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ON A TOPOLOGICAL RELATION BETWEEN A σ-ALGEBRA $\mathbf{A}$ OF SETS AND THE SYSTEM OF ALL $\mathbf{A}$-MEASURABLE FUNCTIONS

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In the present paper the following statement is proved: Every σ-algebra $\mathbf{A}$ of subsets of a given non-void point set X and the system of all $\mathbf{A}$-measurable real functions on X are non-homeomorphic sequentially regular convergence spaces.

In 1952 I put forward [1] the following problem: Are the class of all Borel linear sets and the class of all Baire functions homeomorphic? I solved the question by using notions of sequential regularity and zero-one sequential regularity of a convergence space. Actually more was proved, namely that a σ-algebra $\mathbf{A}$ of subsets of a non-empty set X cannot be homeomorphic to the system of all real-valued $\mathbf{A}$-measurable functions defined on X.

Let X be a non-empty point set, $\mathbf{X}$ the system of all subsets of X and $\mathbf{A}$ a σ-ring of subsets of X. Denote by $\mathfrak{F}$ the system of all real-valued functions defined on X and by $\mathfrak{M}$ the system of all $\mathbf{A}$-measurable functions. Convergence in $\mathbf{X}$ is defined by the well-known condition in the general theory of sets: $\lim A_n = A$ if $\liminf A_n = A = \limsup A_n$, where $\liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$ and $\limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$. Convergence in $\mathfrak{F}$ is defined as point-wise convergence of real functions in X. Both the systems $\mathbf{X}$ and $\mathfrak{F}$ are convergence spaces, their convergences fulfil two Fréchet's axioms $\mathscr{L}_1$, $\mathscr{L}_2$ and the Urysohn's axiom $\mathscr{L}_3$ of convergence [2]:

($\mathscr{L}_1$): if $x_n = x$ for each positive integer n then $\lim x_n = x$.

($\mathscr{L}_2$): if $\lim y_n = y$ then $\lim y_{n_i} = y$ for each subsequence $\{y_{n_i}\}$ of $\{y_n\}$.

($\mathscr{L}_3$): If a sequence $\{z_n\}$ does not converge to a point z then there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ no subsequence of which converges to z.

The closure λA of a subset A in a convergence space L is defined as the set of all limits of sequences of points x_n belonging to the set A. A set A is closed if $A = \lambda A$. It is easy to see that each finite subset of L and the set L itself are closed sets; the topology λ is additive ($\lambda(A \cup B) = \lambda A \cup \lambda B$) and monotone ($A \subset B$ implies $\lambda A \subset \lambda B$); in convergence space, however, the closure of a subset need not be closed. In the sequel we shall always assume that the convergence space L fulfils all three axioms of convergence $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$.

Let (L_1, λ_1) and (L_2, λ_2) be two convergence spaces and φ a map of L_1 into L_2. According to usual definition, the map φ is continuous if $\varphi(\lambda_1 A) \subset \lambda_2 \varphi(A)$ for each set $A \subset L_1$; the map φ is a homeomorphism if it is one-to-one and if $\varphi(\lambda_1 A) = \lambda_2\, \varphi(A)$ for each set $A \subset L_1$. We define the map φ to be sequentially continuous if $\lim x_n = x$ in L_1 implies $\lim \varphi(x_n) = \varphi(x)$ in L_2 for each point $x \in L_1$.

Lemma 1. *Let (L_1, λ_1) and (L_2, λ_2) be two convergence spaces (fulfilling all three axioms of convergence $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$). Let φ be a map of L_1 into L_2. Then φ is continuous if and only if it is sequentially continuous. The map φ is a homeomorphism if and only if it is one-to-one sequentially continuous map of L_1 onto L_2 and if also the inverse map φ^{-1} is sequentially continuous.*

Proof. *Is contained in the book* [4].

Definition 1. A convergence space L (fulfilling all three axioms of convergence $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$) is called *sequentially regular* [3] if for each point $x_0 \in L$ and each sequence of points $x_n \in L$ not converging to x_0 there exists a continuous function f on L such that the sequence of real numbers $f(x_n)$ does not converge to $f(x_0)$.

Lemma 2. *Let (L_1, λ_1) be a sequentially regular convergence space. Let h be a homeomorphism of L_1 onto a convergence space (L_2, λ_2). Then the space L_2 is also sequentially regular.*

Proof. Let $\{y_n\}$ be a sequence of points in L_2 not converging to a point $y_0 \in L_2$. From Lemma 1 it follows that the sequence of points $h^{-1}(y_n)$ fails to converge to the point $h^{-1}(y_0)$ in L_1. Because L_1 is a sequentially regular space, there is a continuous function f on L_1 such that the sequence $\{f(h^{-1}(y_n))\}$ does not converge to the number $f(h^{-1}(y_0))$. Consequently $g = fh^{-1}$ is a continuous function on L_2 such that $g(y_0)$ is not a limit of the sequence $\{g(y_n)\}$.

Definition 2. A convergence space L (fulfilling all three axioms of convergence $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$) is called *zero-one sequentially regular* if for each point $x_0 \in L$ and each sequence of points $x_n \in L$ not converging to the point x_0 there is a two-valued continuous function mapping L into $\{0, 1\}$ such that the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$.

It is possible to prove, in the same way as above, that zero-one sequential regularity of a convergence space is a topological property.

