A problem of G. Choquet [1] is solved. The concept of the relation of completeness is introduced. A (completely regular) space $P$ is a $G_δ$-space or topologically complete in the sense of E. Čech (i.e. $P$ is a $G_δ$-subset of the Stone-Čech compactification of $P$) if and only if there exists a relation of completeness on the space $P$. Analogously, the spaces containing a topologically complete space as a dense subspace are characterized internally.

1. All spaces are assumed to be completely regular. For convenience in this section we shall recall the definitions and some theorems from [2] that are connected with our subject.

**Definition 1.** A space $P$ is said to be topologically complete in the sense of E. Čech (in the terminology of [2] a $G_δ$-space), or merely topologically complete, if $P$ is a $G_δ$-subset of the Stone-Čech compactification $β(P)$ of $P$. A space is said to be almost topologically complete in the sense of E. Čech (in the terminology of [2], an almost $G_δ$-space) or merely an almost topologically complete space, if $P$ contains a topologically complete space as a dense subspace.

If a topologically complete space $P$ is a dense subspace of a space $R$, then $P$ is $G_δ$ in $R$. If a space $P$ is a $G_δ$-subset of a topologically complete space, then $P$ is a topologically complete space.

Using complete sequences of open coverings (almost coverings, respectively), an internal characterization (i.e. without references to larger spaces) of topologically complete (almost topologically complete) spaces is given in [3]. First let us recall that a family $ℳ$ of subsets of a space $P$ is said to be an almost covering (of $P$) if the union of $ℳ$ is a dense subset of $P$.

**Definition 2.** A sequence $(ℳ_n)$ of open coverings (almost coverings, respectively) is said to be complete if, whenever a family $ℳ$ of open subsets has the finite intersection property and $ℳ \cap ℳ_n \neq ∅$ for all $n = 1, 2, \ldots$, then $\bigcap \{A; A ∈ ℳ\} \neq ∅$.

**Theorem 1.** A necessary and sufficient condition that $P$ be a topologically complete (an almost topologically complete) space is that there exist a complete sequence of open coverings (almost coverings, respectively) of the space $P$.

1) For proof see [3], Theorems 2.8 and 4.5.
It is evident that if $\{\mathcal{U}_n\}$ is a complete sequence of open coverings (almost coverings) and if $\mathcal{B}_n$ is an open refinement of $\mathcal{U}_n, n = 1, 2, \ldots$, then $\{\mathcal{B}_n\}$ is also a complete sequence. It may be proved that whenever $\{\mathcal{B}_n\}$ is a complete sequence of open coverings (almost coverings) then $\{\mathcal{B}_n\}$ is a complete one, where $\mathcal{B}_n$ consist of unions of all finite subfamilies of $\mathcal{U}_n$. Thus we have proved the following.

**Theorem 2.** If $P$ is a topologically complete space (an almost topologically complete space) then there exists a complete sequence $\{\mathcal{U}_n\}$ of open coverings (almost coverings, respectively) such that

(i) $\mathcal{U}_n \supset \mathcal{U}_{n+1}, n = 1, 2, \ldots$,
(ii) If $A$ is open and $A \subset B \in \mathcal{U}_n$, then $A \in \mathcal{U}_n$,
(iii) Every $\mathcal{U}_n$ is (finitely) additive, i.e. if both $A$ and $B$ belong to $\mathcal{U}_n$, then $A \cup B$ belongs to $\mathcal{U}_n$.

Finally we shall need the following (see [3], theorem 2.14):

**Theorem 3.** A sequence $\{\mathcal{U}_n\}$ of open coverings of space $P$ is complete if and only if the following two conditions are satisfied:

(i) If $M = \bigcap_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{U}_n$, then $M$ is a compact subspace of $P$.

