Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 14 (1964), No. 1, 39,40–51

Persistent URL: http://dml.cz/dmlcz/100600

RELATIONS BETWEEN THE DIAGONAL ELEMENTS OF TWO MUTUALLY INVERSE POSITIVE DEFINITE MATRICES

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(Received January 23, 1962)

1. Introduction. It is the purpose of this paper to solve the following problem: *To find necessary*[1]) *and sufficient conditions for* $2n$ *numbers* a_{ii}, α_{ii}, $i = 1, \ldots, n$, *to be diagonal elements of an n-rowed positive definite matrix* $A = (a_{ij})$ *and its inverse matrix* $A^{-1} = (\alpha_{ij})$. The complete solution is given in Theorems (3,2) and (3,3). Some applications are added which describe the geometric sense of the conditions.

2. Notation and lemmas. We shall use the well known notions of the theory of matrices and linear algebra. If A is a matrix with complex elements, we shall denote by A^* the conjugate transpose of A. If $C = (c_{ij})$, $i, j = 1, \ldots, n$, is a square matrix then tr C denotes the trace $\sum_{i=1}^{n} c_{ii}$ of C. If D is a diagonal matrix with diagonal elements $d_1, \ldots, d_n$, we shall write simply

$$D = \operatorname{diag}\{d_1, \ldots, d_n\}.$$

Moreover, we shall use the following lemmas:

(2,1) *Let* $n \geqq m \geqq 1$ *be integers,* $d_1, d_2, \ldots, d_n$ *positive numbers. Then,*

$$(\sum_{i=1}^{n} d_i)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \frac{1}{d_i}\right)^{\frac{1}{2}} - n \geqq (\sum_{i=1}^{m} d_i)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \frac{1}{d_i}\right)^{\frac{1}{2}} - m, \tag{1}$$

with equality if and only if

$$d_{m+1} = d_{m+2} = \ldots = d_n = (\sum_{i=1}^{m} d_i)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \frac{1}{d_i}\right)^{-\frac{1}{2}}.$$

[1]) A necessary condition is proved in [2].

Proof. The case $n = m + 1$ follows directly:

$$(\sum_{i=1}^{m} d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{m}\frac{1}{d_i}\right)^{\frac{1}{2}} - m = \left[\sum_{i=1}^{m} d_i \sum_{i=1}^{m}\frac{1}{d_i} + 2\left(\sum_{i=1}^{m} d_i \sum_{i=1}^{m}\frac{1}{d_i}\right)^{\frac{1}{2}} + 1\right]^{\frac{1}{2}} -$$

$$-(m+1) \leqq \left[\sum_{i=1}^{m} d_i \sum_{i=1}^{m}\frac{1}{d_i} + d_{m+1}\sum_{i=1}^{m}\frac{1}{d_i} + \frac{1}{d_{m+1}}\sum_{i=1}^{m} d_i + 1\right]^{\frac{1}{2}} -$$

$$-(m+1) = (\sum_{i=1}^{m+1} d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{m+1}\frac{1}{d_i}\right)^{\frac{1}{2}} - (m+1)\,.$$

From this, (1) follows immediately. Suppose that in (1) equality is reached. Then

$$d_{m+1} = (\sum_{i=1}^{m} d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{m}\frac{1}{d_i}\right)^{-\frac{1}{2}}$$

as well as

$$d_{m+2} = \ldots = d_n = (\sum_{i=1}^{m} d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{m}\frac{1}{d_i}\right)^{-\frac{1}{2}}$$

since the relation is symmetric with respect to $d_{m+1}, d_{m+2}, \ldots, d_n$. It can be easily seen that we really obtain equality in this last case.

(2,2). *Let a_1, a_2, a_3 be real numbers such that $0 < a_1 \leqq a_2 \leqq a_3$. Then,*

(2) $$\frac{a_2}{a_1 + a_3} + \frac{a_2^{-1}}{a_1^{-1} + a_3^{-1}} \leqq 1\,,$$

with equality if and only if $a_2 = a_1$ or $a_2 = a_3$.

Proof. From $(a_3 - a_2)(a_2 - a_1) \geqq 0$ we obtain equivalent inequalities

$$a_2(a_1 + a_3) \geqq a_2^2 + a_1 a_3\,,$$

$$\frac{a_2^2}{a_2(a_1 + a_3)} + \frac{a_1 a_3}{a_2(a_1 + a_3)} \leqq 1\,.$$

Thus, (2) is valid, with equality as asserted.

