

Alexandru Solian

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ON THE EXTENSION OF TRANSI-GROUPS

ALEXANDRU SOLIAN, (Bucharest)

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The author generalizes the theory of extensions of groups to the transi-groups introduced by the author in the paper [5].

In the theory of transi-groups one can construct the quotient-transi-group of a given transi-group with respect to a normal subtransi-group of the latter. As in the theory of groups one can raise the problem of finding transi-groups which are in a certain sense, extensions of two given transi-groups. The present paper treats and solves a slightly modified problem.

Coming now to a more precise formulation we remark that, when giving a transi-group (M, \mathfrak{A}, e) and an equivalence relation \sim in M , imprimitive with respect to \mathfrak{A} , i.e. a normal subtransi-group $(K, \mathfrak{B}, e, \sim)$, the group \mathfrak{B} is not an (effective) transformation group of K but it is homomorphically mapped onto such a group \mathfrak{B}^* , obtaining in this manner the transi-group (K, \mathfrak{B}^*, e) associated with $(K, \mathfrak{B}, e, \sim)$. The group \mathfrak{A} is an extension of \mathfrak{B} by $\mathfrak{A}/\mathfrak{B}$ (which acts as a transformation group of M/\sim). If the quotient-transi-group $(M/\sim, \mathfrak{A}/\mathfrak{B}, [e])$ (where $[e]$ is the equivalence class of e and coincides as a subset of M with K) is transitive, all the equivalence classes of M with respect to \sim have the same cardinal number, so that M can be identified with the cartesian product $K \times (M/\sim)$.

The present paper deals with the problem of *existence* (and not of finding all solutions) of a transi-group (M, \mathfrak{A}, e) when the group \mathfrak{A} is given as an extension of \mathfrak{B} by $\mathfrak{A}/\mathfrak{B}$. To be more precise, a necessary and sufficient condition is given in order that the given extension \mathfrak{A} act as an (effective) transformation group in the cartesian product $K \times (M/\sim)$.

For the questions of the theory of equivalence relations, the reader may refer to [2] and [1], for those of the theory of groups to [3] and for those of the theory of transi-groups, to the papers [4] and [5] where the definitions of technical terms employed are also given.

1. Let K be a set and let \mathfrak{B} be a group for which there exists a group homomorphism ψ from \mathfrak{B} onto an (effective) transformation group \mathfrak{B}^* of K . Let $(M^*, \mathfrak{A}^*, e^*)$

be a transitive transi-group; let \mathfrak{A}_0^* be the subgroup of all α , $\alpha \in \mathfrak{A}^*$, for which $\alpha(e^*) = e^*$ and let α_λ , $\lambda \in A$, be a system of representatives of the left residue classes of \mathfrak{A}^* with respect to \mathfrak{A}_0^* (here the representative of \mathfrak{A}_0^* is meant to be e , the unit element of \mathfrak{A}^*).

For any $x^* \in M^*$, there exists a unique λ , $\lambda \in A$, for which $\alpha_\lambda(e^*) = x^*$.

Let \mathfrak{A} be a given extension of \mathfrak{B} by \mathfrak{A}^* ; let τ_α , $\alpha \in \mathfrak{A}^*$, be a system of representatives of the extension, where $\tau_e = 1$ (here 1 denotes the unit element of \mathfrak{A}); let $\mu_{\alpha,\beta}$, $\alpha, \beta \in \mathfrak{A}^*$, be the factor system associated with the τ_α . Let us denote by \mathfrak{A}_0 the subgroup of all elements of \mathfrak{A} , whose residue classes are in \mathfrak{A}_0^* ; we have $\mathfrak{A}_0 \supset \mathfrak{B}$. We introduce the notations

$$\sigma^{(\alpha)} = \tau_\alpha \sigma \tau_\alpha^{-1}, \quad \sigma^\lambda = \tau_\alpha^{-1} \sigma \tau_\alpha, \quad \alpha \in \mathfrak{A}^*, \quad \sigma \in \mathfrak{B}.$$

The elements of \mathfrak{A} may be uniquely represented in the form $\sigma \tau_\alpha$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$ and the product is made according to

$$(\sigma_1 \tau_\alpha) (\sigma_2 \tau_\beta) = \sigma_1 \sigma_2^{(\alpha)} \mu_{\alpha,\beta} \tau_{\alpha\beta}, \quad \alpha, \beta \in \mathfrak{A}^*, \quad \sigma_1, \sigma_2 \in \mathfrak{B}.$$

Let us denote the cartesian product $K \times M^*$ by M .

