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On bianalytic spaces


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ON BIanalytic SPACES

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Bianalytic spaces are introduced and studied. A metrizable space \(X\) is bianalytic if and only if \(X\) is a separable Borel subset of some complete metrizable space (and consequently, if \(X \subseteq Y\) and \(Y\) is metrizable, then \(X\) is a Borel subset of \(Y\)). An intrinsic characterization of Borel subsets of complete metrizable separable spaces is given.

1. NOTATION AND TERMINOLOGY

1.1. A centered family of sets is a family with the finite intersection property.

1.2. If \(\mathcal{M}\) is a family of sets, then \(\mathcal{M}_a\) and \(\mathcal{M}_b\) will be used to denote the family consisting of all countable unions and countable intersections of sets from \(\mathcal{M}\). The meaning of \(\mathcal{M}_{ab}\) is clear.

1.3. Let \(\mathcal{M}\) be a family of sets. The symbol \(\mathcal{B}(\mathcal{M})\) will be used to denote the smallest family of sets containing \(\mathcal{M}\) and closed under countable unions and countable intersections. Let \(\mathcal{M}\) be the union of \(\mathcal{M}\). The complemented part of \(\mathcal{M}\), denoted by compl. p. \(\mathcal{M}\), is the family \(\{P; P \in \mathcal{M}, (M - P) \in \mathcal{M}\}\).

Finally, the symbol \(\mathcal{B}^*(\mathcal{M})\) will be used to denote the smallest family \(\mathcal{N}\) of sets containing \(\mathcal{M}\), closed under countable unions and intersections and such that \(P \in \mathcal{N}\) implies \((M - P) \in \mathcal{N}\). Clearly

\[
\mathcal{M} \subseteq \text{compl. p. } \mathcal{B}(\mathcal{M}) \subseteq \mathcal{B}(\mathcal{M}) \subseteq \mathcal{B}^*(\mathcal{M})
\]

and

\[
\text{compl. p. } \mathcal{B}^*(\mathcal{M}) = \mathcal{B}^*(\mathcal{M})
\]

1.4. The letter \(N\) always denotes the discrete space of all positive integers. The letter \(S\) always denotes the set of all finite sequences of positive integers. The set of all \(s \in S\) of length \(n\) will be denoted by \(S_n\). The topological product \(N^N\) will be denoted by \(\Sigma\).

If \(\sigma = \{\sigma_1, \sigma_2, \ldots\} \in \Sigma\) and \(s = \{s_1, \ldots, s_n\} \in S_n\) then \(\sigma \supset s\) means that \(s\) is a section of \(\sigma\), i.e. that \(s_i = \sigma_i\) for \(i \leq n\).

1.5. A determining system in a family of sets \(\mathcal{M}\) is a mapping \(M = \{M(s)\}\) of \(S\)
into $\mathcal{M}$. A determining system is regular if always $M(s_1, \ldots, s_{n+1}) \subseteq M(s_1, \ldots, s_n)$. The nucleus of a determining system $M$ is the set
\[ A(M) = \bigcup_{s \in \Sigma} \bigcap_{s \in \sigma} M(s). \]
$A(\mathcal{M})$ denotes the family of $A(M)$, where $M$ varies over all determining systems in $\mathcal{M}$. The sets from $A(\mathcal{M})$ are called $\mathcal{M}$-Souslin, or Souslin with respect to $\mathcal{M}$.

1.6. All topological spaces under consideration are supposed to be completely regular. If $X$ is a space and $\mathcal{M}$ is a family of subsets of $X$, then $\mathcal{M}^X$ or merely $\mathcal{M}$ denotes the family consisting of closures of all sets from $\mathcal{M}$.

1.7. If $X$ is a space, then

1.7.1. $F(X)$ and $G(X)$ denote the family of all closed (all open, respectively) subsets of $X$.

1.7.2. $Z(X)$ denotes the family of all zero-sets of $X$, i.e. the family of all $f^{-1}[0]$, where $f$ varies over all continuous functions on $X$.

1.7.3. $K(X)$ denotes the family of all compact subspaces of $X$.

1.8. A mapping of a space $X$ onto a space $Y$ will be called perfect if $f$ is both continuous and closed and if the inverse images of points are compact.

1.9. $\beta(X)$ will always denote the Čech-Stone compactification of $X$.