(2,3). *Let $d_1, d_2, \ldots, d_n$, $n \geqq 2$, be real numbers for which $0 < d_1 \leqq d_2 \leqq \ldots \leqq d_n$, and $c_1, c_2, \ldots, c_n$ non-negative numbers such that $\sum_{i=1}^{n} c_i = 1$. Then,*

(3) $$(\sum_{i=1}^{n} d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\frac{1}{d_i}\right)^{\frac{1}{2}} \geqq (d_1 + d_n)^{\frac{1}{2}}\left(\frac{1}{d_1} + \frac{1}{d_n}\right)^{\frac{1}{2}} + n - 2 \geqq$$

$$\geqq 2(\sum_{i=1}^{n} c_i d_i)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\frac{c_i}{d_i}\right)^{\frac{1}{2}} + n - 2\,.$$

In the left inequality, equality is attained if and only if $d_2 = d_3 = \ldots = d_{n-1} = (d_1 d_n)^{\frac{1}{2}}$. *In the right inequality,*[2] *equality is valid if* $c_i = 0$ *whenever* $d_1 < d_i < d_n$ *and* $\sum_{k \in M} c_k = \frac{1}{2}$ *where* $M = \{k; d_k = d_1\}$.

Proof. The left inequality is an immediate consequence of (2,1); thus, equality holds if and only if

$$d_2 = \ldots = d_{n-1} = (d_1 + d_n)^{\frac{1}{2}} \left(\frac{1}{d_1} + \frac{1}{d_n} \right)^{-\frac{1}{2}} = (d_1 d_n)^{\frac{1}{2}} .$$

To prove the second inequality in (3), notice first that according to (2,2) and $0 < d_1 \leqq d_i \leqq d_n$

$$\frac{d_i}{d_1 + d_n} + \frac{\frac{1}{d_i}}{\frac{1}{d_1} + \frac{1}{d_n}} \leqq 1 \tag{4}$$

holds with equality if and only if $d_i = d_1$ or $d_i = d_n$. Thus, if $c_i \geqq 0, \sum_{i=1}^{n} c_i = 1$, then

$$\frac{\sum_{i=1}^{n} c_i d_i}{d_1 + d_n} + \frac{\sum_{i=1}^{n} \frac{c_i}{d_i}}{\frac{1}{d_1} + \frac{1}{d_n}} \leqq 1 ,$$

i.e.

$$\frac{\sum_{i=1}^{n} c_i d_i}{d_1 + d_n} \cdot \frac{\sum_{i=1}^{n} \frac{c_i}{d_i}}{\frac{1}{d_1} + \frac{1}{d_n}} \leqq \frac{1}{4} .$$

From this, we obtain easily the second inequality in (3) with equality as asserted.

We shall say further that a matrix A is diagonally congruent to a matrix B if

$$A = DBD^*$$

for a diagonal regular matrix D. It is obvious that this relation of diagonal congruence is an equivalence relation.

(2,4). *If* $A = (a_{ij})$ *is (Hermitian) positive definite, then every diagonally congruent matrix to* A *has this property as well. Morevoer, there exists a matrix* $C = (c_{ij})$ *which is diagonally congruent to* A *and such that*

$$c_{ii} = (a_{ii}\alpha_{ii})^{\frac{1}{2}} = \gamma_{ii} \tag{5}$$

where γ_{ii} *are diagonal elements of* $C^{-1} = (\gamma_{ij})$.

[2]) This is essentially the Kantorovich inequality. See e. g. [3].

Proof. The first part being obvious, let us choose D as a diagonal matrix with diagonal elements $a_{ii}^{-\frac{1}{4}}\alpha_{ii}^{\frac{1}{4}}$ where $A^{-1} = (\alpha_{ij})$. It is then easy to see that $C = DAD^*$ satisfies (5) as asserted.

We shall conclude this section by the following obvious lemma:

(2,5). *Let J be a square matrix whose all elements are equal to 1, and let P be a square matrix of the same order. Then,*

$$JPJ = pJ$$

where the number p is the sum of all elements in P.

3. Results. In this section, we shall prove the main three theorems.

(3,1). Theorem. *Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a (Hermitian) positive definite matrix, $A^{-1} = (\alpha_{ij})$. Let λ_1 be the least, λ_n the greatest proper value of A, $q = \lambda_n/\lambda_1$. Then,*

$$(\operatorname{tr} A \operatorname{tr} A^{-1})^{\frac{1}{2}} \geqq q^{\frac{1}{2}} + q^{-\frac{1}{2}} + n - 2 \geqq 2 \max_{i=1,\ldots,n} \sqrt{(a_{ii}\alpha_{ii})} + n - 2 . \tag{6}$$

In the first inequality, equality is attained if and only if all remaining proper values of A are equal to $(\lambda_1\lambda_n)^{\frac{1}{2}}$; in the second, we obtain equality if and only if $n = 1$ or if $n > 1$ and there exist proper vectors v, w corresponding to λ_1, λ_n resp. such that their coordinates v_j, w_j fulfil the conditions $v_k = w_k$ for $k = 1, \ldots, n$, $k \neq i$, $v_i = - w_i$.

