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EXTENSIONS OF ADDITIVE MAPPINGS

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Let Z be a Boolean ring and $\mathfrak{G}$ an Abelian group. Further, let Λ be a certain class of additive mappings from Z into $\mathfrak{G}$. To each element of Λ we construct an additive extension. By this method the Lebesgue integral can be extended (see [2]).

1. Let Z be a Boolean ring (see, e.g., [1], section 2). We don't suppose that Z has a unit. If $P \subset Z$, $Q \subset Z$, then $P + Q$ is the set of all $x + y$, where $x \in P$, $y \in Q$; the meaning of PQ is defined similarly. (The union, the intersection and the difference of sets S, V will be denoted by $S \cup V$, $S \cap V$ and $S - V$ respectively.)

Further let $\mathfrak{G}$ be an Abelian group. The zeros of Z and $\mathfrak{G}$ will be denoted by the same symbol 0.

A mapping ζ of a set $M \subset Z$ into $\mathfrak{G}$ is called additive, when the implication

$$(x, y, x + y \in M,\ xy = 0) \Rightarrow (\zeta(x + y) = \zeta(x) + \zeta(y))$$

is valid.

If ζ is a mapping of a set $M \subset Z$ and if $z \in Z$, we define mappings ζ_z, ζ'_z in the following way: $\zeta_z(x) = \zeta(zx)$ for all x with $zx \in M$ and $\zeta'_z(x) = \zeta(x + zx)$ for all x with $x + zx \in M$.

2. Let A be a subring of Z and let Θ be a set of mappings ϑ with the following properties: ϑ is defined on a subring $M(\vartheta)$ of Z such that $AM(\vartheta) \subset M(\vartheta)$, $\vartheta(M(\vartheta)) \subset \mathfrak{G}$ and ϑ is additive. Let γ be a transformation of a set $\Lambda \subset \Theta$ into Θ. For each $\lambda \in \Lambda$ put $C(\lambda) = M(\gamma(\lambda))$ and for each $x \in C(\lambda)$ write $(\gamma(\lambda))(x) = \gamma(\lambda, x)$. Instead of "$x \in C(\lambda)$" we shall sometimes write "$\gamma(\lambda, x)$ has a meaning" (or similarly). If we say, e.g., that $\gamma(\lambda, x) = 0$, we mean, of course, that $\lambda \in \Lambda$, $x \in C(\lambda)$. Further, let ω be a homomorphism of $\mathfrak{G}$ into $\mathfrak{G}$. Assume that the following conditions are fulfilled:

R1) If $\lambda \in \Lambda$, $z \in Z$, then $\lambda_z \in \Lambda$, $\lambda'_z \in \Lambda$.

R2) For each $\lambda \in \Lambda$ we have $-\lambda \in \Lambda$, $C(-\lambda) = C(\lambda) \subset A$.

R3) If $\lambda \in \Lambda$, $x \in A \cap M(\lambda)$, then $\gamma(\lambda, x) = \lambda(x)$ (so that $A \cap M(\lambda) \subset C(\lambda)$).

R4) If $\lambda, \mu, \nu \in \Lambda$ and if $\nu(x) = \lambda(x) + \mu(x)$ for each $x \in M(\lambda) \cap M(\mu)$, then $\gamma(\nu, x) = \gamma(\lambda, x) + \gamma(\mu, x)$ for each $x \in C(\lambda) \cap C(\mu)$.

R5) If $\lambda, \mu \in \Lambda$ and if $\mu(x) = \omega\lambda(x)$ for each $x \in M(\lambda)$, then $\gamma(\mu, x) = \omega\gamma(\lambda, x)$ for each $x \in C(\lambda)$.

3. Suppose that a convergence on Z and a convergence on $\mathfrak{G}$, fulfilling the conditions of [1], 3 and 5, are defined. Construct the set Ψ and the transformation β of Ψ into Ψ according to [1], 24. Let ω be a continuous homomorphism of $\mathfrak{G}$ into $\mathfrak{G}$; let Λ be a subset of Ψ such that the implication $\lambda \in \Lambda \Rightarrow -\lambda \in \Lambda$ and the condition R1) are valid. It follows from [1], 20, 29, 25 and 26 that the conditions R2)–R5) are fulfilled, if we put $\gamma = \beta$. (If we use the notation of [1], 24, then, of course, $C(\lambda) = B(\lambda)$.)

4. *For each $\lambda \in \Lambda$ we have $\gamma(-\lambda) = -\gamma(\lambda)$.*

Proof. Choose a $\lambda \in \Lambda$. It follows easily from the additivity of λ that $\lambda(0) = 0$. The mapping $\lambda_0(x) = 0$ $(x \in Z)$ belongs, by R1), to Λ and $\lambda(x) + (-\lambda)(x) = 0 = \lambda_0(x)$ for each $x \in M(\lambda)$. Further, by R3), $\gamma(\lambda_0, x) = \lambda_0(x)$ for each $x \in A = A \cap M(\lambda_0)$. It follows from R4) that $\gamma(\lambda, x) + \gamma(-\lambda, x) = \gamma(\lambda_0, x) = 0$ for each $x \in C(\lambda) = C(-\lambda)$ (see R2)).