1.10. A class $D$ of spaces will be called $A$-closed if $D$ is closed under continuous mappings. $D$ is $A^{-1}$-closed if inverse images under continuous mappings of spaces from $D$ belong to $D$.

A class $C$ of spaces is an $A$-base of $D$ if each space from $D$ is a continuous image of a space from $C$. Using perfect mappings instead of continuous we obtain the definitions of a $P$-closed class, a $P^{-1}$-closed class, and a $P^{-1}$-base, respectively.

2. PRELIMINARIES

If $X$ is a metrizable space then
\[ \mathcal{B}(F(X)) = \mathcal{B}^*(F(X)) = \mathcal{B}(G(X)) \]
because every open set is an $F_\sigma$. In this case the elements of (1) are called Borel sets of $X$. The theory of Borel sets was developed in the case of complete metrizable separable spaces. In this case $M \subset X$ is a Borel set in $X$ if and only if both $M$ and $X - M$ are analytic in the classical sense (that means, both $M$ and $X - M$ are continuous images of the space $\Sigma$ of all irrational numbers of the unit interval $\langle 0, 1 \rangle$ of real numbers). The proofs of the majority of deeper results concerning Borel sets essentially depend on the theory of analytic spaces.

Each of the following families could be considered as a generalization of Borel subsets of metrizable spaces:

- $\mathcal{B}(F(X))$,
- $\mathcal{B}(G(X))$,
- $\mathcal{B}^*(F(X)) = \mathcal{B}^*(G(X))$,
- $\mathcal{B}(Z(X)) = \mathcal{B}^*(Z(X))$,
compl. p. $\mathcal{B}(F(X))$. All these families are identical if $X$ is metrizable. In general all these families are different. The study of each of the above listed families is of certain importance. With the exception of the compl. p. $\mathcal{B}(F(X))$, all of these families has been studied by several authors, usually in connection with measure theory in topological spaces. V. Šneidebor introduced the family $\mathcal{B}(K(X))$ as a generalization of Borel subsets of complete metrizable separable spaces. Continuous images of spaces belonging to $\mathcal{B}(K(X))$ for some $X$, the so-called analytic spaces, were studied by G. Choquet, M. Sion and the author.

In the present note we shall study the above listed families for bianalytic $X$. A space will be called bianalytic if both $X$ and $K-X$ are analytic for some compact space $K$ containing $X$, or equivalently, if $X$ is a Baire set$^1$ of some compact spaces.

In section 3 an interval definition of analytic spaces is given and some older results of G. Choquet, M. Sion and the author are reproved. Moreover certain new theorems are proved.

Section 4 is devoted to a generalization of the first Luzin separation theorem. It is proved that if $\{X_n\}$ is a disjoint sequence of analytic subspaces of a space $Y$, then there exists a disjoint sequence $\{B_n\}$ of Baire sets of $Y$ (i.e. $B_n \in \mathcal{B}(Z(Y))$ with $B_n \supset X_n$.

From this fact two theorems concerning the equality of $\mathcal{B}(F(X))$, $\mathcal{B}(Z(X))$ and the complemented part of $\mathcal{B}(F(X))$ are deduced.

In section 5 bianalytic spaces are introduced and studied.

3. ANALYTIC SPACES

By definition, a space $X$ is an E-space if $X$ is an $F_{\sigma\delta}$ in the Čech-Stone compactification $\beta(X)$ of $X$. If $X$ is a $K_{\sigma\delta}(Y)$ for some $Y \supset X$, then $X$ is an E-space. The continuous images of E-spaces are said to be analytic. By $[4]$, a space $X$ is analytic if and only if there exists an analytic structure in $X$. For convenience, let us recall that an analytic structure in a space $X$ is a complete regular determining system $M$ in $X$ such that $\mathcal{A}(M) = X$, and a complete determining system in a space $X$ is a determining system $M$, where $M(s) \subset X$, such that the following condition is fulfilled: If $\mathcal{M}$ is a centered family of subsets of $X$ and if there exists a $\sigma \in \Sigma$ with

$$s \in S, \quad s < \sigma \Rightarrow M(s) \supset L(s) \in \mathcal{M} \quad 2)$$

then the intersection of $\mathcal{M}$ is non-void.

