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# A CHARACTERIZATION OF TOPOLOGICALLY COMPLETE SPACES IN THE SENSE OF E. ČECH IN TERMS OF CONVERGENCE OF FUNCTIONS

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A characterization of topologically complete spaces (in the sense of E. Čech) analogous to the well known characterization of pseudocompact spaces in terms of convergence of continuous functions.

A space $P$ is said to be topologically complete (in the sense of E. Čech) if $P$ is completely regular and $P$ is a $G_\delta$ in the Čech-Stone compactification $\beta(P)$ of $P$. In the present note, we shall give a characterization of topologically complete spaces analoguous to the following characterization of pseudocompact spaces: If a decreasing sequence $\{f_n\}$ of continuous functions is pointwise convergent to zero, then $\{f_n\}$ is uniformly convergent.

All functions are supposed to be real-valued. If $\mathfrak{F}$ is a family of functions on a set $P$, then the symbol $\mathfrak{F} \downarrow 0$ will be used to express that for every $f_1$ and $f_2$ there exists an $f$ in $\mathfrak{F}$ with $f \leqq \min(f_1, f_2)$ and that for every $x$ in $P$,

$$\inf \{f(x); f \in \mathfrak{F}\} = 0 .$$

All spaces under consideration are supposed to be completely regular. $B(P)$ denotes the family of all bounded continuous functions on a space $P$. The symbol $\alpha(P)$ will be used to denote the family of all subrings $A$ of $B(P)$ satisfying the following two conditions

(1) $f \in A \Rightarrow |f| \in A$.

(2) For every $x$ in $P$ and every neighborhood $U$ of $x$ there exists an $f$ in $A$ such that $0 \leqq f \leqq 1, f(x) = 1, f[P - U] = 0$.

**Definition.** We shall say that a collection $\gamma \subset \alpha(P)$ has the property (V) if the following condition is fulfilled:

If $\mathfrak{F} \subset B(P)$, $\mathfrak{F} \downarrow 0$ and $\mathfrak{F} \cap C \downarrow 0$ for every $C$ in $\gamma$, then for every $\varepsilon > 0$ there exists an $f$ in $\mathfrak{F}$ such that $\|f\| < \varepsilon$, *i.e.*, there exists a sequence in $\mathfrak{F}$ uniformly convergent to zero.

We shall say that a ring $A \in \alpha(P)$ has the property (V), if the collection $(A) \subset \gamma(P)$ has the property (V).

**Example 1.** *A space P is compact if and only if $B(P)$ has the property* (V).

Proof. Evidently the condition is necessary. To prove sufficiency suppose that there exists a maximal centered family $\mathfrak{M}$ of closed subsets with $\bigcap \mathfrak{M} = \emptyset$. Consider the family $\mathfrak{F}$ of all non-negative $f \in B(P)$ for which $f \geqq 1$ on some $M \in \mathfrak{M}$. Clearly $\mathfrak{F} \downarrow 0$ and $\|f\| \geqq 1$ for every $f$ in $\mathfrak{F}$. Thus $B(P)$ does not have the property (V).

**Theorem 1.** *Let* $\mathfrak{m}$ *be a cardinal number. A space P is the intersection of* $\mathfrak{m}$ *open sets in the Čech-Stone compactification $\beta(P)$ of P if and only if there exists a collection $\gamma \subset \alpha(P)$ with the property* (V) *such that the potency of $\gamma$ is at most m.*

Proof. First let us suppose that

$$P = \bigcap \mathfrak{M},$$

where $\mathfrak{M}$ is a family of open subsets of $\beta(P)$ and the potency of $\mathfrak{M}$ is at most $m$. For every $M$ in $\mathfrak{M}$ let $A(M)$ be the family consisting of restrictions to $P$ of all $f \in B(\beta(P))$ with $f[\beta(P) - M] = (0)$. Clearly $A(M) \in \gamma(P)$ for all $M \in \mathfrak{M}$. It is easy to see that the collection $\{A(M); M \in \mathfrak{M}\}$ has the property (V). Indeed, if $\mathfrak{F} \subset \beta(P)$, $\mathfrak{F} \downarrow 0$ and $[\mathfrak{F} \cap A(M)] \downarrow 0$ for all $M \in \mathfrak{M}$, then $\mathfrak{F}^* \downarrow 0$, where $\mathfrak{F}^*$ is the family of continuous extensions to $\beta(P)$ of all $f \in \mathfrak{F}$. Since $B(\beta(P))$ has the property (V), for every $\varepsilon > 0$ there exists a $f^*$ in $\mathfrak{F}^*$ with $\|f^*\| < \varepsilon$. If $f$ is the restriction of $f^*$ to $P$, then $f \in \mathfrak{F}$ and $\|f\| < \varepsilon$, which proves that the collection $\{A(M); M \in \mathfrak{M}\}$ has the property (V).

