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ON THE STRUCTURE OF A CLASS OF COMMUTATIVE SEMIGROUPS

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The purpose of the paper is to clarify the structure of a special type of commutative semigroups to which several authors have been led by studying decompositions of general semigroups.

1. Introduction and summary. In this paper we investigate the structure of a class of semigroups which we call N -semigroups. An N -semigroup is a commutative non-potent archimedean cancellative semigroup. E. HEWITT and H. S. ZUCKERMAN [2] have shown that if G is any commutative semigroup, then there is a maximal separative homomorphic image G' of G , and that a member H_x of the maximal semilattice decomposition of G' is either a group or an N -semigroup. T. TAMURA [5] has given a characterization of N -semigroups. T. TAMURA and N. KIMURA [4] have shown that a member of the maximal semilattice decomposition of an arbitrary commutative semigroup is archimedean and has at most one idempotent. Š. SCHWARZ [6] has established certain properties of decompositions of a semigroup similar to those already mentioned.

In section 2 we define N -semigroups and discuss some properties of commutative semigroups in connection with it. Then in section 3 we establish a property of N -semigroups with a finite number of generators. In section 4 we find the structure of N -semigroups with two generators. Finally in section 5 we give a classification and several examples of N -semigroups.

A semigroup is a non-empty set on which an associative multiplication is defined. We will discuss only commutative semigroups. Throughout the whole paper S will denote an arbitrary commutative semigroup unless stated otherwise. We follow the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1] for all concepts not defined in the paper. By $x^m y^n$ with $m = 0$ and $n > 0$, we mean y^n .

The writer wishes to thank Professor *Edwin Hewitt* for mentioning this problem to him, and Professor *Herbert S. Zuckerman* for his help in preparation of this paper.

2. Definitions and properties. We first define an N -semigroup and then discuss some properties of semigroups in connection with it.

Definition. S will be called an N -semigroup if it has the following properties:

(A) for every $x, y, z \in S$, $xz = yz$ implies $x = y$ (S is cancellative);
 (B) for every $x, y \in S$, $x^n = ay$ for some $a \in S$ and some natural number n (S is archimedean);

(C) S has no idempotents (S is nonpotent).

Proposition 1. *If S satisfies condition (B) and*

(D) *for every $x, y \in S$, $x^2 = y^2 = xy$ implies $x = y$ (S is separative), then S also satisfies (A).*

Proof. Let S satisfy conditions (B) and (D) and suppose that $xz = yz$ for some $x, y, z \in S$. Then $x^m = az$ and $y^n = bz$ for some $a, b \in S$ and some m, n . Hence

$$(1) \quad x^{m+1} = (az)x = a(xz) = a(yz) = (az)y = x^m y,$$

$$(2) \quad y^{n+1} = (bz)y = b(yz) = b(xz) = (bz)x = y^n x.$$

If in (1) $m > 1$, then

$$x^{2m-2}xy = x^{m-2}x^{m+1}y = x^{m-2}x^m y^2 = x^{2m-2}y^2,$$

$$x^{2m-2}xy = x^{m-1}x^m y = x^{m-1}x^{m+1} = x^{2m}.$$

Consequently $(x^{m-1}y)^2 = (x^m)^2 = x^m(x^{m-1}y)$ and thus $x^m = x^{m-1}y$. After $m - 1$ steps, we obtain $x^2 = xy$. Similarly from (2), we obtain $y^2 = xy$ and therefore $x = y$.

Corollary. *In the definition of an N -semigroup, we can substitute condition (A) by the weaker condition (D).*

The proofs of the following statements are either contained in the works mentioned at the beginning of the paper or in [3].

Proposition 2. *If S satisfies (B), then it contains at most one idempotent.*

Proposition 3. *If S satisfies (A) and (B) and does not satisfy (C), then it is a group.*

Theorem 1. *S contains no prime (proper semiprime) ideals if and only if S satisfies (B).*

Theorem 2. *Each member of the maximal semilattice decomposition of S satisfies (B).*

Theorem 3. *S satisfies (D) if and only if S is a semilattice of semigroups each of which satisfies (A).*

3. Finitely generated N -semigroups. The set of positive integers under addition is an N -semigroup generated by the element 1. It is evident that this is the only cyclic N -semigroup. The following theorem establishes a property of N -semigroups with a finite number of generators.

4. N -semigroups with two generators. In this section we find the structure and give a concrete realization of N -semigroups with two generators. We do not consider N -semigroups with more than two generators because the characterization given in this paper becomes too involved in such a case. We first introduce some notations.