**Proposition 1.** Let $M$ be a regular determining system in a space $X$ and put

$$M(\sigma) = \bigcap_{s \sim \sigma} M(s).$$

If $M$ is complete, then all $M(\sigma)$ are compact and the following condition is satisfied:

---

$^1$ A Baire set of $X$ is an element of $\mathcal{B}(Z(X))$.

$^2$ Such a family $\mathcal{M}$ will be called an $M$-Cauchy family.
(*) If \( U \) is an open set containing an \( M(\sigma) \), then there exists a neighborhood \( V \) of \( \sigma \) in \( \Sigma \), such that
\[
\tau \in V \Rightarrow M(\tau) \subset U.
\]

Conversely, if \( M \) is a mapping of \( \Sigma \) to \( \mathcal{K}(X) \) such that the condition (*) is fulfilled and
\[
\bigcup_{\sigma \in \Sigma} M(\sigma) = X,
\]
then \( M = \{M(s)\} \), where
\[
M(s) = \bigcup_{\sigma \supset s} M(\sigma),
\]
is a complete determining system in \( X \) with \( \mathcal{A}(M) = X \).

Proof. The first part of the proposition was proved in [5]. Let \( M = \{M(s)\} \) satisfy the condition of the second part of Proposition 1. Let \( \mathcal{M} \) be a maximal \( M \)-Cauchy family. There exists a \( \sigma \in \Sigma \) such that
\[
s \in \sigma \Rightarrow M(s) \in \mathcal{M}.
\]

To prove \( \bigcap \mathcal{M} \neq \emptyset \), it is sufficient to show that \( \mathcal{M} \cap M(\sigma) \) is a centered family of sets. From condition (*) it follows immediately that if a closed set \( F \) meets each \( M(s) \), then \( F \) meets \( M(\sigma) \). Thus \( \mathcal{M} \cap M(\sigma) \) is centered and the proof is complete.

As a corollary of the preceding Proposition 1 we have:

**Theorem 1.** A space \( X \) is analytic if and only if there exists a mapping \( M \) of \( \Sigma \) to \( \mathcal{K}(X) \) such that the union of all \( M(\sigma), \sigma \in \Sigma, \) is \( X \), and the condition (*) is fulfilled.

Let us recall (for proofs see [5]), that the class of all analytic spaces is \( A \)-closed, \( P^{-1} \)-closed, countably productive\(^3\) and \( F \)-hereditary. Every analytic space is a Lindelöf space, and consequently, a normal space. A metrizable space \( X \) is analytic if and only if \( X \) is analytic in the classical sense, which means that \( X \) is the image under a continuous mapping of the space \( \Sigma \) of irrational numbers of the unit interval of real numbers. Finally, the family of all analytical subspaces of a given space is closed under the operation (\( \mathcal{A} \)), and if \( X \) is an analytic subspace of a space \( Y \), then \( X \in \mathcal{A}(\mathcal{F}(Y)) \).

The following result will not be used in the sequel:

**Theorem 2.** A space \( X \) is the inverse image under a perfect mapping of \( \Sigma \) (i.e. \( X \in P^{-1}(\Sigma) \)) if and only if there exists an analytical structure \( U \) in \( X \) such that the following two conditions are fulfilled:

(a) \( \{U(s); s \in S_n\} \) is disjoint for every \( n \).

(b) all \( U(s) \) are open (and hence closed) and non-void.

Proof. For every \( s \in S \) put
\[
\Sigma(s) = \{\sigma; \sigma \in \Sigma, \sigma \supset s\}.
\]
Clearly $\Sigma = \{\Sigma(s)\}$ is an analytical structure in $\Sigma$ satisfying (a) and (b) reading $\Sigma$ instead of $U$). Let $f$ be a perfect mapping of a space $X$ onto the space $\Sigma$. For every $s \in S$ put
\[ U(s) = f^{-1}[\Sigma(s)]. \]
Clearly the conditions (a) and (b) are fulfilled. By proposition 1, $U$ is an analytic structure in $X$.

