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SOME CLASSES OF COUNTABLY COMPACT SPACES

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The present paper investigates the relations between some classes of countably compact spaces introduced by Z. Frolík [1], [2].

The spaces considered here are always completely regular $T_1$-spaces; $\mathbb{N}$ denotes the set of positive integers. Recently, Z. Frolík introduced the classes $\mathcal{F}$, $\mathcal{F}_p$ and $\mathcal{C}$ which are characterized by the following: A space $X$ belongs to $\mathcal{F}$ (or $\mathcal{C}$) if and only if for every pseudocompact (or countably compact) space $Y$ the topological product $X \times Y$ is pseudocompact (or countably compact); a space $X$ belongs to $\mathcal{F}_p$ if and only if every closed subspace of $X$ belongs to $\mathcal{F}$. Moreover, he gave necessary and sufficient conditions for a space to belong to one of these classes (see § 1 below).

Let $\mathcal{F}_c$ be the subclass of $\mathcal{F}$ consisting of countably compact spaces. It is easy to see that $\mathcal{F} - \mathcal{F}_c$ is not empty: for instance, $X = [1, \omega] \times [1, \Omega] - \{(\omega, \Omega)\}$ belongs to $\mathcal{F} - \mathcal{F}_c$, where $\omega$ and $\Omega$ are the least ordinal numbers of the second and the third classes respectively.

In this paper, we shall give new characterizations of $\mathcal{F}_p$ in § 2, and consider, in § 3, the relations between the classes $\mathcal{F}_c$, $\mathcal{F}_p$ and $\mathcal{C}$, and show that, the three classes $\mathcal{C} - \mathcal{F}_c$, $\mathcal{F}_c - \mathcal{C}$ and $\mathcal{F}_c \cap (\mathcal{C} - \mathcal{F}_p)$ are not empty. Equivalently, i) there is a countably compact space $X$ such that $X \times Y$ is countably compact for every countably compact space $Y$, but $X \times Z$ is not pseudocompact for some pseudocompact space $Z$, ii) there is a countably compact space $X$ such that $X \times Y$ is pseudocompact for every pseudocompact space $Y$ but $X \times Z$ is not countably compact for some countably compact space $Z$, and iii) there is a countably compact space $X$ such that $X \times Y$ is countably compact (or pseudocompact) for every countably compact (or pseudocompact) space $Y$ but $X$ contains some closed subspace $A$ having the property that $A \times B$ is not pseudocompact for some pseudocompact space $B$.

1. Preliminary. In this section, for convenience, we shall state Frolík’s theorems, and transfer the form 1.3 to the forms 1.5 and 1.6.

1.1 [1,3.6]. $\mathcal{F} \ni X$ if and only if $X$ satisfies the following condition: If $\mathcal{H}$ is an infinite disjoint family of non-void open subsets of $X$, then there exists a disjoint sequence
\[ \{U_n\} \text{ in } \mathfrak{U} \text{ such that for every filter } \mathfrak{F} \text{ of infinite subsets of } N \text{ we have} \]
\[ \bigcap_{N \in \mathfrak{F}} \left( \bigcup_{n \in N} U_n \right) \neq \emptyset \quad (\emptyset \text{ denotes the empty set}). \]

1.2 [1,4.2]. \( \mathcal{F}_F \ni X \) if and only if every subset of \( X \) contains an infinite subset with a compact closure in \( X \).

1.3 [2,3.3]. \( \mathcal{G} \ni X \) if and only if \( X \) satisfies the following condition: there exists an infinite discrete subset \( N \) of \( X \) such that for every compactification \( K \) of \( X \) there exists a subset \( S \) of \( K - X \) such that the subspace \( N \cup S \) of \( K \) is countably compact.

Let \( K \) be any compactification of \( X \): Then there is a continuous mapping \( \varphi_K \) of \( \beta X \) onto \( K \) that leaves \( X \) pointwise fixed. We notice that \( \varphi_K(X) = X \) and \( \varphi_K(\beta X - X) = K - X \). Under this notation, we have

1.4 Let \( K \) and \( M \) be any compactifications of \( X \) and let \( N \) be a discrete subset of \( X \). If there is a subset \( S \) of \( K - X \) such that \( N \cup S \) is countably compact, then \( \varphi_K^{-1}(S) \) (or \( \varphi_M \varphi_K^{-1}(S) \)) is a subset of \( \beta X - X \) (of \( M - X \)) such that the subspace \( N \cup \varphi_K^{-1}(S) \) (or \( N \cup \varphi_M \varphi_K^{-1}(S) \)) of \( \beta X \) (of \( M \), respectively) is countably compact.

