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SECTIONS OF DYNAMICAL SYSTEMS IN E^2

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It is shown that every section of an abstract dynamical system in E^2, if it is a locally connected continuum, is an arc or a simple closed curve. Further results concern existence of local sections with special properties, in abstract dynamical systems on uniformisable spaces.

For dynamical systems defined by differential equations on euclidean spaces E^n (and, more generally, on differentiable n-manifolds), the local structure of the trajectories near a noncritical point is described by the "characteristic neighbourhood"; this consists of points on arcs of trajectories which intersect some $(n - 1)$-dimensional non-tangent hypersurface.

Secondly, for abstract dynamical systems on metric spaces (and, more generally, on uniformisable spaces), the local structure near a non-critical point is described by the Whitney-Bebutov theorem, affirming the existence of a closed set or *section*, which plays the role of the non-tangent hypersurface. The situation may be described, up to a homeomorphism, in infinite-dimensional Hilbert space (countably infinite if the carrier space is metric).

Now, possibly one cannot define differential systems on abstract spaces; however, one may easily have abstract dynamical systems (non-differentiable!) on euclidean E^n, and this concept is most useful for practical applications. The Whitney-Bebutov theorem still applies; and then there arise questions as to further properties of the sections: e.g., is a section $(n - 1)$-dimensional? (The answer is affirmative if the section is itself locally euclidean). The present paper is devoted to this class of questions.

Let P be a separated uniformisable space (cf. [1]). A *dynamical system* on P is a mapping $\oplus$ such that

(1) $\oplus: P \times E^1 \to P$ is continuous onto (the value of $\oplus$ at (x, ϑ) will be denoted by $x \oplus \vartheta$), and

(2) $(x \oplus \vartheta_1) \oplus \vartheta_2 = x \oplus (\vartheta_1 + \vartheta_2)$, $x \in P$, $\vartheta_j \in E^1$.

(See also [4], p. 346.) We shall occasionally use $x \oplus \vartheta_1 + \vartheta_2$ to denote the element of (2); and, in a similar sense, $x \oplus - \vartheta$ instead of $x \oplus (-\vartheta)$. For $X \subset P$, $A \subset \mathsf{E}^1$, $X \oplus A$ will denote the set of all $x \oplus \vartheta$ with $x \in X$, $\vartheta \in A$; and also write $X \oplus \vartheta$ in place of $X \oplus \{\vartheta\}$, etc.

Lemma 1. $x \oplus 0 = x$ *for* $x \in P$.

Proof. From (2) we have $(x \oplus \vartheta) \oplus 0 = x \oplus \vartheta + 0 = x \oplus \vartheta$, so that $y \oplus 0 = y$ for all y of the form $x + \vartheta$, i.e. in range $\oplus$. However, from (1), $\oplus$ maps onto P.

It is immediate that, for fixed $\vartheta \in \mathsf{E}^1$, the mappings $\oplus \vartheta : P \to P$ defined by $(x) \oplus \vartheta = x \oplus \vartheta$ form a (topological) group of continuous mappings. Hence each $\oplus \vartheta$ is a homeomorphism of P; and, in particular,

$$(\cup X_\alpha) \oplus \vartheta = \cup (X_\alpha \oplus \vartheta), \quad \overline{X \oplus \vartheta} = \overline{X} \oplus \vartheta$$

etc. Furthermore,

Lemma 2. *If* $X \subset P$, $A \subset \mathsf{E}^1$ *and* A *is compact, then* $\overline{X \oplus A} = \overline{X} \oplus \overline{A}$.

Proof. In any case $\overline{X} \oplus \overline{A} \subset \overline{X \oplus A}$, since $\oplus$ is continuous in $P \times \mathsf{E}^1$. To prove the opposite inclusion, take any $x \in \overline{X \oplus A}$. Then $x_\alpha \oplus \vartheta_\alpha \to x$ for generalized sequences $x_\alpha \in X$ and $\vartheta_\alpha \in A$. Since $\overline{A}$ is compact, there is a convergent subsequence $\vartheta_\beta \to \vartheta_0 \in \overline{A}$, and then

$$x_\beta = (x_\beta \oplus \vartheta_\beta) \oplus - \vartheta_\beta \to x \oplus - \vartheta_0,$$

and necessarily $x \oplus - \vartheta_0 \in \overline{X}$. Thus

$$x = (x \oplus - \vartheta_0) \oplus \vartheta_0 \in \overline{X} \oplus \overline{A}$$

as was to be proved.