Finally we make the following hypothesis:

Hypothesis (\mathcal{S}): If σ , $\sigma \in \mathfrak{B}$, has the property $\psi(\sigma^{\alpha\lambda}) = \psi(1)$ for all $\lambda \in A$, then $\sigma = 1$.

The introduction of this hypothesis will be justified at point 2.

2. To justify the introduction of hypothesis (\mathcal{S}), we make the following considerations:

Suppose that (M, \mathfrak{A}, e) is a transi-group and that $(K, \mathfrak{B}, e, \sim)$ is a normal subtransi-group, the quotient-transi-group with respect to whom, $(M^*, \mathfrak{A}^*, e^*)$ say, is transitive; it is well-known¹⁾ that in this case, all classes in the decomposition of M by \sim are cardinally equivalent; consequently, M can be identified with the cartesian product $K \times M^*$. The identification is made in the following manner (we employ the notations of point 1, though we have to do here with another situation):

If $x \in M$, let x^* be his equivalence class with respect to \sim ; suppose that α_λ , $\lambda \in A$, is such that $\alpha_\lambda(e^*) = x^*$; then

$$(1) \quad \tau_{\alpha_\lambda}^{-1}(x) = k \in K$$

as is easily seen from the definition of quotient-transi-groups; we then identify x , $x \in M$, with the couple $(k, x^*) \in K \times M^*$.

In particular we identify $k \in K$ with (k, e^*) and if $\alpha_\lambda(e^*) = x^*$ we obtain from (1), $\tau_{\alpha_\lambda}(k, e^*) = (k, x^*)$.

¹⁾ See, for instance, [4].

With every $\sigma \in \mathfrak{B}$ there is associated, in a natural way a transformation $\psi(\sigma)$ of K , namely that which satisfies $((\psi\sigma)(k), e^*) = \sigma(k, e^*)$, $k \in K$ ²⁾. Consequently, with the same notations as before, we have

$$\sigma(k, x^*) = \sigma\tau_{\alpha\lambda}(k, e^*) = \tau_{\alpha\lambda}((\psi\sigma^{\alpha\lambda})(k), e^*) = ((\psi\sigma^{\alpha\lambda})(k), x^*).$$

Hence, it is seen that if $\psi\sigma^{\alpha\lambda}$ is the identical transformation of K for all $\lambda \in A$, then $\sigma = 1$.

Thus, hypothesis (\mathcal{S}) is a necessary condition for the construction of the transi-group (M, \mathfrak{A}, e) .

It must be also observed that if \mathfrak{B} coincides with \mathfrak{B}^* and ψ is the identity mapping, hypothesis (\mathcal{S}) is automatically satisfied since among the $\sigma^{\alpha\lambda}$ may be found σ too.

3. With the data and notations of point 1, let us suppose in addition that *there exists a homomorphism φ , $\varphi : \mathfrak{A}_0 \rightarrow \mathfrak{X}(K)$ whose restriction to \mathfrak{B} coincides with ψ* . ³⁾

1) We define a mapping η , $\eta : \mathfrak{A} \rightarrow \mathfrak{X}(M)$ in the following manner: Let $\sigma\tau_\alpha \in \mathfrak{A}$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$ and let $(k, x^*) \in M$, where $k \in K$, $x^* \in M^*$. If the chosen representative of the left residue class with respect to \mathfrak{A}_0^* , which contains α , is α_ρ , $\rho \in A$, then $\alpha_\rho(e^*) = \alpha(e^*)$ hence $\tau_{\alpha_\rho}^{-1}\tau_\alpha \in \mathfrak{A}_0$; put

$$\delta_\alpha = \varphi(\tau_{\alpha_\rho}^{-1}\tau_\alpha);$$

the δ_α , $\alpha \in \mathfrak{A}^*$, define a function which associates with every $\alpha \in \mathfrak{A}^*$ a $\delta_\alpha \in \mathfrak{X}(K)$. Suppose that $x^* = \alpha_\lambda(e^*)$ and that $\alpha_\nu \equiv \alpha\alpha_\lambda \pmod{\mathfrak{A}_{0s}^*}$, ⁴⁾ $\lambda, \nu \in A$.