Proof. The case $n = 1$ being trivial, assume that $n > 1$. Let $A = ULU^*$ where $L = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ ($\lambda_1, \ldots, \lambda_n$ are positive proper values) and $U = (u_{jk})$ is unitary. Then,

$$\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i , \quad \operatorname{tr} A^{-1} = \sum_{i=1}^{n} \lambda_i^{-1} ,$$

$$a_{ii} = \sum_{k=1}^{n} |u_{ik}|^2 \lambda_k , \quad \alpha_{ii} = \sum_{k=1}^{n} |u_{ik}|^2 \lambda_k^{-1} .$$

Since $\sum_{k=1}^{n} |u_{ik}|^2 = 1$, we obtain (6) immediately from (3) in (2,3), $\lambda_1 = \min_{j=1,\ldots,n} \lambda_j$, $\lambda_n = \max_{j=1,\ldots,n} \lambda_j$, $q = \lambda_n/\lambda_1$. By the same theorem, equality in the left inequality is attained if and only if $\lambda_j = (\lambda_1\lambda_n)^{\frac{1}{2}}$ for $j = 2, \ldots, n - 1$. Assume now that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_k < \lambda_{k+1} \leqq \ldots \leqq \lambda_l < \lambda_{l+1} = \ldots = \lambda_n$$

and that equality is valid in the second inequality. According to (2,3), $u_{ij} = 0$ for $j = k + 1, \ldots, l$, $\sum_{p=1}^{k} |u_{ip}|^2 = \sum_{q=l+1}^{n} |u_{iq}|^2 = \frac{1}{2}$. There exist unitary matrices V_1, V_2 (with k rows and $n - l$ rows resp.) such that

$$(u_{i1}, \ldots, u_{ik}) V_1 = (2^{-\frac{1}{2}}, 0, \ldots, 0) ,$$

$$(u_{i,l,1}, \ldots, u_{i,n}) V_2 = (0, 0, \ldots, -2^{-\frac{1}{2}}) .$$

Then, $A = ULU^* = UVLV^*U^* = WLW^*$ where

$$V = \begin{pmatrix} V_1, & 0, & 0 \\ 0, & E, & 0 \\ 0, & 0, & V_2 \end{pmatrix} \quad (E \text{ is an } (l-k)\text{-rowed identity matrix})$$

and $W = UV$ is unitary.

If $W = (w_{ik})$, then $w_{i1} = 2^{-\frac{1}{2}}$, $w_{in} = -2^{-\frac{1}{2}}$; hence,

$$\tfrac{1}{2} = - w_{i1}\overline{w}_{in} = \sum_{\substack{k=1\\k\neq i}}^{n} w_{k1}\overline{w}_{kn} \leqq \Big(\sum_{\substack{k=1\\k\neq i}}^{n} |w_{k1}|^2\Big)^{\frac{1}{2}} \Big(\sum_{\substack{k=1\\k\neq i}}^{n} |w_{kn}|^2\Big)^{\frac{1}{2}} =$$

$$= (1 - |w_{i1}|^2)^{\frac{1}{2}} (1 - |w_{in}|^2)^{\frac{1}{2}} = \tfrac{1}{2}$$

so that $w_{k1} = \sigma w_{kn}$, $k = 1, \ldots, n$, $k \neq i$. But $\sigma = 1$ since $\frac{1}{2} = \sum_{\substack{k=1\\k\neq i}}^{n} w_{k1}\overline{w}_{kn} = \sigma \sum_{\substack{k=1\\k\neq i}}^{n} |w_{kn}|^2 = \frac{1}{2}\sigma$. Consequently, the first column vector v and the last column vector w of W which are proper vectors corresponding to λ_1, λ_n resp. have the property of the theorem. It is easy to see that the converse part is also valid. The proof is complete.

(3,2). Theorem. *Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a Hermitian positive definite matrix, $A^{-1} = (\alpha_{ij})$. Then,*

$$a_{ii} > 0\,, \quad \alpha_{ii} > 0\,, \tag{7}$$

$$a_{ii}\alpha_{ii} \geqq 1\,, \tag{8}$$

$$\sqrt{(a_{ii}\alpha_{ii})} - 1 \leqq \sum_{\substack{j=1\\i\neq i}}^{n} [\sqrt{(a_{jj}\alpha_{jj})} - 1] \tag{9}$$

for $i = 1, \ldots, n$.

Conversely, let a_{ii}, α_{ii} $(i = 1, \ldots, n)$ be $2n$ real numbers which satisfy (7), (8) *and* (9) *for $i = 1, \ldots, n$. Then, there exists a positive definite (even real) matrix $A = (a_{ik})$ such that its diagonal elements coincide with the given numbers a_{ii} and the diagonal elements of its inverse matrix with α_{ii}.*

Remark. In (8), equality holds for a fixed i if and only if $a_{ik} = 0$ for $k \neq i$, $i = 1, \ldots, n$. The case of equality in (9) will be completely solved in (3,3).

