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PRODUCT DECOMPOSITION OF IDEMPOTENT MEASURES ON COMPACT SEMIGROUPS

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In this note the results of paper [2] are used to prove a decomposition theorem for idempotent measures on compact semigroups.

Let S be a compact semigroup, $\mathfrak{M}(S)$ the semigroup of regular probability measures on S, the multiplication being the convolution (see [2]).

If ε is an idempotent $\in \mathfrak{M}(S)$ the support $C(\varepsilon)$ of ε is known to be a closed simple (hence completely simple) subsemigroup T of S.

The purpose of this note is to give a product decomposition of idempotents $\varepsilon \in \mathfrak{M}(S)$.

In the finite case a product decomposition of ε has been given by M. ROSENBLATT in [1]. He uses the fact that a compact simple semigroup T can be represented as the product space $X \times G \times Y$ of a compact topological group G and compact Hausdorff spaces X and Y, where the multiplication is given by $(x, g, y)(x', g', y') = (x, g\varphi(x', y) g', y')$ with φ a continuous function on the product space $X \times Y$ into G. (See A. D. WALLACE [4], [5].) He has proved that ε can be written as a product measure $\varepsilon = \varrho \times \mu \times \tau$, where μ is the normed Haar measure on the finite group G and ϱ, τ are probability measures on finite spaces X and Y respectively. Prof. U. GRENANDER has kindly informed me that in the meantime M. Rosenblatt (in a unpublished paper, which he has given to our disposal) has extended his result to the case that T is compact.

We give here a decomposition of ε in terms of a convolution product in the case that $T = C(\varepsilon)$ is a compact semigroup with a finite number of idempotents. This turns out to be an immediate corollary of the results of paper [2] (see also [3]). The transition to the product measure is sketched at the end of the note.

Though the result itself is now surpassed by that of M. Rosenblatt mentioned above, it seems perhaps to be worth to publish it not only owing to its simplicity but also since our method is quite a different one. We mention also that there may be a special interest in the case that T contains only a finite number of idempotents. For, if T is a closed simple subsemigroup of S there need not exist in general an idempotent $\varepsilon \in \mathfrak{M}(S)$ with $C(\varepsilon) = T$. But such an idempotent always exists if T contains only a finite number of idempotents (see [2]).

A. For simplicity we may and we shall suppose that $T = S$. It is known that S can be written in the form $S = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta$, R_α, L_β running through all minimal right and left ideals of S respectively. Also $R_\alpha L_\beta = R_\alpha \cap L_\beta = G_{\alpha\beta}$ is a closed group, and $S = \bigcup_\alpha \bigcup_\beta G_{\alpha\beta}$.

Let $\mu_{\alpha\beta}$ be the normed Haar measure on the group $G_{\alpha\beta}$ and extend the definition of $\mu_{\alpha\beta}$ to all Borel subsets A of S by putting $\mu_{\alpha\beta}(A) = \mu_{\alpha\beta}(A \cap G_{\alpha\beta})$. In [2] we have proved (the convolution being denoted by juxtaposition):

a) If $g_{\gamma\delta}$ is the point mass at the point $g_{\gamma\delta} \in G_{\gamma\delta} \subset S$, we have $\mu_{\alpha\beta} g_{\gamma\delta} = \mu_{\alpha\delta}$ and $g_{\gamma\delta} \mu_{\alpha\beta} = \mu_{\gamma\beta}$.

b) $\mu_{\alpha\beta}\mu_{\gamma\delta} = \mu_{\alpha\delta}$.

c) If ν is any element $\in \mathfrak{M}(S)$, then $\mu_{\alpha\beta}\nu\mu_{\gamma\delta} = \mu_{\alpha\delta}$.

B. Suppose $1 \in \Lambda_1 \cap \Lambda_2$. Let E_0 be the set of all idempotents $\in S$ and R_1 and L_1 fixed chosen minimal right and left ideals of S respectively. Denote $E = E_0 \cap L_1 = \{e_{\alpha 1} \mid \alpha \in \Lambda_1\}$, the set of all idempotents $\in L_1$. Analogously denote $F = E_0 \cap R_1 = \{e_{1\beta} \mid \beta \in \Lambda_2\}$, the set of all idempotents $\in R_1$. It is easy to see that E and F are closed subsemigroups of S.

In the following theorem it is not necessary to suppose that S contains only a finite number of idempotents.

Theorem 1. *Let ν_l and ν_r be any probability measures on S and μ_{11} the normed Haar measure on G_{11}. Then $\varepsilon = \nu_l \mu_{11} \nu_r$ is an idempotent $\in \mathfrak{M}(S)$. If in particular $C(\nu_l) = E$, $C(\nu_r) = F$, then $C(\varepsilon) = S$.*

Proof. We have $\varepsilon^2 = \nu_l \mu_{11} \nu_r \nu_l \mu_{11} \nu_r = \nu_l \mu_{11} \varrho \mu_{11} \nu_r$, where $\varrho = \nu_r \nu_l$. Since $\mu_{11} \varrho \mu_{11} = \mu_{11}$, we have $\varepsilon^2 = \varepsilon$.