Lemma 3. *Each system $\mathbf{S}$ of subsets of an abstract point set S is a zero-one sequentially regular convergence space.*

Proof. It may be observed that $\mathbf{S}$ is a convergence space fulfilling all three axioms of convergence $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$. Assume that a sequence of sets $S_n \in \mathbf{S}$ does not converge to a set $S \in \boldsymbol{S}$; then there exists a point s belonging either to $\limsup S_n - \liminf S_n$ or to $(S - \lim S_n) \cup (\lim S_n - S)$. Define a set function f on $\boldsymbol{S}$ as follows: $f(A) = 1$ or $f(A) = 0$ according as s belongs to A or not. It is easy to see that the function f is continuous on $\boldsymbol{S}$ and that the sequence $\{f(S_n)\}$ does not converge to $f(S)$.

Lemma 4. *Let X be a point set and $\mathfrak{G}$ a system of real-valued functions on X. Then $\mathfrak{G}$ is a sequentially regular convergence space.*

Proof. First notice that $\mathfrak{G}$ is a convergence space fulfilling all three axioms of convergence $\mathscr{L}_1, \mathscr{L}_2$ and $\mathscr{L}_3$. Let g_0 be an element of $\mathfrak{G}$ and $\{g_n\}$ a sequence of functions $g_n \in \mathfrak{G}$ not converging to g_0. Then there is a point $x_0 \in X$ such that the sequence $\{g_n(x_0)\}$ does not converge to $g(x_0)$. Define a real-valued function h on $\mathfrak{G}$ by $h(f) = f(x_0)$ for each $f \in \mathfrak{G}$; evidently h is a sequentially continuous function such that the sequence $\{h(g_n)\}$ does not converge to $h(g_0)$.

Theorem. *Let X be a non-empty point set and $\boldsymbol{A}$ a σ-algebra of subsets of X. Then the system $\mathfrak{M}$ of all $\boldsymbol{A}$-measurable real-valued functions on X is a sequentially regular convergence space which is not zero-one sequentially regular.*

Proof. The first part of the assertion follows from Lemma 4. In order to prove the second part notice that $X \in \boldsymbol{A}$, so that each constant function c belongs to $\mathfrak{M}$. The value of c will be denoted by c'. Now, choose a sequence $\{c_n\}$ of constant functions $c_n \in \mathfrak{M}$ which does not converge to a constant function $c_0 \in \mathfrak{M}$. Suppose the contrary, that there is a continuous zero-one valued function g on $\mathfrak{M}$ such that the sequence of real numbers $g(c_n)$ does not converge to $g(c_0)$. Then there is a positive integer p such that $g(c_p) \neq g(c_0)$. The mapping $h = \{c' \to c\}$ is a homeomorphism and thus the function gh is continuous on the set of all real numbers. Since gh is a two-valued function, it follows that gh is constant. Thus we have the contradiction, that $g(c_p) = g(c_0)$.

Corollary. *Let X be a non-empty point set and $\boldsymbol{A}$ a σ-algebra of sets of X. Then the convergence space $\boldsymbol{A}$ is not homeomorphic to the convergence space $\mathfrak{M}$ of all $\boldsymbol{A}$-measurable functions defined on X.*

As a matter of fact, according to Lemma 3, the convergence space $\boldsymbol{A}$ is zero-one sequentially regular; on the other hand, by the Theorem, the space $\mathfrak{M}$ is not zero-one sequentially regular. Therefore the spaces $\boldsymbol{A}$ and $\mathfrak{M}$ cannot be homeomorphic, since zero-one sequential regularity is a topological property.

Remark. Since $\lim A_n = A$ in $\boldsymbol{A}$ if and only if $\lim c_{A_n}(x) = c_A(x)$ for each point $x \in X$, and because each characteristic function $c_B(x)$, $B \in \boldsymbol{A}$, is $\boldsymbol{A}$-measurable, it follows that the convergence space $\boldsymbol{A}$ is homeomorphic to a subspace of the system of all $\boldsymbol{A}$-measurable functions.

It is well known [5] that the system $\mathfrak{B}$ of all Baire functions is identical with the system of all Borel measurable functions. Consequently the system of all linear Borel sets is homeomorphic to a subsystem but not to the whole system of all Baire functions.

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Резюме

О ТОПОЛОГИЧЕСКОМ СООТНОШЕНИИ МЕЖДУ σ-АЛГЕБРОЙ **A** МНОЖЕСТВ И СИСТЕМОЙ ВСЕХ **A**-ИЗМЕРИМЫХ ФУНКЦИЙ

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Пространство сходимости L, выполняющее три аксиомы сходимости $\mathcal{L}_1$, $\mathcal{L}_2$ и $\mathcal{L}_3$, называется секвенциално регулярным [нуль один секвенциално регулярным], если для каждой точки $x_0 \in L$ и для каждой последовательности точек $x_n \in L$, которая не сходится к точке x_0, существует непрерывная функция f на L такая, что последовательность действительных чисел $f(x_n)$ не сходится к $f(x_0)$ [$(fx) = 0$ или $= 1$ для каждой точки $x \in L$].

Оба эти свойства будут иметь место при гомеоморфном отображении.

Примером секвенциално регулярных пространств служит σ-алгебра **A** подмножеств данного непустого множества X, где имеется сходимость $\lim A_n = A$, как только $A = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ и система $\mathfrak{M}$ всех **A**-измеримых вещественных функций, определенных на X, где имеется сходимость $\lim f_n = f$ как только $\lim f_n(x) = f(x)$ для каждой точки $x \in X$. При этом пространство **A** является нуль один секвенциално регулярным, в то время как пространство $\mathfrak{M}$ таким свойством не обладает. Поэтому пространства **A** и $\mathfrak{M}$ не являются гомеоморфными. Отсюда вытекает, что система борелевских линейных множеств не является гомеоморфной с системой всех функций Бэра.