(ii) If $\{F_n\}$ is a sequence of closed subsets such that $F_n \supset F_{n+1} \neq \emptyset, (n = 1, 2, \ldots)$ and for some $A_n \in \mathcal{U}_n$ we have $F_n \subset A_n$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

2. Now we are prepared to formulate the definition of the relation of completeness. For convenience we shall use the term “relation” in the following special manner:

**Definition 3.** A relation $r$ on a space $P$ is a binary relation defined for open subsets of $P$ and such that

1. $r(A, B) \Rightarrow A \supset B$.
2. If $r(A, B)$ and both $C$ and $D$ are open, $C \supset A, D \subset B$, then $r(C, D)$.

**Definition 4.** A relation of almost completeness on a space $P$ is a relation $r$ on $P$ satisfying the following two conditions:

3. If a family $\mathcal{U}$ of open sets has the finite intersection property and if for every positive integer $n$ there exist $A_1, \ldots, A_{n+1} \in \mathcal{U}$ such that $r(A_i, A_{i+1}), i = 1, \ldots, n,$ then $\bigcap \{A_i; A_i \in \mathcal{U}\} \neq \emptyset$.
4. If $A$ is a non-void open set then there exists a $B$ with $r(A, B)$.

**Definition 5.** A relation of completeness on a space $P$ is a relation $r$ on $P$ satisfying (3) and

5. For every open set $A$ the family $\{B; r(A, B)\}$ is a base for open subsets of $A$.

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2) For proof see [3], Theorem 2.14.
Note 1. Evidently (5) implies (4). Thus every relation of completeness is a relation of almost completeness. A space $P$ is compact if and only if the inclusion relation $\supset$ is a relation of completeness. A space is locally compact if and only if the following relation is a relation of completeness: $r(A, B)$ if and only if $A \supset B$ and the closure of $B$ is a compact space. A metric space $(P, \varphi)$ is complete if and only if the following relation is a relation of completeness: $r(A, B)$ if and only if $A \supset B$ and the diameter of $B$ is finite and less than the half that of $A$.

First we shall consider the connection between relations of completeness and complete sequences of open coverings.

**Theorem 4.** Let $P$ be a space. There exists a complete sequence of open coverings (almost coverings) if and only if there exists a relation of completeness (almost completeness, respectively).

**Proof.** First let us suppose that $\{ \mathcal{U}_n \}$ is a complete sequence of open coverings (almost coverings) of the space $P$. Without loss of generality we may assume that conditions (i) and (ii) are satisfied. Now if $A$ is an open set which does not belong to $\mathcal{U}_1$, put $n(A) = 0$. In the other case put

$$n(A) = \sup \{ n; \ A \in \mathcal{U}_n \} .$$

Thus $n(A)$ is either an integer $0, 1, 2, \ldots$ or $\infty$. Let us define a relation $r$ on the space $P$ such that $r(A, B)$ if and only if either $n(A) < n(B)$ or $n(A) = n(B) = \infty$ and $B \subset A$.

Evidently the axioms (1) and (2) are fulfilled. Now we shall prove (3). Let us suppose that a family $\mathcal{U}$ of open sets has the finite intersection property and for every positive integer $n$ there exist $A_1, \ldots, A_{n+1} \in \mathcal{U}$ with $r(A_i, A_{i+1})$, $i = 1, 2, \ldots, u$. To prove $\bigcap \{ \overline{A}; \ A \in \mathcal{U} \} \neq \emptyset$, it is sufficient to show $\mathcal{U} \cap \mathcal{U}_n \neq \emptyset$ for all $n = 1, 2, \ldots$. But according to the definition of $r$, if $r(A_i, A_{i+1})$ for $i = 1, 2, \ldots, n$, then $A_{n+1}$ belongs to $\mathcal{U}_{n+1}$. The proof of (3) is complete. It remains to prove that if $\mathcal{U}_n$ are coverings (almost coverings) then the axiom (5) (the axiom (4), respectively) is fulfilled. But this is evident and may be left to the reader.