If $C(\nu_l) = E$, $C(\nu_r) = F$, we have

$$C(\varepsilon) = C(\nu_l)\, C(\mu_{11})\, C(\nu_r) = E\, G_{11}\, F = [\bigcup_\alpha e_{\alpha 1}]\, G_{11} [\bigcup_\beta e_{1\beta}] = \bigcup_\alpha \bigcup_\beta e_{\alpha 1} G_{11} e_{1\beta}\,.$$

Since (with respect to well known properties of completely simple semigroups) $e_{\alpha 1} G_{11} e_{1\beta} = G_{\alpha\beta}$, we have $C(\varepsilon) = \bigcup_\alpha \bigcup_\beta G_{\alpha\beta} = S$. This proves our assertion.

In the special case considered in the paper [2] we prove the converse.

Theorem 2. *With the same notations as above let S be a compact simple semigroup containing a finite number of idempotents: $S = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$. Let ε be any idempotent $\in \mathfrak{M}(S)$ with $C(\varepsilon) = S$. Then $\varepsilon = \nu_l \mu_{11} \nu_r$, where μ_{11} is the normed Haar measure on* *the compact group G_{11} and ν_l, ν_r are suitably chosen probability measures with $C(\nu_l) = E$ and $C(\nu_r) = F$.*

Proof. In the paper [2] we have proved that any idempotent ε with $C(\varepsilon) = S$ can be written in the form $\varepsilon = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta \mu_{\alpha\beta}$, where ξ_α, η_β are positive numbers satisfying $\sum_{\alpha=1}^{s} \xi_\alpha = \sum_{\beta=1}^{r} \eta_\beta = 1$. Since $\mu_{\alpha\beta} = e_{\alpha 1} \mu_{11} e_{1\beta}$, we can write $\varepsilon = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \xi_\alpha \eta_\beta e_{\alpha 1} \mu_{11} e_{1\beta} = (\sum_{\alpha=1}^{s} \xi_\alpha e_{\alpha 1}) \mu_{11} (\sum_{\beta=1}^{r} \eta_\beta e_{1\beta})$. Denoting $\sum_{\alpha=1}^{s} \xi_\alpha e_{\alpha 1} = \nu_l$, $\sum_{\beta=1}^{r} \eta_\beta e_{1\beta} = \nu_r$, we have $\varepsilon = \nu_l \mu_{11} \nu_r$ and $C(\nu_l) = E$, $C(\nu_r) = F$. This proves Theorem 2.

C. We sketch briefly the transition from the convolution product to the product measure.

If p is the map of $S \times S$ into S defined by the multiplication in S and A a Borel subset of S, the convolution product $\varrho\tau(\varrho \in \mathfrak{M}(S), \tau \in \mathfrak{M}(S))$ can be defined by $\varrho\tau(A) = (\varrho \times \tau) [p^{-1}(A)]$. Hereby $p^{-1}(A) = \{(x, y) \mid x \in S, \ y \in S, \ xy \in A\}$ and it is a Borel subset of $S \times S$, if we suppose moreover (as we do in the following) that S is separable.

Let E, F, and G_{11} be as above. Denote $Z = E \times G_{11} \times F$ and define the multiplication in Z by $(e, g, f)(e', g', f') = (e, gfe'g', f')$. Then Z is a semigroup. Define m: $Z \to S$ by $m(e, g, f) = egf$. Then the mapping m is a topological isomorphism. (See [4], [5].)

We have proved that an idempotent $\varepsilon \in \mathfrak{M}(S)$ (under the restriction mentioned in Theorem 2) can be written in the form of a convolution product $\varepsilon = \nu_l \mu_{11} \nu_r$. By definition of the convolution we have for a Borel subset $A \subset S$ $\varepsilon(A) = \nu_l \mu_{11} \nu_r(A) = (\nu_l \times \mu_{11} \times \nu_r) [p^{-1}(A)]$, where $p^{-1}(A) = \{(x, y, z), x \in S, y \in S, z \in S, xyz \in A\}$. Now since $C(\nu_l) = E$, $C(\mu_{11}) = G_{11}$, $C(\nu_r) = F$, this implies $\varepsilon(A) = (\nu_l \times \mu_{11} \times \nu_r) . [m^{-1}(A)]$. Hence any such idempotent ε is representable as a product measure $\nu_l \times \mu_{11} \times \nu_r$, where μ_{11} is the normed Haar measure on G_{11}, ν_l, ν_r are probability measures on E and F respectively.

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Резюме

РАЗЛОЖЕНИЕ ИДЕМПОТЕНТНЫХ МЕР НА БИКОМПАКТНЫХ ПОЛУГРУППАХ

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В этой заметке доказывается одно следствие результатов работы [2], касающееся разложения идемпотентных мер на бикомпактных некоммутативных полугруппах.