The following lemma is well-known.

Lemma 3. *The set of all* $x \in P$ *which have* $x = x \oplus \vartheta$ *for all* $\vartheta \in \mathsf{E}^1$ *(the critical points of* $\oplus$*) is closed.*

The proof is almost trivial. If x is a limit of a generalised sequence of x_α, then also $x_\alpha \oplus \vartheta \to x \oplus \vartheta$; if the x_α are critical, $x_\alpha \oplus \vartheta = x_\alpha \to x$, and hence $x \oplus \vartheta = x$. This holds for all $\vartheta \in \mathsf{E}^1$, i.e. x is critical.

Definition. *A subset* $S \subset P$ *is a section (of length* $\lambda > 0$*) if*

$$x \in S,\ x \oplus \vartheta \in S,\ |\vartheta| \leqq \lambda \quad \textit{implies} \quad \vartheta = 0.$$

(Also see [4], p. 352.) Obviously every subset of a section is itself a section, with any length $\leqq \lambda$. Obviously a section of length λ cannot meet any cycle with (primitive) period $\leqq \lambda$; in particular, it cannot meet a critical point.

Now, the closure of a section need not be a section (of any positive length). Also, if we have two sections S_1, S_2 which are sufficiently far apart (in a certain sense), then $S_1 \cup S_2$ will again be a section. Thus interest centers more around closed connected sections. Finally, an empty set, and also a set consisting of a single non-critical point is trivially a section. We are interested in rather richer sections than these.

Theorem 1. *Let $\oplus$ be a dynamical system on a 2-manifold P. Then every section S which is a locally connected continuum is either a simple arc or a simple closed curve.*

In the proof, two non-trivial parts will be separated out as lemmas. We assume the situation described in the premiss of the theorem.

Lemma 4. *If there is an* arc $\widehat{axb} \subset S$ *then locally at* x, S *is an arc (in fact, the* arc $\widehat{axb}$).

Proof. Let $y = p(\sigma)$, $|\sigma| \leqq \sigma_0 > 0$, be a parametrisation of $\widehat{axb}$, $p(0) = x$. Let $2\lambda_0$ be the length of S. Then the mapping h defined by

$$h(\vartheta, \sigma) = p(\sigma) \oplus \vartheta \quad (|\vartheta| \leqq \lambda_0, |\sigma| \leqq \sigma_0)$$

is continuous and $1-1$ into a 2-manifold P. Thus it is a homeomorphism, and the image of the interval in E^2 is a closed simplex s^2 in P (the Invariance of Domain Theorem); in particular, s^2 is a neighbourhood of x. Obviously S meets s^2 only at $\widehat{axb}$, by choice of λ_0. This proves the lemma.

Lemma 5. *S is a local dendrite.*

Proof. Since P is a 2-manifold, i. e. locally euclidean, it is metrisable. We shall show that there is a positive lower bound to the diameters of simple closed curves in S (this implies our lemma: cf. [3], p. 228). Since S is compact, it suffices to show that any $x \in S$ has a neighbourhood U such that there are no simple closed curves entirely in $U \cap S$.

There is a neighbourhood U_0 of x and a $\lambda \in \mathsf{E}^1$ such that $0 < 2\lambda <$ *length* S and $U_0 \cap (U_0 \oplus \lambda) = \emptyset$; since otherwise there would exist arbitrarily small ϑ with $x_\alpha \to x$, $x_\alpha \oplus \vartheta \to x$, although $x \in S$ is noncritical. There is a homeomorphic image U of an open disc in E^2, with $x \in U \subset U_0$.