Conversely, let $U$ be an analytic structure in $X$ satisfying (a) and (b). Put
\[ U(\sigma) = \bigcap_{s \succ \sigma} U(s). \]
By our assumptions the sets $U(\sigma)$ are compact non-void and disjoint. For $x \in U(\sigma)$ put $f(x) = \sigma$. It is easy to see that $f$ is a perfect mapping of $X$ onto $\Sigma$. The continuity is clear from the facts that the family of all sets $\Sigma(s), s \in S$ is an open base of $\Sigma$ and the sets $U(s) = f^{-1}[\Sigma(s)]$ are open. The sets $f^{-1}[\sigma] = U(\sigma)$ are compact because $U$ is an analytic structure. It remains to prove $f$ is a closed mapping. Let $F$ be closed in $X$ and let $\sigma$ be a point of $\Sigma - f[F]$. Since $F \cap U(\sigma) = \emptyset$, we have by Proposition 1 that there exists a $s \prec \sigma$ with $U(s) \cap F = \emptyset$. It follows that $\Sigma(s) \cap f[F] = \emptyset$ which shows that $\sigma$ is not in the closure of $f[F]$. Thus $f[F]$ is closed. This completes the proof.

4. SEPARATION OF ANALYTIC SPACES

By a classical theorem of Luzin (cf. [7], 393), if $X$ and $Y$ are disjoint analytic subsets of a complete metrizable space $T$, then there exists a Borel set $B$ of $T$ such that $X \subset B \subset T - Y$. This result has the following generalization.

**Theorem 3.** Let $X_1$ and $X_2$ be two disjoint analytic subspaces of a space $X$. There exists a set $B \in \mathcal{B}(Z(X))$ such that
\[ X_1 \subset B \subset X - X_2. \]

**Proof.** For convenience, two subsets $X_1$ and $X_2$ of $X$ will be called $B$-separated if there exists a set $B \in \mathcal{B}(Z(X))$ such that (9) holds. First we shall prove the following simple result:

**Lemma.** If $P = \bigcup_{n=1}^{\infty} P_n$ and $Q = \bigcup_{n=1}^{\infty} Q_n$ are subsets of $X$, and every $P_n$ and $Q_n$ are $B$-separated, then $P$ and $Q$ are separated.

Indeed, if $B_{nm}$ $B$-separates $P_n$ and $Q_n$, then the set
\[ B = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} B_{nm} \]
separates $P$ and $Q$.

3) Countable products of analytic spaces are analytic.
Now let $X_1$ and $X_2$ be two disjoint analytic subspaces of $X$. Let $P = \{P(s)\}$ and $Q = \{Q(s)\}$ be analytic structures in $X_1$ and $X_2$, respectively. For every $t \in S$ put

$$P_1(t) = \bigcup_{\sigma \in \Sigma(t)} \bigcap_{r < \sigma} P(r),$$

$$Q_1(t) = \bigcup_{\sigma \in \Sigma(t)} \bigcap_{r < \sigma} Q(r).$$

Clearly

(11) \quad P_1(s) \subseteq P(s), \quad Q_1(s) \subseteq Q(s)

and

$$P_1(\{s_1, \ldots, s_n\}) = \bigcup_{k=1}^{\infty} P_1(\{s_1, \ldots, s_n, k\}),$$

$$Q_1(\{s_1, \ldots, s_n\}) = \bigcup_{k=1}^{\infty} Q_1(\{s_1, \ldots, s_n, k\}).$$

Now suppose that $X_1$ and $X_2$ are not $B$-separated. Using the above Lemma one can construct by induction $\sigma, \tau \in \Sigma$ such that for every $n = 1, 2, \ldots$ the sets $P_1(\{\sigma_1, \ldots, \sigma_n\})$ and $Q_1(\{\tau_1, \ldots, \tau_n\})$ are not separated. Put

$$P(\sigma) = \bigcap_{s < \sigma} P(s), \quad Q(\tau) = \bigcap_{r < \tau} Q(s).$$

The sets $P(\sigma)$ and $Q(\tau)$ are disjoint (because $X_1$ and $X_2$ are disjoint) and compact because $P$ and $Q$ are analytical structures. It follows that there exists a zero-set $Z$ such that

$$P(\sigma) \subseteq \text{int} \ Z \subset Z \subset X_2 - Q(\tau).$$

By Proposition 1 there exists an $n$ such that

$$P(\{\sigma_1, \ldots, \sigma_n\}) \subset Z,$$

$$Q(\{\tau_1, \ldots, \tau_n\}) \subset X - Z.$$

By (11) $P_1(\{\sigma_1, \ldots, \sigma_n\})$ and $Q_1(\{\tau_1, \ldots, \tau_n\})$ are also $B$-separated which contradicts our construction of $\sigma$ and $\tau$ and completes the proof.

