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RELATIONS BETWEEN THE DIAGONAL ELEMENTS OF TWO MUTUALLY INVERSE POSITIVE DEFINITE MATRICES

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1. Introduction. It is the purpose of this paper to solve the following problem: To find necessary and sufficient conditions for 2n numbers \( a_{ii}, \alpha_{ii}, i = 1, \ldots, n \), to be diagonal elements of an \( n \)-rowed positive definite matrix \( A = (a_{ij}) \) and its inverse matrix \( A^{-1} = (\alpha_{ij}) \). The complete solution is given in Theorems (3,2) and (3,3). Some applications are added which describe the geometric sense of the conditions.

2. Notation and lemmas. We shall use the well known notions of the theory of matrices and linear algebra. If \( A \) is a matrix with complex elements, we shall denote by \( A^* \) the conjugate transpose of \( A \). If \( C = (c_{ij}), i, j = 1, \ldots, n \), is a square matrix then \( \text{tr} C \) denotes the trace \( \sum_{i=1}^{n} c_{ii} \) of \( C \). If \( D \) is a diagonal matrix with diagonal elements \( d_1, \ldots, d_n \), we shall write simply

\[
D = \text{diag} \{d_1, \ldots, d_n\}.
\]

Moreover, we shall use the following lemmas:

(2,1) Let \( n \geq m \geq 1 \) be integers, \( d_1, d_2, \ldots, d_n \) positive numbers. Then,

\[
\left( \sum_{i=1}^{n} d_i \right) \left( \sum_{i=1}^{n} \frac{1}{d_i} \right) - n \geq \left( \sum_{i=1}^{m} d_i \right) \left( \sum_{i=1}^{m} \frac{1}{d_i} \right) - m,
\]

with equality if and only if

\[
d_{m+1} = d_{m+2} = \ldots = d_n = \left( \sum_{i=1}^{m} d_i \right) \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{-\frac{1}{2}}.
\]

1) A necessary condition is proved in [2].
Proof. The case \( n = m + 1 \) follows directly:

\[
\left( \sum_{i=1}^{m} d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{\frac{1}{2}} - m = \left[ \sum_{i=1}^{m} d_i \sum_{i=1}^{m} \frac{1}{d_i} + 2 \left( \sum_{i=1}^{m} d_i \sum_{i=1}^{m} \frac{1}{d_i} \right)^{\frac{1}{2}} + 1 \right]^{\frac{1}{2}} -
\]

\[
- (m + 1) \leq \left[ \sum_{i=1}^{m} d_i \sum_{i=1}^{m} \frac{1}{d_i} + d_{m+1} \sum_{i=1}^{m} \frac{1}{d_i} + \sum_{i=1}^{m} d_i + 1 \right]^{\frac{1}{2}} -
\]

\[
- (m + 1) = \left( \sum_{i=1}^{m+1} d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{\frac{1}{2}} - (m + 1).
\]

From this, (1) follows immediately. Suppose that in (1) equality is reached. Then

\[
d_{m+1} = \left( \sum_{i=1}^{m} d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{\frac{1}{2}}
\]

as well as

\[
d_{m+2} = \ldots = d_n = \left( \sum_{i=1}^{m} d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{m} \frac{1}{d_i} \right)^{\frac{1}{2}}
\]

since the relation is symmetric with respect to \( d_{m+1}, d_{m+2}, \ldots, d_n \). It can be easily seen that we really obtain equality in this last case.

\[\text{(2,2). Let } a_1, a_2, a_3 \text{ be real numbers such that } 0 < a_1 \leq a_2 \leq a_3.\]

Then,

\[
\frac{a_2}{a_1 + a_3} + \frac{a_2^{-1}}{a_1^{-1} + a_3^{-1}} \leq 1,
\]

with equality if and only if \( a_2 = a_1 \) or \( a_2 = a_3 \).

Proof. From \((a_3 - a_2)(a_2 - a_1) \geq 0\) we obtain equivalent inequalities

\[
\frac{a_2}{a_1 + a_3} \geq \frac{a_2^2}{a_1 a_3},
\]

\[
\frac{a_2^2}{a_2(a_1 + a_3)} + \frac{a_1 a_3}{a_2(a_1 + a_3)} \leq 1.
\]

Thus, (2) is valid, with equality as asserted.

\[\text{(2,3). Let } d_1, d_2, \ldots, d_n, n \geq 2, \text{ be real numbers for which } 0 < d_1 \leq d_2 \leq \ldots \leq d_n,\]

and \( c_1, c_2, \ldots, c_n \) non-negative numbers such that \( \sum_{i=1}^{n} c_i = 1. \) Then,

\[
\left( \sum_{i=1}^{n} d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \frac{1}{d_i} \right)^{\frac{1}{2}} \geq \left( \frac{1}{d_1} + \frac{1}{d_n} \right)^{\frac{1}{2}} + n - 2 \geq
\]

\[
2 \left( \sum_{i=1}^{n} c_i d_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \frac{c_i}{d_i} \right)^{\frac{1}{2}} + n - 2.
\]
In the left inequality, equality is attained if and only if \( d_2 = d_3 = \ldots = d_{n-1} = (d_1 d_n)^{\frac{1}{2}} \). In the right inequality, equality is valid if \( c_i = 0 \) whenever \( d_1 < d_i < d_n \) and \( \sum_{k \in M} c_k = \frac{1}{2} \) where \( M = \{ k; d_k = d_1 \} \).