Then, by definition, ⁵⁾

$$(2) \quad (\eta(\sigma\tau_\alpha))(k, x^*) = ((\sigma\mu_{\alpha, \alpha_\lambda})^{\alpha_\lambda} \delta_{\alpha\alpha_\lambda}(k), \alpha(x^*)).$$

We assert that $\eta(\sigma\tau_\alpha)$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}^*$, is a transformation of M : indeed, if $(k', y^*) \in M$, $k' \in K$, $y^* \in M^*$, then, since \mathfrak{A}^* acts transitively on M , there exist $\alpha, \alpha_\lambda, \alpha_\nu \in \mathfrak{A}^*$ where $\lambda, \nu \in A$ such that

$$\alpha(x^*) = y^* = \alpha_\nu(e^*) \quad \text{and} \quad \alpha_\lambda(e^*) = x^*;$$

let $k, k \in K$, be such that

$$(\sigma\mu_{\alpha, \alpha_\lambda})^{\alpha_\nu} \delta_{\alpha\alpha_\lambda}(k) = k';$$

such a k exists since $\delta_{\alpha\alpha_\lambda}$ and $\psi((\sigma\mu_{\alpha, \alpha_\lambda})^{\alpha_\nu})$ are transformations of K ; we have $(\eta(\sigma\tau_\alpha))(k, x^*) = (k', y^*)$.

²⁾ Instead of $\sigma\tau_\alpha$, we have written σ which is the same thing.

³⁾ By $\mathfrak{X}(E)$ we denote, generally, the group of all transformations of the set E .

⁴⁾ That is to say, left congruent with respect to \mathfrak{A}_0^* .

⁵⁾ Here, instead of $(\psi(\sigma_1))(k_1)$ we have written, in order to simplify notations, $\sigma_1(k_1)$, $\sigma_1 \in \mathfrak{B}$, $k_1 \in K$.

If $(\eta(\sigma\tau_\alpha))(k, x^*) = (\eta(\sigma\tau_\alpha))(k_1, x_1^*)$, it is an immediate fact that $x^* = x_1^*$; further it may be easily checked that $k = k_1$.

(2) In order to prove that η is a group homomorphism, let $\sigma_1\tau_\alpha, \sigma_2\tau_\beta \in \mathfrak{A}$ and $(k, x^*) \in M$ where the meaning of notations is obvious; if

$$(3) \quad x^* = \alpha_\lambda(e^*), \quad \alpha(x^*) = \alpha_\nu(e^*), \quad \beta\alpha(x^*) = \alpha_\pi(e^*),$$

where $\lambda, \nu, \pi \in A$, then according to definition (2), we have

$$(4) \quad \begin{aligned} \eta(\sigma_2\tau_\beta)\eta(\sigma_1\tau_\alpha)(k, x^*) &= \eta(\sigma_2\tau_\beta)([\sigma_1\mu_{\alpha, \alpha_\lambda}]^{\alpha\nu} \delta_{\alpha\alpha_\lambda}(k), \alpha(x^*)) = \\ &= ([\sigma_2\mu_{\beta, \alpha_\nu}]^{\alpha\pi} \delta_{\beta\alpha_\nu}([\sigma_1\mu_{\alpha, \alpha_\lambda}]^{\alpha\nu} \delta_{\alpha\alpha_\lambda}(k)), \beta\alpha(x^*)) \end{aligned}$$

According to the same definition we have

$$(5) \quad \eta(\sigma_2\tau_\beta \cdot \sigma_1\tau_\alpha)(k, x^*) = ([\sigma_2\sigma_1^{(\beta)}\mu_{\beta, \alpha}\mu_{\beta\sigma, \alpha_\lambda}]^{\alpha\pi} \delta_{\beta\alpha\alpha_\lambda}(k), \beta\alpha(x^*)).$$

From relations (3) we obtain

$$(6) \quad \delta_{\beta\alpha_\nu} = \varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha_\nu}), \quad \delta_{\alpha\alpha_\lambda} = \varphi(\tau_{\alpha_\nu}^{-1}\tau_{\alpha\alpha_\lambda}), \quad \delta_{\beta\alpha\alpha_\lambda} = \varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha\alpha_\lambda}).$$

Based on the fact that ψ is a restriction of φ and on relations (6), the first component of the second member of (4) respectively (5), is the result of the action on k of the transformation

$$\begin{aligned} \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\mu_{\beta, \alpha_\nu}\tau_{\alpha_\pi})\varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha_\nu})\varphi(\tau_{\alpha_\nu}^{-1}\sigma_1\mu_{\alpha, \alpha_\lambda}\tau_{\alpha_\nu})\varphi(\tau_{\alpha_\pi}^{-1}\tau_{\alpha\alpha_\lambda}) = \\ = \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\alpha\tau_{\alpha_\lambda}) \end{aligned}$$

respectively of the transformation

$$\varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\beta^{-1}\mu_{\beta, \alpha}\mu_{\beta\sigma, \alpha_\lambda}\tau_{\alpha_\pi})\varphi(\tau_{\alpha_\pi}^{-1}\tau_{\beta\alpha\alpha_\lambda}) = \varphi(\tau_{\alpha_\pi}^{-1}\sigma_2\tau_\beta\sigma_1\tau_\alpha\tau_{\alpha_\lambda}).$$

Here we used the definition of $\mu_{\xi, \omega}$ namely

$$\tau_\xi\tau_\omega = \mu_{\xi, \omega}\tau_{\xi, \omega}.$$

So it was proved that η is a group homomorphism.