**Note 1.** In [9] the following result is proved: If $X_1$ and $X_2$ are disjoint subsets of a space $X$ and if $X_1, X_2 \in \mathcal{A}(K(X))$, then there exists a $B \in \mathcal{B}(K(X))$ such that $X_1 \subset B \subset X - X_2$. The proof of this theorem is similar to that of Theorem 3.

**Note 2.** The proof of Theorem 3 yields the following result (in particular, the result from Note 1): Let $P$ and $Q$ be two determining systems in a space $X$ and let $\mathcal{M}$ be a family of subsets of $X$ which is closed under countable unions and intersections. If for every $\sigma \in \Sigma$ and $\tau \in \Sigma$, there exists a positive integer $n$ and an $M \in \mathcal{M}$ such that

$$P(\{\sigma_1, \ldots, \sigma_n\}) \subset M \subset X - Q(\{\tau_1, \ldots, \tau_n\})$$

then there exists an $M$ in $\mathcal{M}$ with

$$\mathcal{A}(P) \subset M \subset X - \mathcal{A}(Q).$$

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Theorem 4. If $\{X_n\}$ is a disjoint sequence of analytic subspaces of a space $X$ then there exists a disjoint sequence $\{B_n\}$ of sets from $\mathcal{B}(Z(X))$ such that $Z_n \subset B_n$.

Proof. By Theorem 3 for every $(n, m)$, $n \neq m$ there exists a $B(n, m)$ such that

$$X_n \subset B(n, m) \subset X - X_m.$$ 

Put

$$B_n = \bigcap_{m=1}^{\infty} B(n, m).$$

Clearly $\{B_n\}$ has the required properties.

Note 3. The preceding theorem will be used essentially in section 5.

Theorem 5. If $X$ is an analytic space then

$$\mathcal{B}(Z(X)) = \text{compl. p. } \mathcal{B}(F(X)).$$

Proof. Since $Z(X) \subset F(X)$ and $\mathcal{B}(Z(X)) = \mathcal{B}^*(Z(X))$ for every space $X$, we have the inclusion $\subset$. If $X$ is an analytic space and both $M \subset X$ and $X - M$ belong to $\mathcal{B}(F(X))$, then both $M$ and $X - M$ are analytic, because closed subspaces of analytic spaces are analytic and the family of all analytic subspaces is closed under the operation $\mathcal{A}$ and clearly $\mathcal{A}(F(X)) \supset \mathcal{B}(F(X))$. By Theorem 3 there exists a $Z \in \mathcal{B}(Z(X))$ with $M \subset Z \subset X - M$. It follows that $Z = M$.

Note 4. I do not know of any reasonable necessary and sufficient condition for (12) to hold.

Theorem 6. If $X$ is an analytic space and

$$\mathcal{B}(F(X)) = \mathcal{B}(Z(X))$$

then $X$ is a perfectly normal space.

Proof. If (13) holds, then every open set is an analytic space, and hence, a Lindelöf space. Thus every open set is an $F_{\sigma}$. Since $X$ is analytic, $X$ is normal. Thus $X$ is perfectly normal.

Note 5. Obviously, if $X$ is a perfectly normal space, then $F(X) = Z(X)$, and consequently (13) holds. I do not know whether (13), implies that $X$ is perfectly normal. (This is an old problem of M. Katětov [6].)

5. BIANTALYTIC SPACES

Definition. A space $X$ will be called bianalytic if both $X$ and $\beta(X) - X$ are analytic.

Theorem 7. If $f$ is a perfect mapping of $X$ onto $Y$, then $X$ is a bianalytic space if and only if $Y$ is such.
Proof. Let $g$ be the Čech-Stone mapping of $\beta(X)$ onto $\beta(Y)$. Since $f$ is perfect, by well known result we have

\[(14) \quad g[\beta(X) - X] = \beta(Y) - Y.\]

Thus if both $X$ and $\beta(X) - X$ are analytic, then also both $Y$ and $\beta(Y) - Y$ are analytic. From (14) it follows at once that the restriction of $g$ to $\beta(X) - X$ is a perfect mapping onto $\beta(Y) - Y$. Since the inverse image under a perfect mapping of an analytic space is analytic if $Y$ is bianalytic then $X$ is analytic.