Proof. The left inequality is an immediate consequence of (2,1); thus, equality holds if and only if

\[
d_2 = \ldots = d_{n-1} = (d_1 + d_n)^{\frac{1}{2}} \left( \frac{1}{d_1} + \frac{1}{d_n} \right)^{\frac{1}{2}} = (d_1 d_n)^{\frac{1}{2}}.
\]

To prove the second inequality in (3), notice first that according to (2,2) and \( 0 < d_1 \leq d_i \leq d_n \)

\[
\frac{d_i}{d_1 + d_n} + \frac{1}{d_1 + d_n} \leq 1
\]

holds with equality if and only if \( d_i = d_1 \) or \( d_i = d_n \). Thus, if \( c_i \geq 0 \), \( \sum_{i=1}^{n} c_i = 1 \), then

\[
\frac{\sum_{i=1}^{n} c_i d_i}{d_1 + d_n} + \frac{\sum_{i=1}^{n} c_i}{d_1 + d_n} \leq 1,
\]

i.e.

\[
\sum_{i=1}^{n} c_i d_i \frac{\sum_{i=1}^{n} c_i}{d_1 + d_n} \leq \frac{1}{4}.
\]

From this, we obtain easily the second inequality in (3) with equality as asserted.

We shall say further that a matrix \( A \) is diagonally congruent to a matrix \( B \) if

\[
A = DBD^*
\]

for a diagonal regular matrix \( D \). It is obvious that this relation of diagonal congruence is an equivalence relation.

**(2,4).** If \( A = (a_{ij}) \) is (Hermitian) positive definite, then every diagonally congruent matrix to \( A \) has this property as well. Moreover, there exists a matrix \( C = (c_{ij}) \) which is diagonally congruent to \( A \) and such that

\[
c_{ii} = (a_{ii} \gamma_{ii})^{\frac{1}{2}} = \gamma_{ii}
\]

where \( \gamma_{ii} \) are diagonal elements of \( B^{-1} = (\gamma_{ij}) \).

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2) This is essentially the Kantorovich inequality. See e. g. [3].
Proof. The first part being obvious, let us choose $D$ as a diagonal matrix with diagonal elements $a^{\frac{1}{2}}_{ii} a^{\frac{1}{2}}_{jj}$ where $A^{-1} = (\alpha_{ij})$. It is then easy to see that $C = DAD^*$ satisfies (5) as asserted.

We shall conclude this section by the following obvious lemma:

(2.5). Let $J$ be a square matrix whose all elements are equal to 1, and let $P$ be a square matrix of the same order. Then,

$$JPJ = pJ$$

where the number $p$ is the sum of all elements in $P$.

3. Results. In this section, we shall prove the main three theorems.

(3,1). Theorem. Let $A = (a_{ij})$, $i, j = 1, \ldots, n$, be a (Hermitian) positive definite matrix, $A^{-1} = (\alpha_{ij})$. Let $\lambda_1$ be the least, $\lambda_n$ the greatest proper value of $A$, $q = \lambda_n/\lambda_1$. Then,

$$\begin{align*}
(6) \quad (\text{tr } A \text{ tr } A^{-1})^\frac{1}{2} & \geq q^\frac{1}{4} + q^{-\frac{1}{4}} + n - 2 \geq 2 \max_{i=1, \ldots, n} \sqrt{(a_{ii} \alpha_{ii})} + n - 2.
\end{align*}$$

In the first inequality, equality is attained if and only if all remaining proper values of $A$ are equal to $(\lambda_1, \lambda_n)\frac{1}{2}$; in the second, we obtain equality if and only if $n = 1$ or if $n > 1$ and there exist proper vectors $v$, $w$ corresponding to $\lambda_1$, $\lambda_n$ resp. such that their coordinates $v_j, w_j$ fulfill the conditions $v_k = w_k$ for $k = 1, \ldots, n$, $k \neq i$, $v_i = -w_i$.

Proof. The case $n = 1$ being trivial, assume that $n > 1$. Let $A = ULU^*$ where $L = \text{diag} \{\lambda_1, \ldots, \lambda_n\}$ ($\lambda_1, \ldots, \lambda_n$ are positive proper values) and $U = (u_{jk})$ is unitary. Then,

$$\begin{align*}
\text{tr } A & = \sum_{i=1}^{n} \lambda_i, \quad \text{tr } A^{-1} = \sum_{i=1}^{n} \lambda_i^{-1}, \\
a_{ii} & = \sum_{k=1}^{n} |u_{ik}|^2 \lambda_k, \quad \alpha_{ii} = \sum_{k=1}^{n} |u_{ik}|^2 \lambda_k^{-1}.
\end{align*}$$