(3) Suppose that $\sigma\tau_\alpha, \sigma\tau_\alpha \in \mathfrak{A}$, has the property that $\eta(\sigma\tau_\alpha)$ is the identical transformation of M ; consequently

$$\eta(\sigma\tau_\alpha)(k, x^*) = (k, x^*)$$

for any $(k, x^*) \in M$.

According to definition (2), this implies, firstly, $\alpha = \varepsilon$. Let $x^* = \alpha_\lambda(e^*)$; since δ_{α_λ} is the identical transformation of K , we obtain from the same definition (2) that $\psi(\sigma^{\alpha_\lambda})$ is the identical transformation of K ; since x^* is arbitrarily chosen in M^* , λ may be considered as an arbitrary element of A , so that we may apply hypothesis (\mathcal{S}) and obtain $\sigma = 1$.

So η is not merely a homomorphism, but a monomorphism.

While identifying $\sigma\tau_\alpha$ with $\eta(\sigma\tau_\alpha)$ and, consequently, \mathfrak{A} with $\eta(\mathfrak{A})$ we may say that \mathfrak{A} is an (effective) transformation group of M . Moreover, if we choose in K a well-determined but otherwise arbitrary element k_0 , we obtain the transi-group (M, \mathfrak{A}, e) where $e = (k_0, e^*)$.

4) We define in M the following equivalence relation which we denote by \sim : let $(k_1, x^*), (k_2, y^*) \in M$; then, by definition $(k_1, x^*) \sim (k_2, y^*)$ if and only if $x^* = y^*$.

Considering the subgroup $\mathfrak{B}_{\mathfrak{A}}(\sim)$ associated with \sim , if $\sigma\tau_\alpha, \sigma\tau_\alpha \in \mathfrak{A}$, belongs to this subgroup, then, according to (2), $\alpha(x^*) = x^*$ for any $x^* \in M^*$ and then $\alpha = \varepsilon$ and $\sigma\tau_\alpha \in \mathfrak{B}$; conversely, every element of \mathfrak{B} may be written in the form $\sigma\tau_\varepsilon$ and, consequently, belongs to $\mathfrak{B}_{\mathfrak{A}}(\sim)$; hence $\mathfrak{B}_{\mathfrak{A}}(\sim) = \mathfrak{B}$.

It is easily seen that the equivalence \sim is imprimitive with respect to \mathfrak{A} .

On the other hand, the equivalence class of e is $K \times \{e^*\}$; if we identify the couple (k, e^*) with k , for any $k \in K$, then $K \times \{e^*\}$ is obtained to be identified with K . So we have to do with the normal subtransi-group $(K, \mathfrak{B}, e, \sim)$ of (M, \mathfrak{A}, e) .

Let $\sigma\tau_\varepsilon \in \mathfrak{B}$; then, from (2) we have

$$(7) \quad \sigma\tau_\varepsilon(k, e^*) = ((\psi\sigma)(k), e^*)$$

so that σ , considered as an element of \mathfrak{A} , induces in K the transformation $\psi(\sigma)$. Consequently, the transi-group associated with $(K, \mathfrak{B}, e, \sim)$ is (K, \mathfrak{B}^*, e) where, namely, to $\sigma, \sigma \in \mathfrak{B}$, we associate $\psi(\sigma)$.

Now, we define a homomorphism of transi-groups, $(h, \chi), (h, \chi) : (M, \mathfrak{A}, e) \rightarrow (M^*, \mathfrak{A}^*, e^*)$, through the relations:

$$h(k, x^*) = x^*, \quad \chi(\sigma\tau_\alpha) = \alpha, \quad k \in K, \quad x^* \in M^*, \quad \sigma \in \mathfrak{B}, \quad \alpha \in \mathfrak{A}^*.$$

It was no error while anticipating on the homomorphism character of (h, χ) , since according to definition (2), we have

$$h(\sigma\tau_\alpha(k, x^*)) = \alpha(x^*) = (\chi(\sigma\tau_\alpha))(h(k, x^*))$$

and there is no difficulty to prove the other properties of transi-group homomorphisms. Obviously, the kernel of (h, χ) is $(K, \mathfrak{B}, e, \sim)$.