Proof. Since (7) as well as (8) written in the form $a_{ii}A_{ii} \geqq \det A$ (A_{ii} is the complementary principal minor of a_{ii} in A) are well known, we shall prove (9) only. Thus, let $C = (c_{ij})$ be the matrix from (5) in (2,4). Then,

$$\sum_{k=1}^{n} \sqrt{(a_{kk}\alpha_{kk})} = \operatorname{tr} C = \operatorname{tr} C^{-1} = (\operatorname{tr} C \operatorname{tr} C^{-1})^{\frac{1}{2}} \geqq$$

$$\geqq 2 \max_{i=1,\ldots,n} \sqrt{(c_{ii}\gamma_{ii})} + n - 2 = 2 \max_{i=1,\ldots,n} \sqrt{(a_{ii}\alpha_{ii})} + n - 2$$

according to (3,1). From this, (9) follows immediately. To prove the converse part, notice that the statement does not depend on the choice of a matrix from the class of diagonally congruent matrices, i.e. according to (2, 4) it is sufficient to prove:

Let $c_1, \ldots, c_n$ be non-negative numbers such that

$$\sum_{i=1}^{n-1} c_i \geqq c_n = \max_{j=1,\ldots,n} c_j . \tag{10}$$

Then, there exists a positive definite matrix $A = (a_{ij})$, $i, j = 1, \ldots, n$ such that its diagonal elements a_{ii} and the diagonal elements of its inverse matrix α_{ii} fulfil the relations

$$\sqrt{(a_{ii}\alpha_{ii})} - 1 = c_i \quad (i = 1, \ldots, n) .$$

This is obvions for $n = 1$. If $n > 1$, let us denote by ϱ_i the numbers

$$\varrho_i = c_i(c_i + 2) \quad (i = 1, \ldots, n)$$

so that $\varrho_n = \max_{j=1,\ldots,n} \varrho_j$.

Let us distinguish two cases:

1° If $\sum_{i=1}^{n-1} \varrho_i = \varrho_n$, put

$$A = \begin{bmatrix} 1, & 0, & \ldots, 0, & \sqrt{\varrho_1} \\ 0, & 1, & \ldots, 0, & \sqrt{\varrho_2} \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, 1, & \sqrt{\varrho_{n-1}} \\ \sqrt{\varrho_1}, & \sqrt{\varrho_2}, & \ldots, \sqrt{\varrho_{n-1}}, & 1 + \varrho_n \end{bmatrix} .$$

Obviously, $\det A = 1 > 0$ and A is positive definite. But diagonal elements α_{jj} of A^{-1} are

$$\begin{aligned} \alpha_{ii} &= 1 + \varrho_i , \quad i = 1, \ldots, n - 1 , \\ \alpha_{nn} &= 1 . \end{aligned}$$

Consequently, $a_{jj}\alpha_{jj} = 1 + \varrho_j = (1 + c_j)^2$ for $j = 1, \ldots, n$.

2° Let now $\sum_{i=1}^{n-1} \varrho_i \neq \varrho_n$ so that $\varrho_n > 0$. Denote by $\varphi_1(x)$, $\varphi_2(x)$ the real functions defined for $x \geqq -\varrho_n^{-1}$

$$\varphi_1(x) = n - 2 - \sum_{k=1}^{n-1} (1 + \varrho_i x)^{\frac{1}{2}} + (1 + \varrho_n x)^{\frac{1}{2}} ,$$

$$\varphi_2(x) = n - 2 - \sum_{k=1}^{n-1} (1 + \varrho_i x)^{\frac{1}{2}} - (1 + \varrho_n x)^{\frac{1}{2}} ,$$

and put further

$$x_0 = -\varrho_n^{-1}, \quad \varepsilon = \operatorname{sgn}(\varrho_n - \sum_{i=1}^{n-1} \varrho_i).$$

Since $\varphi_1(0) = 0$, $\varphi_1'(0) = \frac{1}{2}(\varrho_n - \sum_{i=1}^{n-1} \varrho_i)$,

$$\varphi_1(1) = -(\sum_{i=1}^{n-1} c_i - c_n) \leqq 0, \quad \varphi_1(x_0) = \varphi_2(x_0), \quad \varphi_2(0) = -2 < 0,$$

the following assertion is valid:

If $\varepsilon = 1$, then $\varphi_1(x)$ has a root in $(0, 1\rangle$. If $\varepsilon = -1$ and $\varphi_1(x_0) \geqq 0$, then $\varphi_2(x)$ has a root in $\langle x_0, 0)$. If $\varepsilon = -1$ and $\varphi_1(x_0) < 0$, then $\varphi_1(x)$ has a root in $\langle x_0, 0)$.

Let us denote by ξ such a root (in each case), and by $\sigma, \sigma', d_1, \ldots, d_n$ the real numbers

(11) $$\sigma = 1 + \sqrt{(1 - \xi)}, \quad \sigma' = 1 - \sqrt{(1 - \xi)},$$

(12) $$d_i = \left[\frac{1}{\xi}((1 + \varrho_i \xi)^{\frac{1}{2}} - 1)\right]^{\frac{1}{2}}, \quad i = 1, \ldots, n-1,$$

$$d_n = \left[\frac{\varepsilon}{\xi}(\eta(1 + \varrho_n \xi)^{\frac{1}{2}} + 1)\right]^{\frac{1}{2}}$$

where

$$\eta = 1 \quad \text{if} \quad \varepsilon = 1, \qquad \eta = -\operatorname{sgn} \varphi_1(x_0) \quad \text{if} \quad \varepsilon = -1.$$