Conversely, let $\gamma \subset \alpha(P)$ be a collection with property (V) and let the potency of $\gamma$ be at most $m$. For every $C$ in $\gamma$ let $C^*$ be the family consisting of the continuous extensions to $\beta(P)$ of all $f \in C$. Put

$$K(C) = \{x; x \in \beta(P), f^* \in C^* \Rightarrow f^*(x) = 0\},$$
$$K = \bigcap\{K(C); C \in \gamma\},$$

$K(C)$ are compact subspaces of $\beta(P) - P$, and consequently, it is sufficient to prove

(3) $$K = \beta(P) - P.$$

Clearly $K \subset \beta(P) - P$. Let us suppose that there exists a point $x$ in $\beta(P) - (K \cup P)$. Let $\mathfrak{F}^*$ be the family of all continuous non-negative functions $f^*$ on $\beta(P)$ with $f^*(x) \geqq$ $\geqq 1$ and let $\mathfrak{F}$ be the family consisting of the restrictions to $P$ of all functions from $\mathfrak{F}^*$. Clearly $\mathfrak{F} \downarrow 0$ and $\|f\| \geqq 1$ for every $f$ in $\mathfrak{F}$. Let $C \in \gamma$. By our assumption there exists an $f$ in $C$ with $f^*(x) \neq 0$. Put

(4) $$g = \max(0, f/f^*(x)).$$

Clearly $g \geqq 0$ and $g^*(x) = 1$. If $y \in P$, then there exists a compact neighborhood $F$ of $y$ in $\beta(P)$ with $x$ non $\in F$. According to condition (2) there exists a $h$ in $C$ with $h(y) = 1$, $h(P - F) = (0)$. Consider the function

(5) $$k = \max(0, g - gh).$$

Clearly $k \in C$, $k(y) = 0$ and $k^*(x) = 1$. It follows that $(\mathfrak{F} \cap C) \downarrow 0$. But this is impossible, because $\gamma$ has the property (V) and $\|f\| \geqq 1$ for every $f$ in $\mathfrak{F}$. This contradiction proves (3).

From the proof of the preceding Theorem 1 there follows at once theorem:

**Theorem 2.** *A space $P$ is topologically complete in the sense of E. Čech if and only if there exists a decreasing sequence $\{A_n\}$ in $\alpha(P)$ with the property* (V).

**Theorem 3.** *A Lindelöf space $P$ is topologically complete if and only if there exists a decreasing sequence $\{A_n\}$ in $\alpha(P)$ such that*

$$(6) \qquad f_n \in A_n\,, \quad \{f_n\} \downarrow 0 \Rightarrow \lim_{n\to\infty} \|f_n\| = 0\,,$$

$$(7) \qquad f \in A_n\,, \quad g \in A_{n+1}\,, \quad f \geqq 0\,, \quad g \geqq 0 \Rightarrow \min(f, g) \in A_{n+1}\,.$$

Proof. From the proof of Theorem 1 it follows at once that the condition is necessary. Conversely, let us suppose that there exists a sequence $\{A_n\}$ in $\alpha(P)$ satisfying (6). Let $A_n^*$ be the family consisting of the continuous extensions of all $f \in A_n$ to $\beta(P)$. Put

$$(8) \qquad K_n = \{x;\, x \in \beta(P), f^* \in A_n \Rightarrow f^*(x) = 0\}\,,$$

$$(9) \qquad K = \bigcup_{n=1}^{\infty} K_n\,.$$

The subspaces $K_n$ of $\beta(P)$ being compact, it is sufficient to prove (3). Clearly $K \subset \subset \beta(P) - P$. Suppose that there exists a point $x$ in $\beta(P) - (P \cup K)$. First we shall construct sequences $\{f_k^n\}_{k=1}^{\infty}$ such that

$$(10) \qquad f_k^n \in A_n\,, \quad \{f_k^n\}_{k=1}^{\infty} \downarrow 0 \quad (n = 1, 2, \ldots)\,.$$