Notation. Let S be an N -semigroup with two generators, say a_1 and a_2 . Let m_1 and m_2 be the smallest positive integers such that $a_1^{m_1} = ua_2$ and $a_2^{m_2} = va_1$ for some $u, v \in S$. We suppose that $m_1 \leq m_2$ and denote S by $N(m_1, m_2)$.

Since S is cancellative, a_2 is a generator of S , and m_1 is minimal, we have $u = a_2^{k_2}$ for some $k_2 > 0$, and thus $a_1^{m_1} = a_2^{k_2+1}$. Similarly $a_2^{m_2} = a_1^{k_1+1}$ for some $k_1 > 0$. By minimality of m_1 and m_2 , $m_1 \leq k_1 + 1$ and $m_2 \leq k_2 + 1$. If $k_2 + 1 > m_2$, then

$$a_1^{m_1} = a_2^{k_2+1} = a_2^{m_2} a_2^{k_2+1-m_2} = a_1^{k_1+1} a_2^{k_2+1-m_2}$$

which is impossible since $m_1 \leq k_1 + 1$. Thus $m_2 = k_2 + 1$, that is, $a_1^{m_1} = a_2^{m_2}$.

Notation. Let m_1 and m_2 be integers such that $2 \leq m_1 \leq m_2$. A set S will be denoted by (m_1, m_2) -s.g. if

$$S = \{(k_1, k_2) \mid k_1 = 0, 1, 2, \dots, \quad k_2 = 0, 1, 2, \dots, m_2 - 1, \quad k_1 + k_2 > 0\}$$

with multiplication

$$(k_1, k_2)(l_1, l_2) = (k_1 + l_1 + jm_1, k_2 + l_2 - jm_2)$$

where j is the integer such that $0 \leq k_2 + l_2 - jm_2 < m_2$, and $(k_1, k_2) = (l_1, l_2)$ implies $k_1 = l_1$ and $k_2 = l_2$.

The following theorem gives a simple characterization of N -semigroups with two generators:

Theorem 5. *Let S be a set. Then $S = N(m_1, m_2)$ if and only if $S = (m_1, m_2)$ -s.g.*

Proof. We first prove *necessity*. Thus let $S = N(m_1, m_2)$ with the generators a_1 and a_2 such that $a_1^{m_1} = a_2^{m_2}$. If x is an element of S , then $x = a_1^{k_1} a_2^{k_2}$ for some non-negative integers k_1 and k_2 such that $k_1 + k_2 > 0$. We have $0 \leq k_2 - jm_2 < m_2$ for some non-negative integer j , and hence

$$a_1^{k_1} a_2^{k_2} = a_1^{k_1} a_2^{jm_2 + (k_2 - jm_2)} = a_1^{k_1 + jm_1} a_2^{k_2 - jm_2}.$$

Thus every element of S can be written in the form of an element of (m_1, m_2) -s.g. with a suitable change of notation. One checks similarly that the multiplication of S coincides with that of (m_1, m_2) -s.g. under the restriction that $0 \leq k_2 < m_2$ where $a_1^{k_1} a_2^{k_2} \in S$. Suppose that $a_1^{k_1} a_2^{k_2} = a_1^{l_1} a_2^{l_2}$ with $0 \leq k_2, l_2 < m_2$. If $k_1 > l_1$, then $a_1^{k_1 - l_1} a_2^{k_2} = a_2^{l_2}$ and thus necessarily $k_2 < l_2$. Consequently $a_1^{k_1 - l_1} = a_2^{l_2 - k_2}$ and thus $l_2 - k_2 \geq m_2$ by minimality of m_2 . But this contradicts the hypothesis that $l_2 < m_2$. The case $k_1 < l_1$ is symmetric. Hence $k_1 = l_1$ and consequently $k_2 = l_2$. Therefore $S = (m_1, m_2)$ -s.g.

We next prove *sufficiency*. Let $S = (m_1, m_2)$ -s.g. It is clear that S is closed under its multiplication and is commutative. We verify the postulates for $N(m_1, m_2)$.

Associativity. It is easily seen that both $[(k_1, k_2)(l_1, l_2)](r_1, r_2)$ and $(k_1, k_2) \cdot [(l_1, l_2)(r_1, r_2)]$ are equal to $(k_1 + l_1 + r_1 + im_1, k_2 + l_2 + r_2 - im_2)$ where i is the integer such that $0 \leq k_2 + l_2 + r_2 - im_2 < m_2$. Hence the associative law holds.