According to the definition of ξ

$$1 + \varrho_i \xi > 0 \quad (i = 1, \ldots, n-1), \quad 1 + \varrho_n \xi \geqq 0$$

and it follows easily that all d_j exists since

(13) $$\varepsilon \xi > 0.$$

Moreover,

(14) $$\sum_{i=1}^{n-1} d_i^2 - \varepsilon d_n^2 = -2\xi^{-1}.$$

Now, we shall show that the matrix

$$A = V + \sigma DJD$$

where

$$V = \operatorname{diag}\{1, 1, \ldots, 1, -\varepsilon\},$$

$D = \operatorname{diag}\{d_1, \ldots, d_n\}$ and $J = (j_{ik})$, $j_{ik} = 1$ $(i, k = 1, \ldots, n)$ fulfils our conditions. Really, A is positive definite since

$$\det A = -\varepsilon + \sigma(d_n^2 - \varepsilon \sum_{i=1}^{n-1} d_i^2) = \varepsilon \xi^{-1}(1 + \sqrt{(1 - \xi)})^2 > 0$$

according to (12) and (13) while the principal submatrix consisting of the first $n-1$ rows and columns of A is obviously positive definite itself. Further,

$$a_{ii} = 1 + \sigma d_i^2 \quad (i = 1, \ldots, n-1), \quad a_{nn} = -\varepsilon + \sigma d_n^2 .$$

But, $A^{-1} = V + \sigma' VDJDV$ since $(V + \sigma DJD)(V + \sigma' VDJDV) = E + [\sigma + \sigma' + \sigma\sigma'(\sum_{i=1}^{n-1} d_i^2 - \varepsilon d_n^2)] DJDV = E$ according to (2,5) and (14). Thus,

$$\alpha_{ii} = 1 + \sigma' d_i^2 \quad (i = 1, \ldots, n-1), \quad \alpha_{nn} = -\varepsilon + \sigma' d_n^2 .$$

Hence

$$a_{ii}\alpha_{ii} = 1 + (\sigma + \sigma')\, d_i^2 + \sigma\sigma' d_i^4 = 1 + \varrho_i = (1 + c_i)^2 \quad (i = 1, \ldots, n-1),$$

$$a_{nn}\alpha_{nn} = 1 - \varepsilon(\sigma + \sigma')\, d_n^2 + \sigma\sigma' d_n^4 = 1 + \varrho_n = (1 + c_n)^2 .$$

The proof is complete since both matrices A are even real.

(3,3). Theorem. *Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a (Hermitian) positive definite matrix, $A^{-1} = (\alpha_{ij})$. Then, the following three properties of A are equivalent with each other*:

1° $$\sqrt{(a_{nn}\alpha_{nn})} - 1 = \sum_{i=1}^{n-1} (\sqrt{(a_{ii}\alpha_{ii})} - 1) ;$$

2° $$\frac{a_{ij}}{\sqrt{a_{ii}}\sqrt{a_{jj}}} = \frac{\alpha_{ij}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{jj}}}, \quad i, j = 1, \ldots, n-1,$$

as well as

$$\frac{a_{in}}{\sqrt{a_{ii}}\sqrt{a_{nn}}} = -\frac{\alpha_{in}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{nn}}}, \quad i = 1, \ldots, n-1 ;$$

3° *A is diagonally congruent to a matrix of the form*

$$\begin{pmatrix} E + (a-1)\, bb^* , & b\sqrt{(a^2-1)} \\ b^* \sqrt{(a^2-1)}, & a \end{pmatrix} \tag{15}$$

where E is the $(n-1)$-rowed identity matrix, b is an $(n-1)$-rowed unit vector and $a \geqq 1$.

Remark. The matrix (15) has proper values $a \pm \sqrt{(a^2-1)}, 1, \ldots, 1$ and its inverse is

$$\begin{pmatrix} E + (a-1)\, bb^* , & -b\sqrt{(a^2-1)} \\ -b^* \sqrt{a^2-1}\,, & a \end{pmatrix}. \tag{16}$$

Proof. The assertion being trivially fulfilled if $n = 1$, assume that $n > 1$. We shall show that $1° \to 2° \to 3° \to 1°$. In the first step $1° \to 2°$, we shall use another method (though it is not necessary to prove it in this manner).

Let 1° be fulfilled and denote by $Y = (y_{ij})$, $i, j = 1, \ldots, n$ the matrix with elements

$$y_{ij} = \frac{1}{\sqrt{\alpha_{ii}}\sqrt{a_{jj}}}\left(-\varepsilon_i \frac{a_{ij}}{\sqrt{a_{ii}}\sqrt{a_{jj}}} + \varepsilon_j \frac{\alpha_{ij}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{jj}}}\right),$$