Since $\sum_{k=1}^{n} |u_{ik}|^2 = 1$, we obtain (6) immediately from (3) in (2,3), $\lambda_1 = \min_{j=1, \ldots, n} \lambda_j$, $\lambda_n = \max_{j=1, \ldots, n} \lambda_j$, $q = \lambda_n/\lambda_1$. By the same theorem, equality in the left inequality is attained if and only if $\lambda_j = (\lambda_1 \lambda_n)^\frac{1}{2}$ for $j = 2, \ldots, n - 1$. Assume now that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_k < \lambda_{k+1} \leq \ldots \leq \lambda_l < \lambda_{l+1} = \ldots = \lambda_n$$

and that equality is valid in the second inequality. According to (2,3), $u_{ij} = 0$ for $j = k + 1, \ldots, l$, $\sum_{p=1}^{k} |u_{ip}|^2 = \sum_{q=l+1}^{n} |u_{iq}|^2 = \frac{1}{2}$. There exist unitary matrices $V_1, V_2$ (with $k$ rows and $n - l$ rows resp.) such that

$$(u_{i1}, \ldots, u_{ik}) V_1 = (2^{-\frac{1}{2}}, 0, \ldots, 0),$$

$$(u_{i,l+1}, \ldots, u_{in}) V_2 = (0, 0, \ldots, -2^{-\frac{1}{2}}).$$
Then, $A = U L U^* = U V L V^* U^* = W L W^*$ where

$$V = \begin{pmatrix} V_1 & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & V_2 \end{pmatrix}$$

$E$ is an $(l - k)$-rowed identity matrix)

and $W = U V$ is unitary.

If $W = (w_{ik})$, then $w_{i1} = 2^{-\frac{1}{2}}, w_{in} = -2^{-\frac{1}{2}}$; hence,

$$\frac{1}{2} = -w_{i1} \bar{w}_{in} = \sum_{k=1}^{n} w_{k1} \bar{w}_{kn} \leq \left( \sum_{k=1}^{n} |w_{k1}|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{n} |w_{kn}|^2 \right)^{\frac{1}{2}} =$$

$$= \left( 1 - |w_{i1}|^2 \right)^{\frac{1}{2}} \left( 1 - |w_{in}|^2 \right)^{\frac{1}{2}} = \frac{1}{2}$$

so that $w_{k1} = \sigma w_{kn}, k = 1, \ldots, n, k \neq i$. But $\sigma = 1$ since $\frac{1}{2} = \sum_{k=1}^{n} |w_{k1}|^2 = \sigma \sum_{k=1}^{n} |w_{kn}|^2 = \frac{1}{2}\sigma$. Consequently, the first column vector $v$ and the last column vector $w$ of $W$ which are proper vectors corresponding to $\lambda_1, \lambda_n$ resp. have the property of the theorem. It is easy to see that the converse part is also valid. The proof is complete.

\textbf{(3,2). Theorem.} Let $A = (a_{ij}), i, j = 1, \ldots, n$, be a Hermitian positive definite matrix, $A^{-1} = (\alpha_{ij})$. Then,

\begin{equation}
(7) \quad \alpha_{ii} > 0, \quad \alpha_{ii} > 0,
\end{equation}

\begin{equation}
(8) \quad a_{ii} \alpha_{ii} \geq 1,
\end{equation}

\begin{equation}
(9) \quad \sqrt{(a_{i1} \alpha_{i1})} - 1 \leq \sum_{j=1}^{n} \left[ \sqrt{(a_{ij} \alpha_{jj})} - 1 \right]
\end{equation}

for $i = 1, \ldots, n$.

Conversely, let $a_{ii}, \alpha_{ii} (i = 1, \ldots, n)$ be $2n$ real numbers which satisfy (7), (8) and (9) for $i = 1, \ldots, n$. Then, there exists a positive definite (even real) matrix $A = (a_{ik})$ such that its diagonal elements coincide with the given numbers $a_{ii}$ and the diagonal elements of its inverse matrix with $\alpha_{ii}$.

\textbf{Remark.} In (8), equality holds for a fixed $i$ if and only if $a_{ik} = 0$ for $k \neq i$, $i = 1, \ldots, n$. The case of equality in (9) will be completely solved in (3,3).

\textbf{Proof.} Since (7) as well as (8) written in the form $a_{ii} A_{ii} \geq \det A$ ($A_{ii}$ is the complementary principal minor of $a_{ii}$ in $A$) are well known, we shall prove (9) only. Thus, let $C = (c_{ij})$ be the matrix from (5) in (2,4). Then,

$$\sum_{k=1}^{n} \sqrt{(a_{kk} \alpha_{kk})} = \text{tr } C = \text{tr } C^{-1} = (\text{tr } C \text{ tr } C^{-1})^{\frac{1}{2}} \geq$$

$$\geq 2 \max_{i=1, \ldots, n} \sqrt{(c_{ii} \alpha_{ii})} + n - 2 = 2 \max_{i=1, \ldots, n} \sqrt{(a_{ii} \alpha_{ii})} + n - 2$$