Condition (A). If $(k_1, k_2)(r_1, r_2) = (l_1, l_2)(r_1, r_2)$, then

$$(k_1 + r_1 + im_1, k_2 + r_2 - im_2) = (l_1 + r_1 + jm_1, l_2 + r_2 - jm_2)$$

where i and j are the integers such that $0 \leq k_2 + r_2 - im_2 < m_2$ and $0 \leq l_2 + r_2 - jm_2 < m_2$. Hence

$$(1) \quad k_1 + r_1 + im_1 = l_1 + r_1 + jm_1,$$

$$(2) \quad k_2 + r_2 - im_2 = l_2 + r_2 - jm_2.$$

From (2) we obtain $k_2 - l_2 = (i - j)m_2$ which implies that $i = j$ and consequently $k_2 = l_2$. But $i = j$ in (1) yields $k_1 = l_1$. Therefore $(k_1, k_2) = (l_1, l_2)$ and the cancellation law holds in S .

Condition (B). Now let (k_1, k_2) and (l_1, l_2) be any elements of S . If $k_1 > 0$, then let $n = m_1 + l_1 + 1$ and q be the non-negative integer such that $0 \leq nk_2 - m_2q < m_2$. If $k_1 = 0$, then let $q = m_1 + l_1 + 1$ and n be the integer satisfying the inequality $m_2q/k_2 \leq n < (m_2q/k_2) + 1$ (in this case $k_2 > 0$). Moreover let

$$j = \begin{cases} 1 & \text{if } nk_2 - m_2q - l_2 < 0, \\ 0 & \text{if } nk_2 - m_2q - l_2 \geq 0. \end{cases}$$

It follows easily from the definitions of n , q , and j that

$$(a) \quad nk_1 + m_1q - l_1 - jm_1 > (m_1 + l_1) - l_1 - jm_1 \geq 0;$$

$$(b) \quad nk_1 + m_1q > 0;$$

$$(c) \quad 0 \leq nk_2 - m_2q < m_2;$$

$$(d) \quad 0 \leq nk_2 - m_2q - l_2 + jm_2 < m_2.$$

We therefore have

$$\begin{aligned} (nk_1 + m_1q - l_1 - jm_1, nk_2 - m_2q - l_2 + jm_2)(l_1, l_2) &= \\ &= (nk_1 + m_1q, nk_2 - m_2q) = (k_1, k_2)^n. \end{aligned}$$

Condition (C). Suppose now that $(k_1, k_2)^2 = (k_1, k_2)$ for some $(k_1, k_2) \in S$. Then

$$(2k_1 + jm_1, 2k_2 - jm_2) = (k_1, k_2)$$

where j is the integer such that $0 \leq 2k_2 - jm_2 < m_2$. Hence $2k_1 + jm_1 = k_1$ and $2k_2 - jm_2 = k_2$. Consequently $k_2 - jm_2 = 0$ which implies that $j = 0$ and hence also $k_2 = 0$. But $j = 0$ in the first equation yields $k_1 = 0$, which is impossible. Therefore S has no idempotents.

$S = N(m_1, m_2)$. We have already proved that S is an N -semigroup. It is clear that $(1, 0)$ and $(0, 1)$ are the generators of S . Let n be the smallest positive integer such that $(1, 0)^n = (k_1, k_2)(0, 1)$ for some $(k_1, k_2) \in S$. Then $k_1 = 0$ by minimality of n . Hence $(1, 0)^n = (0, k_2)(0, 1)$ and thus $(n, 0) = (jm_1, k_2 + 1 - jm_2)$ where j is the integer such that $0 \leq k_2 + 1 - jm_2 < m_2$. Therefore $n = jm_1$ and $0 = k_2 + 1 - jm_2$. Consequently $j = 1$ and thus $n = m_1$ and $k_2 = m_2 - 1$.

Similarly we see that if n is the smallest positive integer such that $(0, 1)^n = (k_1, k_2)(1, 0)$ for some $(k_1, k_2) \in S$, then $n = m_2$. Since $m_1 \leq m_2$, we have proved that $S = N(m_1, m_2)$.

Remark. From the definition of (m_1, m_2) -s.g. it follows easily that (m_1, m_2) -s.g. is isomorphic to the free commutative semigroup on two generators, say a_1 and a_2 , with the defining relation $a_1^{m_1} = a_2^{m_2}$. This furnishes a second characterization of $N(m_1, m_2)$ by virtue of Theorem 5.

We next give a concrete realization of (m_1, m_2) -s.g. First we introduce some notation.