Proof. It is known that if \( f \) is a closed mapping from a space \( P \) to a countably compact space \( Q \), then the countable compactness of \( f^{-1}(y) \) for each point \( y \) in \( Q \) implies the countable compactness of \( P \) (c.f., e.g., [2,1.1]). Consider the two sets \( P = \varphi_K^{-1}(N \cap S) \) and \( Q = N \cup S \). Since \( f = \varphi_K \mid P \) is a closed compact mapping of \( P \) onto \( Q \), we have that \( P \) is countably compact and \( \varphi_K^{-1}(S) = P - N \) is a subset of \( \beta X - X \).

The other statement is obvious from the continuity of \( \varphi_M \).

From 1.3 and 1.4 we have

1.5. Theorem. \( \mathcal{G} \ni X \) if and only if \( X \) satisfies the following condition: there is an infinite discrete subset \( N \) of \( X \) such that the subspace \( N \cup S \) of some compactification \( K \) of \( X \) is countably compact for some subset \( S \) of \( K - X \).

From 1.3 and 1.5 we have

1.6. Theorem. The following conditions are equivalent:

i) \( \mathcal{G} \ni X \),

ii) for infinite discrete subset \( N \) of \( X \), there is a compactification \( K \) such that, for every subset \( S \) of \( K - X \), the subspace \( N \cup S \) of \( K \) is not countably compact,

iii) for every infinite discrete subset \( N \) of \( X \), the space \( N \cup S \) is not countably compact, where \( K \) is any compactification of \( X \) and \( S \) is any subset of \( K - X \).

\(^1\) In the following, the left term of this relation will be denoted by \( (\mathfrak{F}, N_1, U_n, X) \) and if \( U_n \) has a form \( \{a_n; n \in N_1\} \), then by \( (\mathfrak{F}, N_1, \{a_n\}, X) \). The symbol "X" denotes the space on which the closure operation is defined.
2. Characterizations of $\Psi_F$. We shall show that the class $\Psi_F$ is contained in $\Psi_c \cap \mathcal{E}$. If $X \in \Psi_F$, then every closed subspace $A$ of $X$ belongs to $\Psi_c$ and hence $A$ must be pseudocompact. Therefore $X$ is countably compact, that is, $\Psi_F \subseteq \Psi_c$. Let $N$ be any infinite discrete subset of $X$. By 1.2 there is a compact subset $F$ of $X$ such that $N \cap F$ is infinite. Then, for every subset $S$ of $\beta X - X$, the set $N \cap F$ has no accumulation points in $N \cup S$. Thus $N \cup S$ is not countably compact and hence, by 1.6 (ii), $X$ belongs to $\mathcal{E}$. Thus we have $\Psi_F \subseteq \Psi_c \cap \mathcal{E}$.

2.1. Theorem. The following conditions are equivalent for any space $X$:

1) $\Psi_F \ni X$,

2) for every infinite discrete subset $N$ of $X$ and for every subset $S$ of $K - X$ where $K$ is some compactification of $X$, the set $N \cup S$ is not pseudocompact,

3) for every infinite discrete sequence $\{a_n\}$ of $X$, there is a subsequence $\{a_{n_i}\}$ such that for every filter $\mathcal{F}$ of infinite subsets of $N$ we have $(\mathcal{F}, N_1, \{a_{n_i}\}, X) \neq 0$.

Proof. 1) $\Rightarrow$ 2). Suppose that $\Psi_F \ni X$, $N$ is any infinite discrete subset of $X$ and $S$ is any subset of $K - X$. By assumption, there is a compact subset $F$ of $X$ such that $F \cap N$ is infinite. Let $\{a_{n_i}\} \subseteq F \cap N$. Then $\{a_{n_i}\}$ is a family of open sets of the space $N \cup S$ and $\{a_{n_i}\}$ has no accumulation points in $N \cup S$. Therefore $\{a_{n_i}\}$ is locally finite in $N \cup S$, and hence $N \cup S$ is not pseudocompact.

2) $\Rightarrow$ 3). Let $N = \{a_n\}$ be any infinite discrete sequence of $X$. $Y = N \cup (\overline{N}(in K) - X)$ is not pseudocompact by assumption. Thus there exists a locally finite family of open sets $\{U_n\}$ of $Y$. Since every point $a_n$ is open in $Y$ and $N$ is dense in $Y$, each $U_n$ contains a point $a_{n_i}$ of $N$. Then for every filter $\mathcal{F}$ of infinite subsets of $N$, we have $B = (\mathcal{F}, N_1, \{a_{n_i}\}, X) \neq 0$. For, if $B = 0$, then we have $(\mathcal{F}, N_1, \{a_{n_i}\}, Y) = (\mathcal{F}, N_1, \{a_{n_i}\}, K) \neq 0$, and hence $\{U_n\}$ is not locally finite in $Y$. This is a contradiction.