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according to (3,1). From this, (9) follows immediately. To prove the converse part, notice that the statement does not depend on the choice of a matrix from the class of diagonally congruent matrices, i.e. according to (2, 4) it is sufficient to prove:

Let \( c_1, \ldots, c_n \) be non-negative numbers such that

\[
\sum_{i=1}^{n-1} c_i \geq c_n = \max_{j=1, \ldots, n} c_j.
\]

Then, there exists a positive definite matrix \( A = (a_{ij}), i, j = 1, \ldots, n \) such that its diagonal elements \( a_{ii} \) and the diagonal elements of its inverse matrix \( a_{ii} \) fulfil the relations

\[
\sqrt{(a_{ii}a_{ii})} - 1 = c_i \quad (i = 1, \ldots, n).
\]

This is obvious for \( n = 1 \). If \( n > 1 \), let us denote by \( q_i \) the numbers

\[
q_i = c_i(c_i + 2) \quad (i = 1, \ldots, n)
\]

so that \( q_n = \max_{j=1, \ldots, n} q_j \).

Let us distinguish two cases:

1° If \( \sum_{i=1}^{n-1} q_i = q_n \), put

\[
A = \begin{bmatrix}
1, & 0, & \ldots, & 0, & \sqrt{q_1} \\
0, & 1, & \ldots, & 0, & \sqrt{q_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0, & 0, & \ldots, & 1, & \sqrt{q_{n-1}} \\
\sqrt{q_1}, & \sqrt{q_2}, & \ldots, & \sqrt{q_{n-1}}, & 1 + q_n
\end{bmatrix}.
\]

Obviously, \( \det A = 1 > 0 \) and \( A \) is positive definite. But diagonal elements \( a_{jj} \) of \( A^{-1} \) are

\[
a_{ii} = 1 + q_i, \quad i = 1, \ldots, n - 1, \\
a_{nn} = 1.
\]

Consequently, \( a_{jj}a_{jj} = 1 + q_j = (1 + c_j)^2 \) for \( j = 1, \ldots, n \).

2° Let now \( \sum_{i=1}^{n-1} q_i \neq q_n \) so that \( q_n > 0 \). Denote by \( \varphi_1(x), \varphi_2(x) \) the real functions defined for \( x \geq -q_n^{-1} \)

\[
\varphi_1(x) = n - 2 - \sum_{k=1}^{n-1} (1 + q_kx)^{\frac{1}{2}} + (1 + q_nx)^{\frac{1}{2}},
\]

\[
\varphi_2(x) = n - 2 - \sum_{k=1}^{n-1} (1 + q_kx)^{\frac{1}{2}} - (1 + q_nx)^{\frac{1}{2}},
\]

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and put further
\[ x_0 = -q_n^{-1}, \quad e = \text{sgn} \left( q_n - \sum_{i=1}^{n-1} q_i \right). \]

Since \( \varphi_1(0) = 0 \), \( \varphi_1'(0) = \frac{1}{2} \left( q_n - \sum_{i=1}^{n-1} q_i \right) \),
\[ \varphi_1(1) = - \left( \sum_{i=1}^{n-1} c_i - c_n \right) \leq 0, \quad \varphi_1(x_0) = \varphi_2(x_0), \quad \varphi_2(0) = -2 < 0, \]
the following assertion is valid:

If \( e = 1 \), then \( \varphi_1(x) \) has a root in \((0, 1)\). If \( e = -1 \) and \( \varphi_1(x_0) \geq 0 \), then \( \varphi_2(x) \) has a root in \( (x_0, 0) \). If \( e = -1 \) and \( \varphi_1(x_0) < 0 \), then \( \varphi_1(x) \) has a root in \( (x_0, 0) \).

Let us denote by \( \xi \) such a root (in each case), and by \( \sigma, \sigma', d_1, \ldots, d_n \) the real numbers

\[
\sigma = 1 + \sqrt{(1 - \xi)}, \quad \sigma' = 1 - \sqrt{(1 - \xi)},
\]

\[
d_i = \left[ \frac{1}{\xi} \left( (1 + q_i \xi)^{\frac{1}{2}} - 1 \right) \right]^2, \quad i = 1, \ldots, n - 1,
\]

\[
d_n = \left[ \frac{\varepsilon}{\xi} \left( \eta (1 + q_n \xi)^{\frac{1}{2}} + 1 \right) \right]^2
\]
where
\[
\eta = 1 \quad \text{if} \quad e = 1, \quad \eta = -\text{sgn} \varphi_1(x_0) \quad \text{if} \quad e = -1.
\]

According to the definition of \( \xi \)
\[
1 + q_i \xi > 0 \quad (i = 1, \ldots, n - 1), \quad 1 + q_n \xi \geq 0
\]
and it follows easily that all \( d_j \) exists since

\[
\varepsilon \xi > 0.
\]

Moreover,

\[
\sum_{i=1}^{n-1} d_i^2 - \varepsilon d_n^2 = -2 \xi^{-1}.
\]