Conversely, let $r$ be a relation of completeness (almost completeness), respectively, on the space $P$. Let $\mathcal{U}_1$ be the family of all nonvoid open subsets of $P$. By induction, put

$$\mathcal{U}_{n+1} = \{ A; r(B, A) \text{ for some } B \in \mathcal{U}_n \} .$$

We shall prove that $\{ \mathcal{U}_n \}$ is a complete sequence of open coverings (almost coverings). Let us suppose that a family of open subsets of $P$ has the finite intersection property and $\mathcal{U}_n \cap \mathcal{U} \neq \emptyset$ for all $n = 1, 2, \ldots$. Without loss of generality we may assume

$$B \text{ open, } \ B > A \in \mathcal{U} \Rightarrow B \in \mathcal{U} .$$

It follows that if $A_{n+1} \in \mathcal{U}_{n+1} \cap \mathcal{U}$, then there exist $A_1, \ldots, A_n \in \mathcal{U}$ such that $r(A_i, A_{i+1})$, $i = 1, \ldots, n$. In consequence, by (3) we have $\bigcap \{ \overline{A}; \ A \in \mathcal{U} \} \neq \emptyset$. It remains to prove that $\mathcal{U}_n$ are coverings or almost coverings provided that the condition (5) or (4), respectively, is fulfilled by $r$. But this is evident.
Note 2. The sequence \( \{\mathcal{U}_n\} \) from the second part (i.e. the "if" part) of the proof of the preceding theorem satisfies the conditions (i) and (ii) of theorem 2.

As a consequence of the preceding theorem and theorem 1 we have at once

**Theorem 5.** A necessary and sufficient condition that \( P \) be a topologically complete space (almost topologically complete space) is that there exists a relation of completeness (almost completeness, respectively) on the space \( P \).

In the following section we shall prove the preceding theorem 5 directly (i.e. without reference to theorems 1 and 4). We shall also prove a characterization of complete sequences in theorem 3.

3. **Proposition 1.** Let us suppose that \( P \) is a dense subspace of a space \( K \) and that there exists a relation \( r \) of completeness (almost completeness) on the space \( P \). Then \( P \) is a \( G_\sigma \)-subset of \( P \) \((P \text{ contains a dense } G_\sigma \text{-subset of } K, \text{ respectively})\).

*Proof.* We shall prove the assertion concerning the relation of completeness only. For every open subset \( A \) of \( P \) let \( A' \) be the union of all open \( U \subset K \) with \( U \cap P = A \). Thus we have \( A' \cap P = A \). Let \( U_n \) be the union of all \( A' \) for which there exist sets \( A_1, \ldots, A_n \) open in \( P \) such that \( r(A_n, A) \) and

\[
r(A_i, A_{i+1}) \quad (i = 1, \ldots, n - 1).
\]

Put

\[
G = \bigcap_{n=1}^{\infty} U_n.
\]

Clearly \( G \supset P \). To prove the converse inclusion, let us suppose that there exists a point \( x \) in \( G - P \). Let \( \mathcal{B} \) be the family of all open neighborhoods of the point \( x \) and let \( \mathcal{A} \) be the family of all \( A = B \cap P \) where \( B \in \mathcal{B} \). Since \( P \) is dense in \( K \), the family \( \mathcal{A} \) has the finite intersection property. Clearly the assumption of (3) is satisfied, and hence

\[
\bigcap \{ A^p; \ A \in \mathcal{A} \} \neq \emptyset.
\]

Choosing a point \( y \) in this intersection, we have \( y \neq x \). But this is impossible since

\[
\bigcap \{ B^k; \ B \in \mathcal{B} \} = (x).
\]

**Proposition 2.** Let us suppose that \( P \) is dense and \( G_\delta \) in a space \( K \) and that there exists a relation of completeness (almost completeness) on \( K \). Then there exists a relation of completeness (almost completeness, respectively) on \( P \).

*Proof.* Again we shall prove the assertion concerning the relation of completeness only. Let \( r \) be a relation of completeness on \( K \) and let

\[
P = \bigcap \{ U_n; \ n = 1, 2, \ldots \},
\]

where \( U_n \) are open subsets of \( K \) and \( U_n \supset U_{n+1} \). Let us define a relation \( r_1 \) of \( P \) as follows:

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For every open $A \subset P$ put $u(A) = 0$ if the closure of $A$ in $K$ is not contained in $U_1$. In the other case put

$$n(A) = \sup \{n; \overline{A}^K \subset U_n\}.$$  

Thus $n(A)$ is 0, 1, ... or $\infty$. Now we shall define $r(A, B)$ if and only if $r(A', B')$ where $A'$ is the interior of the closure of $A$ in $K$ and either $n(B) = n(A) = \infty$ or $n(A) < n(B)$.