$i, j = 1, \ldots, n$, where $\varepsilon_i = 1$ for $i = 1, \ldots, n-1$, $\varepsilon_n = -1$. According to (3,2), the function $\Phi(X) = \sum_{i=1}^{n} \varepsilon_i \sqrt{(x_{ii}\xi_{ii})}$ defined on the open set $\mathfrak{M}_n$ of all n-rowed positive definite matrices $X = (x_{ij})$ with $X^{-1} = (\xi_{ij})$ attains its minimum $n - 2$ for the matrix A. We shall prove 2° by showing that if $Y \neq 0$ then there exists a matrix $C \in \mathfrak{M}_n$ for which $\Phi(c) < \Phi(A)$. To prove this, put

$$C = (E + \varepsilon Y)\, A(E + \varepsilon Y^*)$$

where ε is a sufficiently small positive number. For a moment, we shall use the following notation:

If $\varphi_1(\varepsilon)$, $\varphi_2(\varepsilon)$ are functions or matrix functions of ε, we shall denote by $\varphi_1(\varepsilon) \approx \approx \varphi_2(\varepsilon)$ the fact that $\varphi_1(\varepsilon) - \varphi_2(\varepsilon)$ is $O(\varepsilon^2)$ for $\varepsilon \to 0$.

Thus,

$$C \approx A + \varepsilon(YA + AY^*), \quad C^{-1} \approx A^{-1} - \varepsilon(Y^*A^{-1} + A^{-1}Y).$$

If $C = (c_{ij})$, $C^{-1} = (\gamma_{ij})$, then

$$\Phi(C) = \sum_{i=1}^{n} \varepsilon_i \sqrt{(c_{ii}\gamma_{ii})} \approx \sum_{i=1}^{n} \varepsilon_i (a_{ii} + 2\varepsilon \operatorname{Re} \sum_{j=1}^{n} y_{ij} a_{ji})^{\frac{1}{2}} (\alpha_{ii} - 2\varepsilon \operatorname{Re} \sum_{j=1}^{n} \alpha_{ij} y_{ji})^{\frac{1}{2}} \approx$$

$$\approx \sum_{i=1}^{n} \varepsilon_i \sqrt{(a_{ii}\alpha_{ii})} \left(1 + \varepsilon \operatorname{Re} \sum_{j=1}^{n} \frac{a_{ji} y_{ij}}{a_{ii}}\right)\left(1 - \varepsilon \operatorname{Re} \sum_{j=1}^{n} \frac{\alpha_{ij} y_{ji}}{\alpha_{ii}}\right) \approx$$

$$\approx \Phi(A) + \varepsilon \sum_{i=1}^{n} \varepsilon_i \left[\operatorname{Re} \sum_{j=1}^{n} y_{ij} a_{ji} \sqrt{\left(\frac{\alpha_{ii}}{a_{ii}}\right)} - \operatorname{Re} \sum_{j=1}^{n} y_{ji} \alpha_{ij} \sqrt{\left(\frac{a_{ii}}{\alpha_{ii}}\right)}\right] =$$

$$= \Phi(A) + \varepsilon \operatorname{Re} \sum_{i,j=1}^{n} \left[y_{ij} \varepsilon_i a_{ji} \sqrt{\left(\frac{\alpha_{ii}}{a_{ii}}\right)} - y_{ij} \varepsilon_j \alpha_{ji} \sqrt{\left(\frac{a_{jj}}{\alpha_{jj}}\right)}\right] =$$

$$= \Phi(A) - \varepsilon \sum_{i,j=1}^{n} \left|\varepsilon_i \frac{a_{ij}}{\sqrt{a_{ii}}\sqrt{a_{jj}}} - \varepsilon_j \frac{\alpha_{ij}}{\sqrt{\alpha_{ii}}\sqrt{\alpha_{jj}}}\right|^2 < \Phi(A).$$

To prove 2° → 3°, let $C = (c_{ij})$ be the matrix in (2,4) satisfying (5). According to 2°, C is of the partitioned form

$$C = \begin{pmatrix} C_1, & c \\ c^*, & \gamma \end{pmatrix} \quad \text{while} \quad C^{-1} = \begin{pmatrix} C_1, & -c \\ -c^*, & \gamma \end{pmatrix}$$

(C_1 is an $(n-1)$-rowed square matrix, c is a column $(n-1)$-rowed vector).

Consequently,

$$C_1^2 - cc^* = E_1 \quad (E_1 \text{ is identity matrix}), \tag{17}$$

$$-c^*c + \gamma^2 = 1\,, \tag{18}$$

$$C_1 c = \gamma c\,. \tag{19}$$

From (18) it follows that $\gamma \geqq 1$. If $\gamma = 1$, then $c = 0$, $C_1^2 = E_1$ so that $C_1 = E_1$ since C_1 is positive definite. Thus, A satisfies 3°.

Let thus $\gamma > 1$ so that $c \neq 0$. From (17) it follows easily that, since $C_1^2 = E_1 + cc^*$ and C_1 is positive definite,

$$C_1 = E_1 + \frac{1}{c^*c}\left[(1 + c^*c)^{\frac{1}{2}} - 1\right] cc^*\,.$$

Consequently, if we put $a = \gamma$, $b = c(\gamma^2 - 1)^{-\frac{1}{2}}$, we obtain C in the form (16).

The implication 3° → 1° is a very simple consequence of (15), (16) and the fact that both properties are invariant under diagonal congruency. The proof is complete.