Now, we shall show that the matrix
\[
A = V + \sigma DJD
\]
where
\[
V = \text{diag} \{1, 1, \ldots, 1, -\varepsilon\},
\]
\[
D = \text{diag} \{d_1, \ldots, d_n\} \quad \text{and} \quad J = (j_{ik}), \quad j_{ik} = 1 \quad (i, k = 1, \ldots, n) \quad \text{fulfils our conditions.}
\]

Really, \( A \) is positive definite since
\[
det A = -\varepsilon + \sigma (d_n^2 - \varepsilon \sum_{i=1}^{n-1} d_i^2) = \varepsilon \xi^{-1}(1 + \sqrt{(1 - \xi)})^2 > 0
\]
according to (12) and (13) while the principal submatrix consisting of the first \( n - 1 \) rows and columns of \( A \) is obviously positive definite itself. Further,

\[
a_{ii} = 1 + \sigma d_i^2 \quad (i = 1, \ldots, n - 1), \quad a_{nn} = -\varepsilon + \sigma d_n^2.
\]

But, \( A^{-1} = V + \sigma' VDJDV \) since \( (V + \sigma DJD)(V + \sigma' VDJDV) = E + [\sigma + \sigma' + + \sigma'(\sum d_i^2) - \varepsilon d_n^2]) DJDV = E \) according to (2,5) and (14). Thus,

\[
\alpha_{ii} = 1 + \sigma d_i^2 \quad (i = 1, \ldots, n - 1), \quad \alpha_{nn} = -\varepsilon + \sigma' d_n^2.
\]

Hence

\[
a_{ii} \alpha_{ii} = 1 + (\sigma + \sigma') d_i^2 + \sigma \sigma' d_i^4 = 1 + \varrho_i = (1 + c_i)^2 \quad (i = 1, \ldots, n - 1),
\]

\[
a_{nn} \alpha_{nn} = 1 - \varepsilon (\sigma + \sigma') d_n^2 + \sigma \sigma' d_n^4 = 1 + \varrho_n = (1 + c_n)^2.
\]

The proof is complete since both matrices \( A \) are even real.

\((3,3). \) \textbf{Theorem.} \textit{Let} \( A = (a_{ij}), \ i, j = 1, \ldots, n, \text{ be a (Hermitian) positive definite matrix,} \ A^{-1} = (\alpha_{ij}). \text{ Then, the following three properties of} \ A \text{ are equivalent with each other:}

1° \quad \sqrt{(a_{nn} \alpha_{nn})} - 1 = \sum_{i=1}^{n-1} (\sqrt{(a_{ii} \alpha_{ii})} - 1);

2° \quad \frac{a_{ij}}{\sqrt{a_{ii} \sqrt{a_{jj}}}} = \frac{\alpha_{ij}}{\sqrt{\alpha_{ii} \sqrt{\alpha_{jj}}}}, \quad i, j = 1, \ldots, n - 1,

\text{as well as}

\quad \frac{a_{in}}{\sqrt{a_{ii} \sqrt{a_{nn}}}} = -\frac{\alpha_{in}}{\sqrt{\alpha_{ii} \sqrt{\alpha_{nn}}}}, \quad i = 1, \ldots, n - 1;

3° \quad A \text{ is diagonally congruent to a matrix of the form}

\[
(E + (a - 1) bb^*, \ b \sqrt{(a^2 - 1)}, \ b^* \sqrt{(a^2 - 1)} , \ a)
\]

\text{where} \ E \text{ is the} (n - 1)-\text{rowed identity matrix,} \ b \text{ is an} (n - 1)-\text{rowed unit vector}

\text{and} \ a \geq 1.

\textbf{Remark.} \textit{The matrix} (15) \textit{has proper values} \ a \pm \sqrt{(a^2 - 1)}, 1, \ldots, 1 \text{ and its inverse is}

\[
(E + (a - 1) bb^*, \ -b \sqrt{(a^2 - 1)} , \ -b^* \sqrt{a^2 - 1} , \ a)
\]

\textbf{Proof.} \textit{The assertion being trivially fulfilled if} \ n = 1, \text{ assume that} \ n > 1. \text{ We shall show that} \ 1° \rightarrow 2° \rightarrow 3° \rightarrow 1°. \text{ In the first step} \ 1° \rightarrow 2°, \text{ we shall use another method (though it is not necessary to prove it in this manner).}
Let $1^\circ$ be fulfilled and denote by $Y = (y_{ij})$, $i, j = 1, \ldots, n$ the matrix with elements

$$y_{ij} = \frac{1}{\sqrt{\alpha_{ii} \sqrt{d_{jj}}}} \left( - \varepsilon_i \frac{a_{ij}}{\sqrt{\alpha_{ii}} \sqrt{d_{jj}}} + \varepsilon_j \frac{\alpha_{ij}}{\sqrt{\alpha_{ii} \alpha_{jj}}} \right),$$