**Theorem 8.** The following conditions on a space $X$ are equivalent:

1. $X$ is a bianalytic space.
2. There exists a compactification $K$ of $X$ such that both $X$ and $K - X$ are analytic spaces.
3. $X$ is analytic and for every compactification $K$ of $X$ the space $K - X$ is analytic.
4. $X \in \mathcal{B}(Z(\beta(X)))$.
5. For some compactification $K$ of $X$ we have $X \in \mathcal{B}(Z(K))$.
6. For every space $Y \supset X$, $\bar{X} = Y$, we have $X \in \mathcal{B}(Z(Y))$.

Proof. The equivalence of conditions (1)–(3) follows from Theorem 7. From Theorem 6 it follows at once that (1) implies (4). Clearly (4) implies (5). If $K$ is an analytic space, then every set from $\mathcal{A}(F(K))$ is an analytic space. Since $\mathcal{A}(F(K))$ contains $\mathcal{B}(Z(K))$, we have that (5) implies (2). It remains to prove that (6) is equivalent with (1)–(5). Obviously (6) implies (4). Finally, suppose (3). Let $Y \supset X$, $\bar{X} = Y$. Let $K$ be a compactification of $Y$. By (2), $X \in \mathcal{B}(Z(K))$. Obviously $X \in \mathcal{B}(Z(Y))$. This completes the proof.

**Theorem 9.** Closed subspaces of bianalytic spaces are bianalytic.

Proof. Let $X$ be closed in a bianalytic space $Y$. Then the space

$$Z = \bar{X}^{\beta(Y)}X$$

is a closed subspace of the analytic space $\beta(Y) - Y$ and consequently $Z$ is an analytic space. By Theorem 8, $X$ is a bianalytic space.

**Theorem 10.** The topological product of a countable number of bianalytic spaces is a bianalytic space.

Proof. Let $X_n, n \in \mathbb{N}$, be analytic and let $X$ be the topological product of all $X_n$. Let $K$ be the topological product of all $\beta(X_n)$. Since the topological product of analytic spaces is analytic, it is easy to see that $K - X$ is the union of a countable number of analytic spaces. Since $\mathcal{A}(K)$ is closed under Souslin’s operation $\mathcal{A}$, in particular under countable unions, $K - X$ is an analytic space. By theorem 8 the space $X$ is bianalytic.
Proposition 2. If $Y$ is a bianalytic space and both $X \subseteq Y$ and $Y - X$ are analytic, then $X$ is a bianalytic space.

Proof. Consider the space

$$Z = \overline{X^c(Y)} - X.$$  

We have

$$Z = (Z, Y) \cup (Z \cap (\beta(Y) - Y)).$$

The first term of the right side of the above equality is closed in the analytic space $Y - X$ and hence it is analytic. The second term is closed in the analytic space $\beta(Y) - Y$ and hence it is also analytic. Thus $Z$ is analytic, and finally by Theorem 6, the space $X$ is bianalytic.

If $X$ is a bianalytic subspace of a space $Y$, then $Y - X$ may fail to be an analytic space. Moreover, open subspaces of compact spaces, in general, are not analytic. For example, if $M$ is an uncountable discrete space and $K$ is a compactification of $M$, then $M$ is open in $K$, but $M$ is not an analytic space because $M$ is not a Lindelöf space. On the other hand we shall prove the following result.

Proposition 3. If $X \subseteq Y$, $Y - X$ is dense in $Y$ and both $Y$ and $Y - X$ are bianalytic, then $X$ is analytic (and by Proposition 2 bianalytic.) In particular, if $Y$ is a bianalytic space and both $X \subseteq Y$ and $Y - X$ are dense in $Y$, then $X$ is a bianalytic space if and only if $Y$ is such.

Proof. Let $K$ be a compactification of $Y$. Obviously

$$X = [K - (Y - X)] \cap Y.$$  

Since $Y$ is bianalytic and $Y - X$ is dense in $Y$ and hence in $K$, the first member of the right side is an analytic space. Since $Y$ is (by our assumption) analytic, the space $X$ is also analytic.

Theorem 11. A subspace $X$ of a bianalytic space $Y$ is bianalytic if and only if

$$X \in \text{compl. p. } \mathcal{B}(F(Y)) = \mathcal{B}(Z(X^c)).$$

The proof follows at once from propositions 2 and 3. As an immediate consequence of the preceding result we have the following assertion.