4. Applications. We shall show first that the conditions (7)–(9) involve necessary and sufficient conditions for the lengths of $2n$ vectors forming a biorthogonal basis in a (real or complex) n-dimensional unitary vector space X_n. Here, we denote by (x, y) the scalar product of vectors x and y and by $|x|$ the length $(x, x)^{\frac{1}{2}}$ of the vector x. Two bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are said to form a biorthogonal system[3]) if $(a_i, b_j) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol) for $i, j = 1, \ldots, n$ or, equivalently, if the basis $a_1, \ldots, a_n$ is an image of an orthonormal basis $e_1, \ldots, e_n$ by a regular linear mapping C while the basis $b_1, \ldots, b_n$ is image of $e_1, \ldots, e_n$ by the inverse adjoint mapping C^{*-1}:

$$a_i = Ce_i,\; b_i = C^{*-1}e_i, \quad i = 1, \ldots, n\,. \tag{20}$$

It is well known that to any basis $a_1, \ldots, a_n$ in X_n there exists a (single) basis $b_1, \ldots, b_n$ forming with the preceding basis a biorthogonal system.

(4,1). Theorem. *Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be a biorthogonal system in a unitary n-dimensional vector space X_n. Then, the lengths $\alpha_i = |a_i|$, $\beta_j = |b_j|$, $i, j = 1, \ldots, n$, fulfil the inequalities*

$$\alpha_i\beta_i \geqq 1\,, \qquad (i = 1, \ldots, n)\,, \tag{21}$$

$$\alpha_i\beta_i - 1 \leqq \sum_{\substack{j=1 \\ j \neq i}}^{n} (\alpha_j\beta_j - 1) \quad (i = 1, \ldots, n)\,. \tag{22}$$

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[3]) See e. g. [1].

Conversely, if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ *are* $2n$ *non-negative numbers satisfying* (21), (22), *then there exists a biorthogonal system* $a_1, \ldots, a_n, b_1, \ldots, b_n$ *such that* $|a_i| = \alpha_i$, $|b_j| = \beta_j$, $i, j = 1, \ldots, n$.

Further, equality is attained in (21) *if and only if* a_i *and* b_i *are linearly dependent. In* (22), *equality holds if and only if the angles of the vectors* a_j, a_k $(j, k = 1, \ldots, \ldots, n, j \neq i \neq k)$ *are equal to the corresponding angles between* b_j, b_k *while the anles between* a_i *and* a_k $(k = 1, \ldots, n, k \neq i)$ *are equal to the corresponding angles between* $-b_i$ *and* b_k.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis in X_n and C such a linear mapping that (20) holds. Then,

$$\alpha_i^2 = (Ce_i, Ce_i) = (C^*Ce_i, e_i) = (Ae_i, e_i)$$

where $A = C^*C$ is a positive definite mapping in X_n. Analogously,

$$\beta_i^2 = (C^{*-1}e_i, C^{*-1}e_i) = (C^{-1}C^{*-1}e_i, e_i) = (A^{-1}e_i, e_i) .$$

Thus, (21) and (22) are identical with conditions (8) and (9) for the corresponding matrix with elements $a_{ij} = (Ae_i, e_j)$.

Conversely, if (21) and (22) hold, there exists a positive definite matrix (a_{ij}) (which may be chosen real) such that relations $a_{ii} = \alpha_i^2$, $\alpha_{ii} = \beta_i^2$ $(i = 1, \ldots, n)$ are satisfied where α_{ii} are diagonal elements of the inverse matrix (α_{ij}) to (a_{ij}). Consequently, there exists a positive definite mapping A fulfilling $a_{ij} = (Ae_i, e_j)$, $\alpha_{ij} = (A^{-1}e_i, e_j)$. If we choose a mapping C such that $C^*C = A$ (and it is possible to choose C positive definite), then the vectors $a_1, \ldots, a_n, b_1, \ldots, b_n$ from (20) form a biorthogonal system. Since then

$$a_{ij} = (Ce_i, Ce_j) = (a_i, a_j)$$

and

$$\alpha_{ij} = (C^{*-1}e_i, C^{*-1}e_j) = (b_i, b_j) ,$$

it follows easily from (3,2) that in (21) equality holds if and only if $(a_i, a_j) = 0$ for $j = 1, \ldots, n, j \neq i$, i.e. if $a_i = \lambda b_i$ (since $(b_i, a_j) = 0$ for $j = 1, \ldots, n, j \neq i$, as well). According to the equivalence of 1° and 2° in (3,3), equality in (22) is attained if and only if

$$\frac{(a_k, a_l)}{|a_k|\,|a_l|} = \frac{(b_k, b_l)}{|b_k|\,|b_l|} \quad \text{for} \quad k, l = 1, \ldots, n\,,\ k \neq i \neq l,$$

while

$$\frac{(a_i, a_j)}{|a_i|\,|a_j|} = -\frac{(b_i, b_j)}{|b_i|\,|b_j|} \quad \text{for} \quad j = 1, \ldots, n,\ j \neq i\,.$$

The proof is complete.