It is easy to see that $r_1$ is a relation of completeness on the space $P$. Indeed, (5) is evident and if $\mathcal{A}$ satisfies the assumption of (3) with respect to $r_1$, then the family of all $A', A \in \mathcal{A}$, satisfies the assumptions of (3) with respect to $r$. Thus

$$F = \bigcap \{\overline{A}^K; A \in \mathcal{A}\} \neq \emptyset.$$  

But from the definition of $r_1$ we have that $F \subset U_n$ for every $n$ and hence $F \subset P$. The proof is complete.

**Proposition 3.** Let $P$ be a closed subspace of a space $K$. If there exists a relation of completeness on $K$, then there exists one on $P$ also.

**Proof.** Let $r$ be a relation of completeness on the space $K$. For every pair of open subsets $A$ and $B$ of $P$ pur $r_1(A, B)$ if and only if there exist open subsets $A'$ and $B'$ of $K$ such that $r(A', B')$ and $A' \cap P = A, B' \cap P = B$. Evidently $r_1$ satisfies (1), (2) and (5). To prove the condition (3), it is sufficient to prove the following.

**Proposition 4.** If $r$ is a relation of completeness on a space $P$, then the following condition (3') is satisfied:

(3') If a family $\mathcal{M}$ of subsets of $P$ has the finite intersection property and if for every positive integer $n$ there exist $A_1, \ldots, A_{n+1} \in \mathcal{M}$ with $r(A_i, A_{i+1})$, $(i = 1, \ldots, n)$ then \( \bigcap \{\overline{M}; M \in \mathcal{M}\} \neq \emptyset \).

**Proof.** Let $\mathcal{A}$ be the family of all open subsets $A$ of $P$ containing a set $M \in \mathcal{M}$. Evidently $\mathcal{A}$ has the finite intersection property and the assumptions of (3) are satisfied. Thus we have

$$F = \bigcap \{\overline{A}; A \in \mathcal{A}\} \neq \emptyset.$$  

The space $P$ being regular, every closed subset $K$ of $P$ is the intersection of closures of all open sets containing $K$. Thus for every $M$ in $\mathcal{M}$ we have $F \subset M$ and consequently $F \subset \bigcap \{\overline{M}; M \in \mathcal{M}\}$ which completes the proof of proposition 4 and also that of proposition 3.

**Note.** For almost relations the analogue of proposition 3 does not hold.

As an immediate consequence of the preceding propositions $1-4$ and theorem 4 we have the following theorem:

**Theorem 6.** The following conditions on a space $P$ are equivalent:

1. $P$ is $G_3$ in the Čech-Stone compactification of $P$.
2. $P$ is $G_3$ in some compactification of $P$.
3. There exists a relation of completeness of the space $P$.
4. There exists a complete sequence of open coverings of the space $P$. 

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Theorem 7. The following conditions on a space $P$ are equivalent:

1. There exists a dense $G_δ$-subset $R$ of the Stone-Čech compactification of $P$ such that $R \subset P$.
2. There exists a dense $G_δ$-subset $S$ of some compactification of $P$ with $S \subset P$.
3. There exists a relation of almost completeness of $P$.
4. There exists a complete sequence of open almost coverings of the space $P$.

Finally, we shall prove the following analogue of theorem 3:

Theorem 8. Let $r$ be a relation on a space $P$ such that condition (5) is satisfied. Then $r$ is a relation of completeness if and only if the following two conditions are satisfied:

(k) Let $M$ be a subset of $P$ such that for every positive integer $n$ there exist $A_1, \ldots, A_{n+1}$ such that $M \subset A_{n+1}$ and $r(A_i, A_{i+1})$, $i = 1, 2, \ldots, n$. Then the closure of $M$ is a compact subspace of $P$.