Let $n$ be a fixed positive integer. There exists an $f$ in $A_n$ such that $f^*(x) \neq 1$. Let $g$ be the function defined by (4). For every $y$ in $P$ choose a compact neighborhood $F(y)$ of $y$ in $\beta(P)$ with $x$ non $\in F$. There exists a $h_y \in A_n$ such that $h_y(y) = 1$, $h_y[P - F(y)] = (0)$. Put

$$r_y = \max(0, g - gh_y)\,.$$

Clearly $r_y^*(x) = 1$, $r_y(y) = 0$ and $r_y \in A_n$. Since $P$ is a Lindelöf space, there exists, for every $\varepsilon > 0$, a countable set $Y(\varepsilon) \subset P$ such that for any $y \in P$ there is a point $z \in Y(\varepsilon)$ with $r_z(y) < \varepsilon$. Let every $Y(1/j)$, $j = 1, 2, \ldots$, be arranged in a sequence $\{z_i^j\}_{i=1}^{\infty}$; for $z = z_i^j$, denote $r_z$ by $r_i^j$, and put

$$f_k^n = \min_{i,p \leqq K} r_i^j \quad (k = 1, 2, \ldots)\,.$$

Clearly $f_k^n \in A_n$, $\{f_k^n\}_{k=1}^{\infty} \downarrow 0$.

We have proved that for every $n = 1, 2, \ldots$ there exists a sequence $\{f_k^n\}_{k=1}^{\infty}$ in $A_n$ with $\{f_k^n\}_{k=1}^{\infty} \downarrow 0$. Now put

$$f_n = \min_{i,j \leqq n} f_i^j \quad (n = 1, 2, \ldots)\,.$$

According to (7), $f_n \in A_n$, and by construction $\{f_n\} \downarrow 0$ and $\|f_n\| = 1$, which contradicts (6). Thus (3) holds and $P$ is topologically complete.

## Резюме

# ХАРАКТЕРИЗАЦИЯ ТОПОЛОГИЧЕСКИ ПОЛНЫХ ПРОСТРАНСТВ ПРИ ПОМОЩИ СХОДИМОСТИ ФУНКЦИЙ

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Если $\mathfrak{F}$ — множество непрерывных вещественных функций на пространстве $P$, то символ $\mathfrak{F} \downarrow 0$ обозначает, что

(1) Если $f_1, f_2 \in \mathfrak{F}$, то существует $f \in \mathfrak{F}$ так, что

$$f \leqq \min (f_1, f_2) .$$

(2) Для всякой точки $x \in P$

$$\inf \{f(x);\ f \in \mathfrak{F}\} = 0 .$$

Через $B(P)$ обозначается множество всех ограниченных непрерывных вещественных функций на $P$; $\alpha(P)$ обозначает множество всех подколец $A$ кольца $B(P)$, имеющих следующие два свойства:

(a) $f \in A \Rightarrow |f| \in A$;

(б) Для всякой окрестности $U$ всякой точки $x \in P$ существует $f \in A$ так, что $0 \leqq f \leqq 1$, $f(x) = 1$, $f[P - U] = (0)$.

Определение. Семейство $\gamma < \alpha(P)$ имеет свойство (V), если выполняется следующее условие:

Если $\mathfrak{F} \subset B(P)$, $\mathfrak{F} \downarrow 0$ и также $(\mathfrak{F} \cap C) \downarrow 0$ для всякого $C \in \gamma$, то для всякого $\varepsilon > 0$ существует $f \in \mathfrak{F}$ так, что $\|f\| < \varepsilon$.

Доказываются следующие теоремы:

**Теорема 1.** *Вполне регулярное пространство $P$ является пересечением $m$ открытых множеств в чеховском компактном расширении тогда, и только тогда, если существует семейство $\gamma \subset \alpha(P)$ со свойством* (V), *имеющее мощность $\leqq m$.*

**Теорема 2.** *Линделефовское пространство $P$ является топологически полным в смысле Э. Чеха тогда, и только тогда, если существует невозрастающая последовательность $\{A_n\}$ в $\alpha(P)$ так, что*

(1) $f_n \in A_n$, $\{f_n\} \downarrow 0 \Rightarrow \lim \|f_n\| = 0$,

(2) $f \in A_n$, $g \in A_{n+1}$, $f \geqq 0$, $g \geqq 0 \Rightarrow \min (f, g) \in A_{n+1}$.