(4,2). Theorem. *If $a_1, \dots, a_n$, $b_1, \dots, b_n$ is a biorthogonal system in a unitary space, then the angles ω_i, $0 \leqq \omega_i < \frac{1}{2}\pi$, between the corresponding vectors a_i and b_i fulfil inequalities*

(23) $$\sec \omega_i - 1 \leqq \sum_{\substack{j=1 \\ i \neq i}}^{n} (\sec \omega_j - 1), \quad i = 1, \dots, n.$$

Conversely, if $\omega_1, \dots, \omega_n$ are zero or acute angles satisfying (23) (*or the single inequality* (23) *for that i for which $\alpha_i = \max_{k=1,\dots,n} \alpha_k$*), *then there exists a biorthogonal system $a_1, \dots, a_n$, $b_1, \dots, b_n$ such that ω_j is the angle between a_j and b_j, $j = 1, \dots, n$.*

Proof. Follows immediately from the preceding theorem since

$$\cos \omega_i = \frac{(a_i, b_i)}{|a_i|\,|b_i|} = \frac{1}{|a_i|\,|b_i|}.$$

In the next application, we shall use the notion of the spherical m-simplex. This will mean essentially a system of $m + 1$ linearly independent directions $\delta_1, \dots, \delta_{m+1}$ in a Euclidean $(m + 1)$-space E_{m+1}. We shall call altitude-angle of the spherical m-simplex corresponding to the vertex-direction δ_i the angle (acute or right) φ_i between δ_i and the hyperplane in E_{m+1} which contains all δ_j for $j \neq i$. It is easy to see that, if $a_1, \dots, a_{m+1}$ are any non-zero vectors such that a_i $(i = 1, \dots, m + 1)$ is of direction δ_i and $b_1, \dots, b_{m+1}$ are vectors forming together with $a_1, \dots, a_{m+1}$ a biorthogonal system in E_{m+1}, then $\omega_i = \frac{1}{2}\pi - \varphi_i$ is the angle between a_i and b_i $(i = 1, \dots, m + 1)$.

Conversely, if $a_1, \dots, a_{m+1}$, $b_1, \dots, b_{m+1}$ is a biorthogonal system in E_{m+1} and ω_i is the angle between a_i and b_i $(i = 1, \dots, m + 1)$, then $\varphi_i = \frac{1}{2}\pi - \omega_i$ is the altitude-angle corresponding to the direction of a_i in the spherical m-simplex whose vertex-directions are the directions of $a_1, \dots, a_{m+1}$.

From this observation and (4,2) the following theorem follows immediately:

(4,3) Theorem. *Let $\varphi_1, \dots, \varphi_{m+1}$ be the altitude-angles of a spherical m-simplex. Then,*

(24) $$\operatorname{cosec} \varphi_i - 1 \leqq \sum_{\substack{j=1 \\ j \neq i}}^{m+1} (\operatorname{cosec} \varphi_j - 1), \quad i = 1, \dots, m + 1.$$

Conversely, if $\varphi_1, \dots, \varphi_{m+1}$ are acute or right angles satisfying (24) (*or the single inequality* (24) *with such i that $\varphi_i = \min_{k=1,\dots,m+1} \varphi_k$*), *then there exists a spherical m-simplex whose altitude-angles are φ_i $(i = 1, \dots, m + 1)$.*

Remark. It can be proved that, if equality in (24) is attained, the corresponding m-simplex is orthocentric (i.e., his "altitudes" have a common direction) and satisfies a further condition.

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Резюме

СООТНОШЕНИЯ МЕЖДУ ДИАГОНАЛЬНЫМИ ЭЛЕМЕНТАМИ ДВУХ ВЗАИМНО ОБРАТНЫХ ПОЛОЖИТЕЛЬНО ОПРЕДЕЛЕННЫХ МАТРИЦ

МИРОСЛАВ ФИДЛЕР (Miroslav Fiedler), Прага

Доказывается следующая теорема: *Необходимым и достаточным условием для того, чтобы* $2n$ *действительных чисел* $a_{11}, \ldots, a_{nn}, \alpha_{11}, \ldots, \alpha_{nn}$ *образовали системы диагональных элементов положительно определенной матрицы* $A = (a_{ik})$ *и диагональных элементов обратной к ней матрицы* $A^{-1} = (\alpha_{ik})$, *является одновременное выполнение следующих неравенств для* $i = 1, \ldots, n$:

$$a_{ii} > 0,\ \alpha_{ii} > 0,\ a_{ii}\alpha_{ii} \geqq 1\,,$$

$$\sqrt{(a_{ii}\alpha_{ii})} - 1 \leqq \sum_{j=1, j\neq i}^{n} [\sqrt{(a_{jj}\alpha_{jj})} - 1]\,.$$

Далее характеризуются случаи равенства и дается геометрическое истолкование этой теоремы как условия, налагаемого на длины векторов биортогональной системы, на углы соответствующих векторов биортогональной системы или на высоты сферического симплекса.