Now assume there exists a closed curve $C \subset U \cap S$. Then S separates U; let x_0 be a point in the interior of C. By choice of U, $x_0 \in U$. Since $U \subset U_0$, $U_0 \cap (U_0 \oplus \lambda) = \emptyset$, $x_0 \oplus \lambda$ is not in the interior of C, and there is a ϑ_1 with $0 < \vartheta_1 < \lambda$, $x_0 \oplus \vartheta_1 \in C \subset S$. We also have $(U_0 \oplus - \lambda) \cap U_0 = \emptyset$, so that analoguously there is a ϑ_{-1} with $0 > \vartheta_{-1} > - \lambda$, $x_0 \oplus \vartheta_{-1} \in C \subset S$. But this contradicts $2\lambda <$ *length* S and proves our lemma.

Proof of theorem 1. For this it suffices to show that S is an arc locally at each $x \in S$; since then it is a routine matter to prove that S, compact connected, is an arc or simple closed curve.

We already have lemma 5; furthermore, from lemma 4 it follows that S, and hence also any subcontinuum of S, has no branch points. Hence it remains to prove that any non-degenerate dendrite without branch points is an arc or simple closed curve. This is rather obvious; and may be proved directly, using the basic property of dendrites D, that distinct points x, y of D are on a subarc $\widehat{xy} \subset D$ uniquely determined by its end points [3, p. 225]. (A possible proof proceeds via the Zorn lemma to obtain a saturated arc in D.)

The idea of this theorem (but not its proof) goes back to [5]. The situation considered there concerned a single differential equation $y' = f(x, y)$ with $x, y, f(x, y) \in \in \mathsf{E}^1$ and f continuous, but without unicity assumptions. A set $S \subset \mathsf{E}^2$ was called trajectory-intersecting if there is no solution through distinct points of S. Then [5, § 6, lemma] every such S may be continuously completely ordered; and thus [l.c., theorem 21] a trajectory-intersecting continuum is a simple arc.

Returning to the general situation treated in theorem 1, the topological structure of the dynamical system near S is then completely described by the homeomorphism h constructed in lemma 4. It remains to show that there exist sufficiently many sections-continua. This may be done in a far more general setting than 2-manifolds. The construction, a mild generalisation (from metric to separated uniformisable spaces) of the Whitney-Bebutov construction, will be performed in detail, since further consequences will be drawn from it. (Lemma 7 is then the Whitney-Bebutov theorem [4, pp. 352–357].)

Construction. Let $\oplus$ be a dynamical system on a separated uniformisable space P; thus P is completely regular [1, chap. IX, p. 9]. Let $x_0 \in P$ be a noncritical point.

There is a $\vartheta \neq 0$ with $x_0 \neq x_0 \oplus \vartheta$; we may assume, say, $\vartheta = 1$. There is a continuous real-valued function ψ on P with

$$\psi(x_0) = 0 < \psi(x_0 \oplus 1) = 1$$

(if P is metrisable one may take $\psi(x) = \varrho(x, x_0)/\varrho(x_0 \oplus 1, x_0)$). Define φ by

$$\varphi(x) = \int_0^1 \psi(x \oplus \sigma)\, \mathrm{d}\sigma\ ;$$

Obviously then

$$\varphi(x \oplus \vartheta) = \int_{\vartheta}^{\vartheta + 1} \psi(x \oplus \sigma)\, \mathrm{d}\sigma \quad \text{and} \quad \frac{\partial}{\partial \vartheta}\, \varphi(x \oplus \vartheta) = \psi(x \oplus \vartheta + 1) - \psi(x \oplus \vartheta)$$

are both continuous functions of (x, ϑ). Since $(\partial/\partial\vartheta)\, \varphi(x \oplus \vartheta) = 1$ for $x = x_0$, $\vartheta = 0$, there is a neighbourhood U_1 of x_0 and a $\lambda > 0$ such that

$$(1) \qquad \frac{\partial}{\partial \vartheta}\, \varphi(x \oplus \vartheta) > 0 \quad \text{for} \quad x \in U_1,\ |\vartheta| \leqq 2\lambda\,.$$

In particular, $\varphi(x_0 \oplus \lambda) > \varphi(x_0) > \varphi(x_0 \oplus -\lambda)$, and thus there is a neighbourhood U_2 of x_0 such that