5) To summarize, we have constructed a transi-group (M, \mathfrak{A}, e) with the following properties:

i) Denoting by \sim the equivalence defined as at 4), the subtransi-group generated by \sim is $(K, \mathfrak{B}, e, \sim)$ where K appears through identification with $K \times \{e^*\}$, the identification having been done as at 4).

ii) For any $\sigma \in \mathfrak{B}, k \in K$, we have (7), hence the transi-group associated with $(K, \mathfrak{B}, e, \sim)$ is (K, \mathfrak{B}^*, e) where to $\sigma, \sigma \in \mathfrak{B}$, we associate $\psi(\sigma)$.

iii) The couple (h, χ) as defined at 4) is a transi-group homomorphism $(h, \chi) : (M, \mathfrak{A}, e) \rightarrow (M^*, \mathfrak{A}^*, e^*)$, whose kernel is $(K, \mathfrak{B}, e, \sim)$.

4. With the data and notations of point 1, let us suppose that *there exists a transi-group* (M, \mathfrak{A}, e) , where $e = (k_0, e^*)$ such that conditions i)–iii) from 3.5) are fulfilled.

Let $\sigma_{\tau_\alpha} = \mathfrak{A}_0$, where $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_0^*$; if $(k, e^*) \in K$ then, according to condition iii) from 3.5), we have

$$h(\sigma_{\tau_\alpha}(k, e^*)) = \alpha(e^*) = e^*.$$

Hence σ_{τ_α} maps any element of K in an element of K . Since \mathfrak{A}_0 is a group, it is not difficult to prove that σ_{τ_α} induces quite a transformation $\varphi(\sigma_{\tau_\alpha})$ of K . So, we have constructed a mapping $\varphi, \varphi : \mathfrak{A}_0 \rightarrow \mathfrak{X}(K)$, which is obviously a group homomorphism.

If $\sigma_{\tau_\alpha} \in \mathfrak{B}$, then $\alpha \doteq \varepsilon$ and, by definition, $\varphi(\sigma_{\tau_\alpha})(k, e^*) = \sigma_{\tau_\alpha}(k, e^*)$ (where in the second member we mean the result of the action of σ_{τ_α} on (k, e^*) as in (M, \mathfrak{A}, e)). But according to condition ii) from 3.5) this implies that

$$(8) \quad \varphi(\sigma_{\tau_\alpha})(k, e^*) = ([\psi\sigma](k), e^*).$$

With the convention of identification (see i) from 3.5)) and on account of the fact that $\tau_\varepsilon = 1$, relation (8) becomes

$$(\varphi\sigma)(k) = (\psi\sigma)(k);$$

as this is valid for any $k \in K$, it follows that ψ is the restriction of φ to \mathfrak{B} .

5. We are now quite ready to formulate our fundamental result (we are given the data and notations of point 1):

Theorem. *In order that a transi-group* (M, \mathfrak{A}, e) (where $e = (k_0, e^*)$, $k_0 \in K$) should exist, so that conditions i)–iii) from 3.5) be satisfied, it is necessary and sufficient that there should exist a group homomorphism $\varphi, \varphi : \mathfrak{A}_0 \rightarrow \mathfrak{X}(M)$, whose restriction to \mathfrak{B} be ψ .

6. As particular cases we may consider the following quite trivial ones. Suppose in addition to the conditions stipulated at point 1, that:

1) \mathfrak{A}^* is a regular⁶⁾ transformation group of M^* ; in this case \mathfrak{A}_0^* is the identity subgroup, hence $\mathfrak{A}_0 = \mathfrak{B}$ and we may take $\varphi = \psi$.

2) \mathfrak{A} is the direct product $\mathfrak{B} \times \mathfrak{A}^*$; in this case we may define $\varphi(\sigma_{\tau_\alpha}) = \psi(\sigma)$, $\sigma \in \mathfrak{B}$, $\alpha \in \mathfrak{A}_0^*$.

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⁶⁾ For the definition see, for instance, [3], p. 36.

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Резюме

О РАСШИРЕНИИ ТРАНЗИГРУПП

АЛЕКСАНДРУ СОЛИАН (Alexandru Solian), Бухарест

Автор обобщает теорию расширения групп на случай транзигрупп, введенных им в работе [5].