$i, j = 1, \ldots, n$, where $\varepsilon_i = 1$ for $i = 1, \ldots, n - 1$, $\varepsilon_n = -1$. According to (3,2), the function $\Phi(X) = \sum_{i=1}^{n} \varepsilon_i \sqrt{x_{ii} \xi_{ii}}$ defined on the open set $\mathbb{M}_n$ of all $n$-rowed positive definite matrices $X = (x_{ij})$ with $X^{-1} = (\xi_{ij})$ attains its minimum $n - 2$ for the matrix $A$. We shall prove $2^\circ$ by showing that if $Y \neq 0$ then there exists a matrix $C \in \mathbb{M}_n$ for which $\Phi(C) < \Phi(A)$. To prove this, put

$$C = (E + \varepsilon Y) A (E + \varepsilon Y^*)$$

where $\varepsilon$ is a sufficiently small positive number. For a moment, we shall use the following notation:

If $\varphi_1(\varepsilon)$, $\varphi_2(\varepsilon)$ are functions or matrix functions of $\varepsilon$, we shall denote by $\varphi_1(\varepsilon) \approx \approx \varphi_2(\varepsilon)$ the fact that $\varphi_1(\varepsilon) - \varphi_2(\varepsilon)$ is $O(\varepsilon^2)$ for $\varepsilon \to 0$.

Thus,

$$C \approx A + \varepsilon(YA + AY^*), \quad C^{-1} \approx A^{-1} - \varepsilon(Y^*A^{-1} + A^{-1}Y).$$

If $C = (c_{ij})$, $C^{-1} = (y_{ij})$, then

$$\Phi(C) = \sum_{i=1}^{n} \varepsilon_i \sqrt{(c_{ii}y_{ii})} \approx \sum_{i=1}^{n} \varepsilon_i (a_{ii} + 2\varepsilon \text{Re} \sum_{j=1}^{n} y_{ij}a_{ji})^\frac{\alpha_{ij}}{\alpha_{ii}} \approx$$

$$\approx \sum_{i=1}^{n} \varepsilon_i (a_{ii} - 2\varepsilon \text{Re} \sum_{j=1}^{n} \alpha_{ij}y_{ji})^\frac{\alpha_{ij}}{\alpha_{ii}} \approx \Phi(A) + \varepsilon \sum_{i=1}^{n} \varepsilon_i \left[ \text{Re} \sum_{j=1}^{n} y_{ij}a_{ji} \sqrt{\frac{\alpha_{ii}}{\alpha_{ii}}} - \text{Re} \sum_{j=1}^{n} y_{ij}\alpha_{ij} \sqrt{\frac{\alpha_{ii}}{\alpha_{ii}}} \right] =$$

$$= \Phi(A) + \varepsilon \text{Re} \sum_{i,j=1}^{n} \left[ y_{ij}\varepsilon_i a_{ji} \sqrt{\frac{\alpha_{ii}}{\alpha_{ii}}} - y_{ij}\varepsilon_i \alpha_{ij} \sqrt{\frac{\alpha_{ii}}{\alpha_{ii}}} \right] =$$

$$= \Phi(A) - \varepsilon \sum_{i,j=1}^{n} \left| \frac{\varepsilon_i a_{ij} \sqrt{\alpha_{ii}} \sqrt{\alpha_{jj}} - \varepsilon_j \alpha_{ij} \sqrt{\alpha_{ii}} \sqrt{\alpha_{jj}}}{\sqrt{\alpha_{ii} \alpha_{jj}}} \right|^2 < \Phi(A).$$

To prove $2^\circ \to 3^\circ$, let $C = (c_{ij})$ be the matrix in (2,4) satisfying (5). According to $2^\circ$, $C$ is of the partitioned form

$$C = \begin{pmatrix} C_1 & c \\ e^* & \gamma \end{pmatrix} \quad \text{while} \quad C^{-1} = \begin{pmatrix} C_1^* & -c \\ -e^* & \gamma \end{pmatrix}$$

($C_1$ is an $(n - 1)$-rowed square matrix, $c$ is a column $(n - 1)$-rowed vector).
Consequently,

(17) \[ C_1^2 - cc^* = E_1 \quad (E_1 \text{ is identity matrix}), \]

(18) \[ -c^*c + \gamma^2 = 1, \]

(19) \[ C_1c = \gamma c. \]

From (18) it follows that \( \gamma \geq 1 \). If \( \gamma = 1 \), then \( c = 0, \) \( C_1^2 = E_1 \) so that \( C_1 = E_1 \) since \( C_1 \) is positive definite. Thus, \( A \) satisfies \( 3° \).

Let thus \( \gamma > 1 \) so that \( c \neq 0 \). From (17) it follows easily that, since \( C_1^2 = E_1 + cc^* \) and \( C_1 \) is positive definite,

\[ C_1 = E_1 + \frac{1}{c^*c} [(1 + c^*c)^{1/2} - 1] cc^*. \]

Consequently, if we put \( a = \gamma, b = c(\gamma^2 - 1)^{-1/2} \), we obtain \( C \) in the form (16).

The implication \( 3° \to 1° \) is a very simple consequence of (15), (16) and the fact that both properties are invariant under diagonal congruency. The proof is complete.