(2) $$\varphi(x \oplus \lambda) > \varphi(x_0) > \varphi(x \oplus -\lambda) \quad \text{for} \quad x \in U_2 .$$

Now take any neighbourhood U of x_0 with $\bar{U} \subset U_1 \cap U_2$ (particular choices of this U will, subsequently, imply diverse properties of the section to be constructed). Finally, set

(3) $$S = \{x \in P: \varphi(x) = \varphi(x_0)\} \cap (\bar{U} \oplus \langle -\lambda, \lambda \rangle) ,$$

(4) $$F = S \oplus \langle -\lambda, \lambda \rangle .$$

We will now consider some properties of the objects just obtained.

Lemma 6. *Both S, F are closed, and $x_0 \in S$, $\bar{U} \subset F$. The relations*

(5) $$x \in \bar{U} , \quad p(x) = x \oplus \vartheta \in S , \quad |\vartheta| \leqq \lambda$$

define a continuous closed map p of $\bar{U}$ onto S.

Proof. From continuity of φ and lemma 2, S and hence F are closed. Obviously $x_0 \in S$. Take any $x \in \bar{U}$; from (2), $\varphi(x \oplus \vartheta) = \varphi(x_0)$ for some $\vartheta \in \langle -\lambda, \lambda \rangle$; thus $x \oplus \vartheta \in S$ and therefore $x \in F$; this proves $\bar{U} \subset F$.

As concerns the map p, it has just been shown that to any given $x \in \bar{U}$ there is a $\vartheta \in \langle -\lambda, \lambda \rangle$ with $x \oplus \vartheta \in S$, i.e. with $\varphi(x \oplus \vartheta) = \varphi(x_0)$. This ϑ is determined uniquely, since $\partial\varphi/\partial\vartheta > 0$ on the arc $x \oplus \langle -2\lambda, 2\lambda \rangle$. Thus (5) indeed define a map p: $\bar{U} \to S$. From (3), p is onto.

Continuity: Let $x_\alpha \to x$ in $\bar{U}$ (generalised sequence), let $px_\alpha = x_\alpha \oplus \vartheta_\alpha$, $px = x \oplus \vartheta$. Take any confinal x_β, and from ϑ_β select a confinal convergent $\vartheta_\gamma \to \vartheta'$. Obviously then

$$px_\gamma = x_\gamma \oplus \vartheta_\gamma \to x \oplus \vartheta' \in S ,$$

by closedness of S, and $|\vartheta'| \leqq \lambda$ since $|\vartheta_\alpha| \leqq \lambda$. From unicity, $\vartheta' = \vartheta$, so that $px_\gamma \to$ $\to px$. This proves $px_\alpha \to px$, i.e., continuity.

Closedness: Let $px_\alpha \to y$ in S; then $px_\alpha = x_\alpha \oplus \vartheta_\alpha$ with $|\vartheta_\alpha| \leqq \lambda$, and one may take a convergent confinal $\vartheta_\beta \to \vartheta'$. Necessarily, $|\vartheta'| \leqq \lambda$, and

$$x_\beta = (x_\beta \oplus \vartheta_\beta) \oplus -\vartheta_\beta \to y \oplus -\vartheta' \in \bar{U} ,$$

since $\bar{U}$ is closed. Obviously $p(y \oplus -\vartheta') = y$; thus, closedness.

Lemma 7. *S is a closed section of length λ, and generates F, a neighbourhood of x_0.*

Proof. F is a neighbourhood of x_0 since U is such and $\bar{U} \subset F$; both S, F are closed (lemma 6).

Assume S is not a section of length λ, and aim at a contradiction. Then there exist $x, x \oplus \vartheta$ in S with $0 < \vartheta \leqq \lambda$. From (3), $S \subset \bar{U} \oplus \langle -\lambda, \lambda \rangle$, so that there is a $\vartheta_0 \in \in \langle -\lambda, \lambda \rangle$ with $x \oplus \vartheta_0 \in \bar{U} \subset \bar{U}_1$. Now apply (1): on the arc

$$(x \oplus \vartheta_0) \oplus \langle -2\lambda, 2\lambda \rangle ,$$

and this includes both x and $x \oplus \vartheta$, there is $\partial\varphi/\partial\vartheta > 0$; thus one cannot have both $\varphi(x) = \varphi(x_0) = \varphi(x \oplus \vartheta)$; this contradiction completes the proof.