3) $\Rightarrow$ 1). Let $N = \{a_n\}$ be an infinite discrete sequence of $X$. By assumption, there is a subsequence $N' = \{a_{n_i}\}$ satisfying the relation in (3). For any point $a$ in $\overline{N'}(in K)$, we take a base $\{U_a\}$ of neighborhoods (in $K$) of $a$ and put $N_a = \{a_{n_i}; a_{n_i} \in U_a \cap N'\}$. Then $\{N_a\}$ is a filter $\mathcal{F}$. Thus by assumption we have $(\mathcal{F}, N_1, \{a_{n_i}\}, X) = \{a\}$, that is, the closure (in $X$) of $N'$ is compact and hence $X$ belongs to $\Psi_F$.

From 2.1 we have

2.2. Theorem. $X$ belongs to $\mathcal{E} - \Psi_F$ if and only if for every infinite discrete subset $N$ of $X$, $N \cup S$ is not countably compact for every subset $S$ of $K - X$ but $T \cup N$ is pseudocompact for some subset $T$ of $K - X$ where $K$ is some compactification of $X$.

2.3. Corollary. If $\Psi \ni X$ and $Y$ is a dense subset of $X$ every point of which is isolated in $X$, then any infinite subset of $Y$ contains a subset with a compact closure in $X$. 

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2.4. Remark. In 2.1(2) and 2.2, we may replace the word "some" compactification $J^\circ$ of $X$ by the word "any".

3. Examples. In this section we assume the continuum hypothesis.

3.1. Example. Let $M$ be the set of all $P$-points of $\beta N - N$ and let $X = \beta N - M$ We shall prove that $X$ belongs to $\tilde{\mathcal{C}} - \tilde{\mathcal{H}}_c$.

1) $\beta X = \beta N(= K)$ 2) This is obvious.

2) $X$ does not belong to $\tilde{\mathcal{H}}_c$. Every subset $L$ of $N$ has no subset with compact closure in $X$ by [3;9M2], and hence, by 2.3, $\tilde{\mathcal{H}}_c \not\subseteq X$.

3) $X$ belongs to $\tilde{\mathcal{C}}$. Suppose that $\tilde{\mathcal{C}} \not\subseteq X$, that is, there is, by 1.4 and 1.5, an infinite discrete subset $N$ of $X$ such that the space $N \cup S$ is countably compact for some subset $S$ of $\beta X - X = M$. Since $N$ is discrete, either $N \cap N$ or $N \cap (X - N)$ contains copies $N_n$ of $N$ which are mutually disjoint $(n = 1, 2, \ldots)$. Thus $S$ contains an accumulation point $y_n$ of $N_n$ for every $n$. $M$ being a $P$-space and $S$ being a subset of $M$, $\{y_n\}$ has no accumulation points in $S$. On the other hand, $N \cup S$ is countably compact, and hence $\{y_n\}$ has an accumulation point in $N$. But this contradicts the fact that $N$ is discrete.

3.2. Example. Let $A$ be a copy of $\mathcal{N}$ contained in $\beta N - N$ such that every point of $A$ is a $P$-point of $\beta N - N$ and $\beta A \subset \beta N - N$. We shall prove that $X = \beta N - N - M$ belongs to $\tilde{\mathcal{C}} - (\tilde{\mathcal{H}}_c \cap (\mathcal{C} - \tilde{\mathcal{H}}_c))$ where $M$ is a set of all $P$-points of $\beta A - A$. We notice that $\beta N - N(= K)^2$ is a compactification of $X$ and no point of $M$ is a $P$-point of $\beta N - N$.

1) $X$ does not belong to $\tilde{\mathcal{H}}_c$. The copy $A$ of $\mathcal{N}$ has no subsets with compact closure in $X$. For let $B$ be any infinite subset of $A$, then by [3,9M2] and by the method of construction of $M$, $B$ does not have a compact closure in $X$.

2) $X$ belong to $\tilde{\mathcal{C}}$. Let $N$ be any infinite discrete subset of $X$ and $S$ any subset of $M$. To prove 3), it is sufficient, by 1.6 (ii), to show that $N \cup S$ is not countably compact. Since $N$ is a discrete subset of $\beta A - \beta A - A$ and all accumulation points of $N$ are contained in $\beta A$, $N$ has a copy of $\mathcal{N}$ by [3,9.10]. Thus, similarly to 3.1(3), we have $\tilde{\mathcal{C}} \not\subseteq X$.