Notation. For integers m_1 and m_2 such that $2 \leq m_1 \leq m_2$, let $C(m_1, m_2)$ be the subsemigroup of the group of non-zero complex numbers, generated by the two elements

$$a_1 = 2^{1/m_1} e^{(2\pi i)/m_1} \quad \text{and} \quad a_2 = 2^{1/m_2} e^{(4\pi i)/m_2}.$$

Then we have the following result:

Theorem 6. *The semigroups $C(m_1, m_2)$ and (m_1, m_2) -s.g. are isomorphic.*

Proof. First note that $a_1 \neq a_2$ because $2 \leq m_1 \leq m_2$. Since $C(m_1, m_2)$ is a sub-semigroup of a commutative group (non-zero complex numbers), by the remark above it suffices to show that $a_1^{k_1} = a_2^{k_2}$ implies $k_1/k_2 = m_1/m_2$ and $k_1 \geq m_1$. If $a_1^{k_1} = a_2^{k_2}$, that is,

$$2^{k_1/m_1} e^{[(2\pi i)/(m_1)]k_1} = 2^{k_2/m_2} e^{[(4\pi i)/(m_2)]k_2},$$

then by equating moduli, we obtain $k_1/m_1 = k_2/m_2$. But then

$$e^{2\pi i[2(k_1/m_1) - (k_1/m_1)]} = 1$$

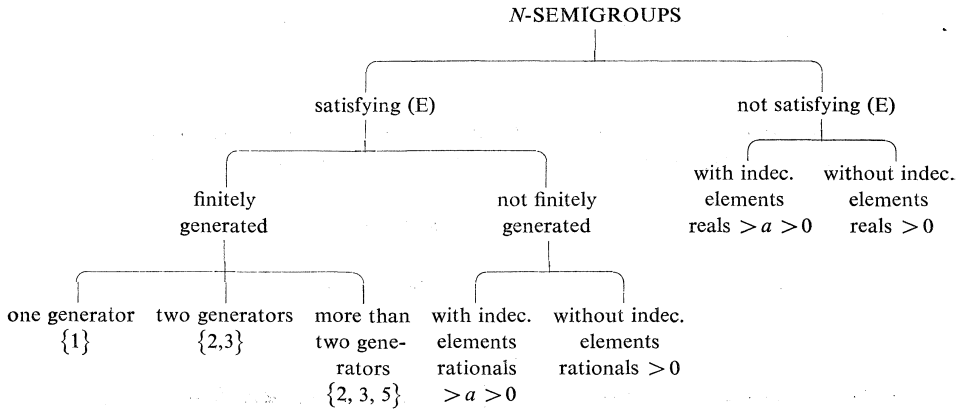
and consequently $k_1 \geq m_1$. The theorem follows.

The following is essentially a resume of some results of this section:

Theorem 7. *The semigroup $C(m_1, m_2)$ is an N -semigroup with two-generators and $C(m_1, m_2) = C(m'_1, m'_2)$ only if $m_1 = m'_1$ and $m_2 = m'_2$. Conversely, every N -semigroup with two generators is isomorphic to the semigroup $C(m_1, m_2)$ for some (unique) integers m_1 and m_2 , $2 \leq m_1 \leq m_2$.*

5. Classification and examples of N -semigroups. We classify N -semigroups according to whether they satisfy condition (E) of Theorem 4, whether they are finitely generated, and whether they contain indecomposable elements. We say that an element x of S is indecomposable if $x \neq yz$ for all $y, z \in S$. We also give an example

for each group of N -semigroups. All the examples given are subsemigroups of the additive semigroup of positive real numbers. The numbers in brackets denote the generators of S ; the letter a denotes any positive real number.



References

- [1] Clifford A. H. and G. B. Preston: The Algebraic Theory of Semigroups. Vol. I. American Math. Society, 1961.
- [2] Hewitt Edwin and H. S. Zuckerman: The l_1 -algebra of a commutative semigroup. Trans. A.M.S., 83 (1956), 70—97.
- [3] Petrich Mario: The greatest semilattice decomposition of a semigroup (to appear).
- [4] Tamura T. and N. Kimura: On decompositions of a commutative semigroup. Kōdai Math. Sem. Rep., 4 (1954), 109—112.
- [5] Tamura T.: Commutative nonpotent archimedean semigroup with cancellation law I. J. Gakugei Tokushima Univ., 8 (1957), 5—11.
- [6] Schwarz Štefan: O pologrupách splňujúcich zoslabené pravidla krátania. Matematicko-Fyzikálny Časopis, S.A.V. 6 (1956), 149—158.

Резюме

О СТРОЕНИИ ОПРЕДЕЛЕННОГО КЛАССА КОММУТАТИВНЫХ ПОЛУГРУПП

МАРИО ПЕТРИХ (Mario Petrich), Мариланд (США)

Коммутативная полугруппа S называется N -полугруппой, если 1. В S имеет место правило сокращения; 2. для каждой пары $x, y \in S$ существует $a \in S$ и целое число $n > 0$ так, что $x^n = ay$; 3. S не имеет идемпотентов. Работа посвящена изучению строения N -полугрупп. Именно, описана структура всех N -полугрупп, обладающих двумя генераторами.