4. Applications. We shall show first that the conditions (7)–(9) involve necessary and sufficient conditions for the lengths of \( 2n \) vectors forming a biorthogonal basis in a (real or complex) \( n \)-dimensional unitary vector space \( X_n \). Here, we denote by \((x, y)\) the scalar product of vectors \( x \) and \( y \) and by \(|x|\) the length \((x, x)^{1/2}\) of the vector \( x \).

Two bases \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) are said to form a biorthogonal system\(^3\) if \((a_i, b_j) = \delta_{ij} \) (\( \delta_{ij} \) is the Kronecker symbol) for \( i, j = 1, \ldots, n \) or, equivalently, if the basis \( a_1, \ldots, a_n \) is an image of an orthonormal basis \( e_1, \ldots, e_n \) by a regular linear mapping \( C \) while the basis \( b_1, \ldots, b_n \) is image of \( e_1, \ldots, e_n \) by the inverse adjoint mapping \( C^{-1} \): \( a_i = Ce_i, b_i = C^{-1}e_i, \quad i = 1, \ldots, n \).

It is well known that to any basis \( a_1, \ldots, a_n \) in \( X_n \) there exists a (single) basis \( b_1, \ldots, b_n \) forming with the preceding basis a biorthogonal system.

(4,1). Theorem. Let \( a_1, \ldots, a_n, b_1, \ldots, b_n \) be a biorthogonal system in a unitary \( n \)-dimensional vector space \( X_n \). Then, the lengths \( \alpha_i = |a_i|, \beta_j = |b_j|, i, j = 1, \ldots, n, \) fulfill the inequalities

(21) \[ \alpha_i \beta_i \geq 1, \quad (i = 1, \ldots, n), \]

(22) \[ \alpha_i \beta_i - 1 \leq \sum_{j=1}^{n} (\alpha_j \beta_j - 1) \quad (i = 1, \ldots, n). \]

\(^3\) See e. g. [1].
Conversely, if $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ are $2n$ non-negative numbers satisfying (21), (22), then there exists a biorthogonal system $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $|a_i| = \alpha_i$, $|b_j| = \beta_j$, $i, j = 1, \ldots, n$.

Further, equality is attained in (21) if and only if $a_i$ and $b_i$ are linearly dependent. In (22), equality holds if and only if the angles of the vectors $a_j, \beta_k$ ($j, k = 1, \ldots, n, j \neq i \neq k$) are equal to the corresponding angles between $b_j, b_k$ while the angles between $a_i$ and $a_k$ ($k = 1, \ldots, n, k \neq i$) are equal to the corresponding angles between $-b_i$ and $b_k$.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis in $X_n$ and $C$ such a linear mapping that (20) holds. Then,

$$\alpha_i^2 = (Ce_i, Ce_i) = (C^*Ce_i, e_i) = (Ae_i, e_i)$$

where $A = C^*C$ is a positive definite mapping in $X_n$. Analogously,

$$\beta_i^2 = (C^{-1}e_i, C^{-1}e_i) = (C^{-1}C^*e_i, e_i) = (A^{-1}e_i, e_i).$$

Thus, (21) and (22) are identical with conditions (8) and (9) for the corresponding matrix with elements $a_{ij} = (Ae_i, e_j)$.

Conversely, if (21) and (22) hold, there exists a positive definite matrix $(a_{ij})$ (which may be chosen real) such that relations $a_{ii} = \alpha_i^2$, $a_{ii} = \beta_i^2$ ($i = 1, \ldots, n$) are satisfied where $\alpha_{ii}$ are diagonal elements of the inverse matrix $(\alpha_{ij})$ to $(a_{ij})$. Consequently, there exists a positive definite mapping $A$ fulfilling $a_{ij} = (Ae_i, e_j)$, $a_{ij} = (A^{-1}e_i, e_j)$. If we choose a mapping $C$ such that $C^*C = A$ (and it is possible to choose $C$ positive definite), then the vectors $a_1, \ldots, a_n, b_1, \ldots, b_n$ from (20) form a biorthogonal system. Since then

$$a_{ij} = (Ce_i, Ce_j) = (a_i, a_j)$$

and

$$\alpha_{ij} = (C^{-1}e_i, C^{-1}e_j) = (b_i, b_j),$$

it follows easily from (3,2) that in (21) equality holds if and only if $(a_i, a_j) = 0$ for $j = 1, \ldots, n, j \neq i$, i.e. if $a_i = \lambda b_i$ (since $(b_i, a_j) = 0$ for $j = 1, \ldots, n, j \neq i$, as well). According to the equivalence of $1^\circ$ and $2^\circ$ in (3,3), equality in (22) is attained if and only if

$$\frac{(a_k, a_l)}{|a_k| |a_l|} = \frac{(b_k, b_l)}{|b_k| |b_l|}$$

for $k, l = 1, \ldots, n$, $k \neq i \neq l$,

while

$$\frac{(a_i, a_j)}{|a_i| |a_j|} = -\frac{(b_i, b_j)}{|b_i| |b_j|}$$

for $j = 1, \ldots, n, j \neq i$.