Theorem 12. A metrizable space $X$ is bianalytic if and only if $X$ is separable and an absolute Borel set, i.e., if $Y$ is a separable metrizable space and $X \subseteq Y$, then $X \in \mathcal{B}(F(Y))$.

Note 6. The union of two bianalytic subspaces of a given space may fail to be bianalytic. Indeed, $N$ is a bianalytic space and every one-point set is bianalytic. However, $X = N \cup \{x\} \subseteq \beta(N)$, where $x \in \beta(N) - N$ is not bianalytic, because $\beta(N) = \beta(X)$ and $\beta(X) - X$ is not a Lindelöf space.
Note 7. One-to-one continuous images of a bianalytic space may fail to be bianalytic. Indeed, the space $X$ from Note 6 is a one-to-one continuous image of $\mathbb{N}$.

**Theorem 13.** The intersection of a countable number of bianalytic subspaces of a given space is a bianalytic space.

**Proof.** Let $X_n$, $n \in \mathbb{N}$ be bianalytic subspaces of $Y$ and let $X$ be the intersection of all $X_n$. Without a loss of generality we may assume that $Y$ is compact. Let $K$ be the closure of $X$ in $Y$. Clearly $Y_n = K \cap X_n$ are also bianalytic. Since $X$ is dense in $K$ and $X \subseteq Y_n \subseteq K$, the space $K$ is a compactification of each $Y_n$. Thus $K - Y_n$ are analytic, and consequently, the set

$$K - X = \bigcup_{n=1}^{\infty} (K - Y_n)$$

is analytic.

6. INTERNAL CHARACTERIZATION OF METRIZABLE BIANALYTIC SPACES

By a well-known classical theorem the image under a one-to-one continuous mapping of an absolute Borel set is an absolute Borel set.

By Note 6 of Section 5 the image under a one-to-one continuous mapping of a bianalytic space may fail to be a bianalytic space. In this section a class of spaces invariant under one-to-one continuous mappings is defined, such that the metrizable spaces from this class are precisely the absolute Borel separable sets.

**Proposition 4.** Let $X$ be a subspace of a space $Y$. Let there exists an analytic structure $M$ in $X$ such that

(a) $\{M(s); s \in S_n\}$ are disjoint,

(b) every $M(s)$ is an analytic space.

Then $X \in \mathcal{B}(F(X))$. If, in addition, the closures of $M(s)$ in $Y$ are zero-sets, then $X \in \mathcal{B}(Z(X))$.

**Proof.** By Theorem 4 there exist sets $Z(s) \in \mathcal{B}(Z(X))$ such that $Z(s) \supseteq M(s)$ and that the families

(18)

$$\{Z(s); s \in S_n\}$$

are disjoint. We may assume $Z(i_1, \ldots, i_{n+1}) \subseteq Z(i_1, \ldots, i_n)$. Put

$$F(s) = Z(s) \cap M(s)^Y.$$

Since the families (18) and hence also the families $\{F(s), s \in S_n\}$ are disjoint, we have

(19)

$$\mathcal{A}(F) = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} F(s).$$

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But
\[ X = \mathcal{A}(M) \supset \mathcal{A}(F) \supset X. \]

Thus
\[ X = \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} F(s). \]

Clearly \( M(s) \in \mathcal{B}(F(Y)) \) and hence \( X \in \mathcal{B}(F(Y)). \) If in addition the sets \( \overline{M(s)^Y} \) are zero-sets in \( Y, \) or more generally, if \( \overline{M(s)^Y} \in \mathcal{B}(Z(Y)) \), then \( F(s) \in \mathcal{B}(Z(Y)) \). This completes the proof.

As an immediate consequence of Proposition 4 we have the following results:

**Theorem 14.** If there exists an analytic structure \( M \) in a space \( X \) such that the conditions (a) and (b) from Proposition 4 are fulfilled, and if exists a perfectly normal compactification of \( X \) (in particular, if \( X \) is metrizable), then \( X \) is a bianaalytic space.