3) $X$ belongs to $\tilde{\mathcal{H}}$. Let $\tilde{\mathcal{A}}$ be an infinite family of open sets of $X$ such that for every subfamily $\{U_n\}$ of $\tilde{\mathcal{A}}$ there is a filter $\tilde{\mathcal{F}}$ of infinite subsets of $\mathcal{N}$ such that $(\tilde{\mathcal{A}}, N_1, U_n, X) = \emptyset$. Since $M$ is itself a $P$-space, $M$ does not contain an infinite countably compact subset. Therefore, $M$, as a subspace of $\beta N - N$, has no inner points. Thus every set $U_n - A$ contains a $P$-point $x_n$ of $\beta N - N$ by [3,9M3]. Every point of $\beta A(\subset X \cup M)$ is contained in a closure (in $\beta A$) of a (countable) subset of $A$. Since $A$ is a copy of $\mathcal{N}$, we have $x_n \not\in \beta A$ for every $n$. Moreover we lose no generality by assuming that $\{x_n\}$ is a copy of $\mathcal{N}$ by [3,9.10]. Thus we have two sets $A(= \{a_n\})$ and $B = \{x_n\}$ of $P$-points of $\beta N - N$, and $A \cup B$ is a discrete subset of $\beta N$. Since $A \cup B$ is countable, there are open sets $V_n$ and $W_n$ of $\beta N$ such that $V_n \ni x_n$, $W_n \ni a_n$.

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2) In this section, if $K$ is a (fixed) compact space and $X$ is dense, then we use the phrase "a subset $P$ of $X$ is a copy of $\mathcal{N}$" if $P(in K) - P = \beta P - P = \beta N - N \subset K.$
$V_n \cap W_m = \emptyset$, $V_n \cap V_m = \emptyset$ and $W_n \cap W_m = \emptyset(n \neq m)$. Then $V(A) = \bigcup_{n} V_n$ and $V(B) = \bigcup_{n} W_n$ are disjoint open sets of $\beta N$. Put $N(A) = N \cap V(A)$ and $N(B) = N \cap W(B)$. Since $N$ is dense in $\beta N$, both $N(A)$ and $N(B)$ are infinite and we have that $\beta N \supset \beta N(A) \supset A$ and $\beta N \supset \beta N(B) \supset B$. On the other hand, $\beta N(A) \cap \beta N(B) = \emptyset$ because $N(A)$ and $N(B)$ are subspaces of $N$. Now suppose that $(\tilde{\mathcal{U}}, N_1, U_n, X) = (\tilde{\mathcal{V}}, N_1, V_n, \beta N - N) \neq \emptyset$. Since $\beta N - N$ is compact, we have $C = (\tilde{\mathcal{U}}, N_1, U_n, \beta N - N) \neq \emptyset$. Since $C$ is a compact subset of $M$ and $M$ is itself a $P$-space, we can assume that $C$ consists of only one point $a$. Thus we have $\{a\} = (\tilde{\mathcal{U}}, N_1, \{x_n\}, \beta N - N)$ and hence $\beta N(B) \ni a$. On the other hand we have that $\beta N(A) \ni a \supset M \ni a$. This is a contradiction.

3.3. Example. By [2,2.9] there is a countably compact subset $R$, containing $N$, of $\beta N$ whose cardinality is $\leq c$. Let $\{x_n\}$ be a discrete set in $R - N$. Then we shall show that $X = \beta N - (R - \{x_n\})$ belongs to $\beta \mathfrak{c} - \mathfrak{c}$. It is obvious that $\beta N - N(= K)$ is a compactification of $X$.

1) $X$ is countably compact. If $X$ is not countably compact, there is a countable discrete closed subset having a copy of $N$ as subset, and hence $M = R - \{x_n\}$ contains a compact subset with cardinality $2^c$. This is a contradiction.

2) $X$ does not belong to $\mathfrak{c}$. This follows from 1.5 and the countable compactness of $(R - \{x_n\}) \cup \{x_n\}$.

3) $X$ belongs to $\mathfrak{c}$. Let $\tilde{\mathcal{U}}$ be an infinite disjoint family of open sets of $X$ such that for every subfamily $\{U_n\}$ of $A$, there is a filter $\tilde{\mathcal{F}}$ of infinite subsets of $N$ such that $(\tilde{\mathcal{F}}, N_1, U_n, X) = \emptyset$. Since $\beta N - N$ is compact, $C = (\tilde{\mathcal{F}}, N_1, U_n, \beta N - N)$ is a compact subset of $\beta N - N$, and hence $C$ is a compact subset of $R$. Since the cardinality of $R$ is $\leq c$, $C$ must be a finite set by [3,9.11]. Thus we can assume that $C$ consists of only one point $a$. Every $U_n$ contains distinct $P$-points $x_n, y_n$ of $\beta N - N$. From this, we obtain a contradiction as in the proof of (3) in 3.2.

References


Резюме

НЕКОТОРЫЕ КЛАССЫ СЧЕТНО КОМПАКТНЫХ ПРОСТРАНСТВ

ТАКЕСИ ИСИВАТА (Takesi Isiwata), Токио

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