The proof is complete.
(4.2) Theorem. If \(a_1,\ldots,a_n, b_1,\ldots,b_n\) is a biorthogonal system in a unitary space, then the angles \(\omega_i, 0 \leq \omega_i < \frac{1}{2}\pi\), between the corresponding vectors \(a_i\) and \(b_i\) fulfill inequalities

\[
\sec \omega_i - 1 \leq \sum_{j=1, j\neq i}^{n} (\sec \omega_j - 1), \quad i = 1, \ldots, n.
\]

Conversely, if \(\omega_1, \ldots, \omega_n\) are zero or acute angles satisfying (23) (or the single inequality (23) for that \(i\) for which \(a_i = \max_{k=1,\ldots,n} a_k\)), then there exists a biorthogonal system \(a_1, \ldots, a_n, b_1, \ldots, b_n\) such that \(\omega_j\) is the angle between \(a_j\) and \(b_j, j = 1, \ldots, n\).

Proof. Follows immediately from the preceding theorem since

\[
\cos \omega_i = \frac{(a_i, b_i)}{|a_i||b_i|} = \frac{1}{|a_i||b_i|}.
\]

In the next application, we shall use the notion of the spherical \(m\)-simplex. This will mean essentially a system of \(m + 1\) linearly independent directions \(\delta_1, \ldots, \delta_{m+1}\) in a Euclidean \((m + 1)\)-space \(E_{m+1}\). We shall call altitude-angle of the spherical \(m\)-simplex corresponding to the vertex-direction \(\delta_i\) the angle (acute or right) \(\varphi_i\) between \(\delta_i\) and the hyperplane in \(E_{m+1}\) which contains all \(\delta_j\) for \(j \neq i\). It is easy to see that, if \(a_1, \ldots, a_{m+1}\) are any non-zero vectors such that \(a_i (i = 1, \ldots, m+1)\) is of direction \(\delta_i\) and \(b_1, \ldots, b_{m+1}\) are vectors forming together with \(a_1, \ldots, a_{m+1}\) a biorthogonal system in \(E_{m+1}\), then \(\omega_i = \frac{1}{2}\pi - \varphi_i\) is the angle between \(a_i\) and \(b_i\) \((i = 1, \ldots, m+1)\).

Conversely, if \(a_1, \ldots, a_{m+1}, b_1, \ldots, b_{m+1}\) is a biorthogonal system in \(E_{m+1}\) and \(\omega_i\) is the angle between \(a_i\) and \(b_i\) \((i = 1, \ldots, m+1)\), then \(\varphi_i = \frac{1}{2}\pi - \omega_i\) is the altitude-angle corresponding to the direction of \(a_i\) in the spherical \(m\)-simplex whose vertex-directions are the directions of \(a_1, \ldots, a_{m+1}\).

From this observation and (4.2) the following theorem follows immediately:

(4.3) Theorem. Let \(\varphi_1, \ldots, \varphi_{m+1}\) be the altitude-angles of a spherical \(m\)-simplex. Then,

\[
cosec \varphi_i - 1 \leq \sum_{j=1, j\neq i}^{m+1} (\cosec \varphi_j - 1), \quad i = 1, \ldots, m + 1.
\]

Conversely, if \(\varphi_1, \ldots, \varphi_{m+1}\) are acute or right angles satisfying (24) (or the single inequality (24) with such \(i\) that \(\varphi_i = \min_k \varphi_k\)), then there exists a spherical \(m\)-simplex whose altitude-angles are \(\varphi_i (i = 1, \ldots, m + 1)\).

Remark. It can be proved that, if equality in (24) is attained, the corresponding \(m\)-simplex is orthocentric (i.e., his “altitudes” have a common direction) and satisfies a further condition.
Резюме

СООТНОШЕНИЯ МЕЖДУ ДИАГОНАЛЬНЫМИ ЭЛЕМЕНТАМИ ДВУХ ВЗАИМНО ОБРАТНЫХ ПОЛОЖИТЕЛЬНО ОПРЕДЕЛЕННЫХ МАТРИЦ

МИРОСЛАВ ФИДЛЕР (Miroslav Fiedler), Прага

Доказывается следующая теорема: Необходимым и достаточным условием для того, чтобы 2n действительных чисел $a_{11}, ..., a_{nn}, a_{11}, ..., a_{nn}$ образовали системы диагональных элементов положительно определенной матрицы $A = (a_{ii})$ и диагональных элементов обратной к ней матрицы $A^{-1} = (a_{ij})$, является одновременное выполнение следующих неравенств для $i = 1, ..., n$:

$$a_{ii} > 0, \quad a_{ii} > 0, \quad a_{ii} a_{ii} \geq 1,$$

$$\sqrt{a_{ii} a_{ii}} - 1 \leq \sum_{j=1, j \neq i}^{n} \left[ \sqrt{(a_{jj} a_{jj})} - 1 \right].$$

Далее характеризуются случаи равенства и дается геометрическое истолкование этой теоремы как условия, налагаемого на длины векторов биортогональной системы, на углы соответствующих векторов биортогональной системы или на высоты сферического симплекса.