Theorem 2. *Given a dynamical system on a separated uniformisable space P, and a noncritical $x_0 \in P$, there exist sections $S \ni x_0$ generating arbitrarily small neighbourhoods of x_0. If P is locally compact and /or locally connected, then S may be taken compact and /or connected, respectively. Furthermore, if P is metrisable and has property $\mathscr{S}$, then S may also be taken locally connected.*

Proof. Preserve the preceding notation. The first assertion follows from lemma 7; if λ, U are taken small, then

$$F = S \oplus \langle -\lambda, \lambda \rangle \subset \bar{U} \oplus \langle -2\lambda, 2\lambda \rangle$$

may be made as small as one pleases.

If P is locally compact, one may take $\bar{U}$ compact; from lemma 6, p maps $\bar{U}$ continously onto S, which is then also compact. Similarly for connectedness. If P has property $\mathscr{S}$, one may take $\bar{U}$ with property $\mathscr{S}$ and hence locally connected [6, p. 217, 215, 212]; since p is a closed continuous mapping onto, S is also locally connected [*l.c.*, p. 200].

Now consider several dynamical systems on the same uniform space P, $\mathscr{U}$. For one of these, say $\oplus_0$, perform the construction of S as above; for any other $\oplus$, define F by (4). Do lemmas 6, 7 hold, at least for $\oplus$ sufficiently near to $\oplus_0$? The answer is affirmative, if "sufficiently near" is reasonably interpreted.

Theorem 3. *Assume the above construction for a dynamical system $\oplus_0$. Then the set of all dynamical system $\oplus$ on P, $\mathscr{U}$ such that* (i) *S is a closed section of length λ, and* (ii) *$\bar{U} \subset S \oplus \langle -\lambda, \lambda \rangle$, is a neighbourhood of $\oplus_0$ in the topology of uniform convergence if P is locally compact.*

Proof. It may be noticed that the proofs of lemmas 6 and 7 (corresponding to (i) and (ii) depend, in essence, only on the relations (1) and (2), respectively. Now consider the topology of uniform convergence [2, p. 5] on the set of all continuous maps $P \times \mathsf{E}^1 \to P$, and the its subspace consisting of dynamical systems on P, $\mathscr{U}$. The set of all dynamical systems $\oplus$ on P, $\mathscr{U}$ such that, on taking U_1 compact,

$$\psi(x \oplus \vartheta + 1) - \psi(x \oplus \vartheta) > 0 \quad \text{for} \quad x \in U_1,\ |\vartheta| \leqq 2\lambda$$

is open, and contains $\oplus_0$ by construction. Similarly, the set of $\oplus$ with

$$\int_0^1 \psi((x \oplus \lambda) \oplus_0 \sigma) \, d\sigma > \varphi(x_0) > \int_0^1 \psi((x \oplus -\lambda) \oplus_0 \sigma) \, d\sigma \quad \text{for} \quad x \in U_2$$

is open and contains $\oplus_0$. This concludes the proof of theorem 3.

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Резюме

СЕЧЕНИЯ ДИНАМИЧЕСКИХ СИСТЕМ В E^2

ОТОМАР ГАЕК (Otomar Hájek), Прага

Изучаются топологические свойства локальных сечений абстрактных динамических систем в тихоновском пространстве (см. [4] в случае метрических пространств). Основная конструкция Уитней-Бебутова — в несколько упрощеном виде — используется для перенесения некоторых локальных свойств пространства на конструированное сечение.

Если несущее топологическое многообразие размерности два, то всякое локально связное континуум-сечение является простой дугой или простой замкнутой кривой (теорема 1). Из этих теорем следует существование трансверсальных крывых, проходящих через наперед заданную некритическую точку, в абстрактных системах на плоских многообразиях.

Наконец, показано (теорема 3), что сечение из т. 2 остается сечением при малых изменениях динамической системы (в равномерной топологии).