(kk) Let $\{F_n\}$ be a sequence of non-empty closed subsets of $P$ such that $F_n < F_{n+1}$ and that for every positive integer $n$ there exist $A_1, \ldots, A_{n+1}$ such that $A_{n+1} \subset F_{n+1}$ and $r(A_i, A_{i+1})$, $i = 1, \ldots, n$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. First let us suppose that $r$ is a relation of completeness. The condition (kk) follows at once from (3). To prove (k) it is sufficient to notice that if $\mathcal{M}$ is a family of subsets of $M$ with the finite intersection property, then the assumptions of proposition 4 are satisfied and hence $\bigcap \{N; \: N \in \mathcal{M}\} \neq \emptyset$, which proves the compactness of $\bar{M}$ and completes the first part of the proof.

Conversely, suppose (k) and (kk) and let the family $\mathcal{M}$ satisfy the assumptions of (3). Let $\mathcal{M}$ be a maximal family of subsets of $P$ with the finite intersection property and containing $\mathcal{M}$. It is easy to construct by induction a sequence $\{F_n\}$ of closed subsets of $P$, satisfying the assumptions of (kk). Put $F = \bigcap_{n=1}^{\infty} F_n$. By (kk) we have $F \neq \emptyset$, and by (k) the set $F$ is compact. Now it remains to prove that the family of all $F \cap \bar{M}$, $M \in \mathcal{M}$, has the finite intersection property. According to the maximality of $\mathcal{M}$ it is sufficient to prove $\bar{M} \cap F \neq \emptyset$ for all $M \in \mathcal{M}$. But this is evident, since

$$\bar{M} \cap F = \bigcap_{n=1}^{\infty} \bar{M} \cap F_n,$$

and the sequence $\{\bar{M} \cap F_n\}$ satisfies the condition (kk). The proof is complete.

References

Резюме

ОТНОШЕНИЯ ПОЛНОТЫ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Пусть дано пространство $P$ (рассматриваются только вполне регулярные топологические пространства). „Отношением на $P$“ мы будем называть бинарное отношение $r$, определенное для открытых подмножеств $P$ и такое, что

(1) если $r(A, B)$, то $A \supseteq B$;

(2) если $r(A, B)$, $C$ и $D$ открыты, $C \supseteq A$, $\emptyset \neq D \subseteq B$, то $r(C, D)$.

Отношение $r$ на $P$ называется отношением почти-полноты, если

(3) для любой центрированной системы $\mathcal{A}$ открытых множеств такой, что при любом натуральном $n$ найдутся $A_i \in \mathcal{A}$, для которых $r(A_i, A_{i+1}), i = 1, \ldots, n$, пересечение всех $\overline{A}$, $A \in \mathcal{A}$, непусто;

(4) для любого открытого $A \subseteq P$, $A \neq \emptyset$, существует $B$ так, что $r(A, B)$.

Если же удовлетворяется условия (3) и

(5) для любого открытого $A$ система $\{B; r(A, B)\}$ является открытой базой $A$, то $r$ называется отношением полноты.

Основным результатом являются следующие теоремы (встречающееся в них понятие полной последовательности покрытий или „почти-покрытий“ определено в работе [3]).

Теорема. Следующие свойства пространства $P$ эквивалентны:

(1) $P$ является $G_\delta$-множеством в своем чеховском расширении $\beta P$;

(2) $P$ является $G_\delta$-множеством в одном из своих компактных расширений;

(3) существует отношение полноты на $P$;

(4) существует полная последовательность открытых покрытий $P$.

Теорема. Следующие свойства пространства $P$ эквивалентны:

(1) существует $R \subseteq P$, являющееся плотным $G_\delta$-подмножеством пространства $\beta P$;

(2) существует $R \subseteq P$, являющееся плотным $G_\delta$-множеством в одном из компактных расширений пространства $P$;

(3) существует отношение почти-полноты на $P$;

(4) существует полная последовательность открытых почти-покрытий пространства $P$. 

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