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Czechoslovak Mathematical Journal, Vol. 14 (1964), No. 2, 322–323

Persistent URL: http://dml.cz/dmlcz/100622

Чехословацкий математический журнал, т. 14 (88) 1964, Прага

A MEAN VALUE THEOREM FOR STRONGLY CONTINUOUS VECTOR VALUED FUNCTIONS

(Preliminary communication)

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(Received November 27, 1963)

1. The purpose of this brief announcement is to state a mean value theorem for strongly continuous vector valued functions, which is a generalization of the mean value theorem for strongly continuous and strongly differentiable functions of A. K. AZIZ and J. B. DIAZ ([1], p. 261). It is planned that a complete discussion, with full details, will appear in the journal "Contributions to Differential Equations", which is published jointly by R. I. A. S. and the University of Maryland.

The mean value theorem of [1], for strongly differentiable vector valued functions, may be stated as follows, for convenience of the present exposition:

Mean value theorem. If 1) *the vector valued function* $x(t)$, *of one real variable* t, *is defined for all* t *such that* $a \leqq t \leqq b$, *where* $a < b$, *both* a *and* b *being finite, and its values are in a normed vector space with the norm* $\| \ \|$; 2) *the function* $x(t)$ *is strongly continuous for* $a \leqq t \leqq b$; 3) *the strong derivative* $x'(t)$ *exists, and is finite whenever* $a < t < b$; then *there is a number* c, *with* $a < c < b$, *such that*

$$\|(x(b) - x(a))/(b - a)\| \leqq \|x'(c)\| . \tag{1}$$

Remark 1. Simple examples, involving complex valued functions of a real variable, show that the strict inequality can actually occur in (1) for each c with $a < c < b$.

Remark 2. The mean value theorem for strongly differentiable functions yields uniqueness theorems for differential equations, which are stronger than the classical ones. This was proved in [1].

*) The research of this author was supported by the Air Force Office of Scientific Research, under Grant AFOSR 40063 with the University of Maryland.

**) The research of this author was supported by the U. S. National Science Foundation, under Grant NSF GP-2067 to the University of Maryland.

2. For real valued functions, there is a well known generalization of the classical mean value theorem, namely, the theorem of W. H. and G. C. YOUNG [2] concerning the Dini derivates. This raises the natural question as to whether there are possible generalizations, to vector valued functions, of the theorem of W. H. and G. C. YOUNG. An affirmative answer to this question is given by the following theorem:

Theorem. *Suppose that hypotheses* 1) *and* 2) *of the mean value theorem above hold. Then, there always exists a number c, with $a < c < b$, such that* either, *whenever both $h > 0$ and $a \leqq c + h \leqq b$, one has*

$$\|(x(b) - x(a))/(b - a)\| \leqq \|(x(c + h) - x(c))/h\| , \tag{2}$$

or, *whenever both $h > 0$ and $a \leqq c - h \leqq b$, one has*

$$\|(x(b) - x(a))/(b - a)\| \leqq \|(x(c) - x(c - h))/h\| . \tag{3}$$

Remark 3. Notice that inequalities (2) and (3) imply that either

$$\|(x(b) - x(a))/(b - a)\| \leqq \liminf_{h \to +0} \|(x(c + h) - x(c))/h\| , \tag{4}$$

or

$$\|(x(b) - x(a))/(b - a)\| \leqq \liminf_{h \to +0} \|(x(c) - x(c - h))/h\| . \tag{5}$$

Remark 4. If, further, hypothesis 3) of the mean value theorem above is valid, then the inequality (1) of its conclusion is an immediate consequence of inequalities (4) and (5) (or of (2) and (3)). More generally, if instead of hypothesis 3) one supposes that, whenever $a < t < b$, both the right and the left hand derivatives $x'_+(t)$ and $x'_-(t)$ exist, then the inequalities (2) and (3) imply the existence of an intermediate value c such that either

$$\|(x(b) - x(a))/(b - a)\| \leqq \|x'_+(c)\| , \text{ or } \|(x(b) - x(a))/(b - a)\| \leqq \|x'_-(c)\| .$$

References

[1] *A. K. Aziz* and *J. B. Diaz*: On a mean-value theorem of the differential calculus of vector-valued functions, and uniqueness theorems for ordinary differential equations in a linear normed space. U. S. Naval Ordnance Laboratory, White Oak, Maryland, NOLTR 61-22, May 16, 1961; Contributions to Differential Equations, Vol. *I*, No. 2, 1963, 251—269.

[2] *W. H.* and *G. C. Young*: On derivatives and the theorem of the mean, Quarterly Journal of Pure and Applied Mathematics, Vol. *40*, 1909, 1—26.

[3] *J. B. Diaz* and *R. Výborný*: On mean value theorems for strongly continuous vector valued functions. To appear in Contributions to Differential Equations.