Note 8. If \( M = \{ M(s) \} \) is an analytic structure in \( X \) such that the families \( \mathcal{M}_n = \{ M(s); s \in S_n \} \) are disjoint, then \( \{ \mathcal{M}_n \} \) is a complete sequence\(^4\) of countable disjoint coverings of \( X \) such that \( \mathcal{M}_{n+1} \) refines \( \mathcal{M}_n. \) Conversely, if \( \{ \mathcal{M}_n \} \) is a complete sequence of countable disjoint coverings of \( X \) such that \( \mathcal{M}_{n+1} \) refines \( \mathcal{M}_n, \) then there exists an analytical structure \( M \) in \( X \) such that
\[ \mathcal{M}_n = \{ M(s); s \in S_n \}. \]

**Theorem 15.** A metrizable space \( X \) is bianaalytic (absolutely Borel separable space) if and only if there exists a complete sequence \( \{ \mathcal{M}_n \} \) of countable disjoint coverings of \( X \) such that all sets from \( \bigcup_{n=1}^{\infty} \mathcal{M}_n \) are analytic.

Proof. By Theorem 14 and the preceding Note 8, the condition is sufficient. Conversely, let \( X \) be bianaalytic. By a well-known classical theorem \( X \) is a disjoint union of a countable set \( X_1 \) and a set \( X_2 \) which is a one-to-one continuous image of \( \Sigma. \) Denoting this mapping by \( f, \) let \( \mathcal{M}_s \) be the covering of \( X \) consisting of all one-point sets \( (x), \) \( x \in X_1 \) and all \( f[\Sigma(s)]; s \in S_n. \) Clearly \( \{ \mathcal{M}_n \} \) is a complete sequence of countable disjoint coverings of \( X, \) the sets from \( \bigcup_{n=1}^{\infty} \mathcal{M}_n \) are analytic and \( \mathcal{M}_{n+1} \) refines \( \mathcal{M}_n. \) This completes the proof.

We have proved that any bianaalytic metrizable space has a complete sequence \( \{ \mathcal{M}_n \} \) of countable disjoint coverings such that all \( M \in \bigcup_{n=1}^{\infty} \mathcal{M}_n \) are analytic. All one-to-one images of inverse images under perfect mappings (see Theorem 2) also have such complete sequences. For the sake of completeness we shall prove the following result:

\(^4\) For definition see [5], [4] or [3].
Theorem 16. A space $X$ is the one-to-one continuous image of the inverse image under a perfect mapping of the space $\Sigma$ (of all irrational numbers) if and only if there exists an analytical structure $M$ in $X$ such that

(a) the coverings $\{M(s) ; s \in S_{\sigma}\}$ are disjoint, and

(b) the sets $M(\sigma) = \bigcap_{s \in \sigma} M(s)$ are non-void and disjoint.

Proof. By Theorem 2 the condition is necessary. Conversely, let $M$ be an analytic structure in $X$ such that the conditions (a) and (b) are fulfilled. Let us define a new topology in $X$ such that $M(\sigma)$ are subspaces and the sets $M(s)$ are open. Denote this space by $Y$. It is easy to see that $M$ is an analytic structure in $Y$ satisfying the conditions (a) and (b) from Theorem 2. Thus $Y$ is the inverse image under a perfect mapping of $\Sigma$. This completes the proof.

References


Резюме

О БИАНАЛИТИЧЕСКИХ ПРОСТРАНСТВАХ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolik), Прага

Если $X$ — пространство, то $Z(X)$ обозначает совокупность всех множеств вида $f^{-1}(0)$, где $f$ — вещественная непрерывная функция на $X$. Наименьшая система множеств, содержащая данную систему $\mathcal{M}$ и замкнутая по отношению к счетным пересечениям и соединениям, обозначается через $\mathcal{B}(\mathcal{M})$. Следуя
М. Катетову, множества, принадлежащие системе $\mathcal{B}(\mathcal{Z}(X))$, называются множествами Бэра пространства $X$.

В статье рассматриваются пространства, так наз. бианалитические, которые являются множествами Бэра в некотором компактном пространстве. Оказывается, что вполне регулярное пространство $X$ является бианалитическим, если и только если для одного и, следовательно, для всякого компактного $K$, содержащего $X$ как плотное множество, пространства $X$ и $K - X$ являются аналитическими пространствами (в смысле Шюке). Доказательства основаны на обобщении первой теоремы Лузина об отделимости аналитических пространств.

В заключение дается внутренняя характеристика борелевских подмножеств полнот метризуемых сепарабельных пространств.