5. *Suppose that $a, s \in Z$, $as = a$. Then $\gamma(\lambda, a) = \gamma(\lambda_s, a)$, whenever at least one side of this equality has a meaning.*

Proof. We may assume that $a \in A$. Since $a + sa = 0$, we have $\lambda_s'(a) = 0$, whence $\gamma(\lambda_s', a) = \gamma(-\lambda_s', a) = 0$. Evidently $\lambda_s(x) + \lambda_s'(x) = \lambda(x)$, $\lambda(x) + (-\lambda_s')(x) = \lambda_s(x)$, whenever the corresponding sum has a meaning. Now, our assertion follows easily from R4).

6. If $z \in Z$ and $\lambda \in \Lambda$, put

$$H(\lambda, z) = \{a \in C(\lambda_z);\; z + az \in M(\lambda)\}\,.$$

For each $a \in H(\lambda, z)$ write

$$\alpha(\lambda, a, z) = \gamma(\lambda_z, a) + \lambda(z + az)\,.$$

We see that $H(\lambda, z)$ is the set of all a such that $\alpha(\lambda, a, z)$ has a meaning. Further, let $S(\lambda)$ be the set of all s such that $H(\lambda, s) \neq \emptyset$.

Remark. Let f be a function on the Euclidean space E_r and let z be a measurable set in E_r. Let f_z be a function that coincides with f on z and equals zero on $E_r - z$. Let, further, λ be the (indefinite) Lebesgue integral of f and let $\gamma(\lambda)$ be a suitable "improper integral" of f. Then λ_z is the Lebesgue integral of f_z. Suppose that there exists a set a such that $\alpha(\lambda, a, z)$ has a meaning. In the next section we show that $\alpha(\lambda, a, z)$ does not depend on the choice of a; the number $\sigma(\lambda, z)$, defined in 8, is then a certain generalized integral of f over z (see [2]).

7. *If $a \in A$, $b, c \in H(\lambda, s)$, $ab = b$, then $a \in H(\lambda, s)$, $\alpha(\lambda, b, s) = \alpha(\lambda, c, s)$.*

Proof. Since $s(a + b) = a(s + bs) \in AM(\lambda) \subset M(\lambda)$ and $a + b \in A$, we have $\lambda(s(a + b)) = \lambda_s(a + b) = \gamma(\lambda_s, a + b)$; now, from the relations $b + (a + b) = a$, $b(a + b) = 0$ we infer that

$$\gamma(\lambda_s, a) = \gamma(\lambda_s, b) + \gamma(\lambda_s, a + b) = \gamma(\lambda_s, b) + \lambda(s(a + b)) . \tag{1}$$

Clearly $(s(a + b))(s + as) = 0$, $s(a + b) + (s + as) = s + bs$ and so $\lambda(s(a + b)) + \lambda(s + as) = \lambda(s + bs)$. Hence it follows from (1) that $\alpha(\lambda, a, s) = \alpha(\lambda, b, s)$. If we choose $a = b + c + bc$, we have $ab = b$, $ac = c$ and so $\alpha(\lambda, c, s) = \alpha(\lambda, a, s) = \alpha(\lambda, b, s)$.

8. For each $s \in S(\lambda)$ we may put, according to 7, $\sigma(\lambda, s) = \alpha(\lambda, a, s)$, where a is an arbitrary element of $H(\lambda, s)$.

9. *The mapping $\sigma(\lambda, .)$ is an extension of both mappings λ, $\gamma(\lambda)$.*

Proof. Choose a $c \in C(\lambda)$ and an $m \in M(\lambda)$. By 5 we have $\gamma(\lambda, c) = \gamma(\lambda_c, c)$, so that $\gamma(\lambda, c) = \alpha(\lambda, c, c)$; clearly $\lambda(m) = \alpha(\lambda, 0, m)$.

10. *Suppose that $\lambda, \lambda^{(1)}, \lambda^{(2)} \in \Lambda$, $s \in S(\lambda^{(1)}) \cap S(\lambda^{(2)})$ and that $\lambda(x) = \sum \lambda^{(i)}(x)$ $(\sum = \sum_{i=1}^{2})$ for each $x \in M(\lambda^{(1)}) \cap M(\lambda^{(2)})$ with $sx = x$. Then $\sigma(\lambda, s) = \sum \sigma(\lambda^{(i)}, s)$.*

Proof. Choose $a_i \in H(\lambda^{(i)}, s)$ and put $a = a_1 + a_2 + a_1 a_2$. By 7 we have $a \in H(\lambda^{(i)})$, whence

$$\sigma(\lambda^{(i)}, s) = \gamma(\lambda_s^{(i)}, a) + \lambda^{(i)}(s + as) \quad (i = 1, 2) . \tag{2}$$

If $x \in M(\lambda_s^{(1)}) \cap M(\lambda_s^{(2)})$, then, by assumption, $\sum \lambda_s^{(i)}(x) = \sum \lambda^{(i)}(sx) = \lambda(sx) = \lambda_s(x)$ and so, on account of R4), $\sum \gamma(\lambda_s^{(i)}, a) = \gamma(\lambda_s, a)$. Now, it follows from (2) that $\sum \sigma(\lambda^{(i)}, s) = \gamma(\lambda_s, a) + \lambda(s + as) = \sigma(\lambda, s)$.

11. *Suppose that $\lambda, \mu \in \Lambda$, $s \in S(\lambda)$ and that $\mu(x) = \omega\, \lambda(x)$ for each $x \in M(\lambda)$ with $xs = x$. Then $\sigma(\mu, s) = \omega\, \sigma(\lambda, s)$.*

(This follows easily from R5).)

12. *We have $\sigma(-\lambda, x) = -\sigma(\lambda, x)$, whenever at least one side of this equality has a meaning.*

(This follows easily from 4.)

13. *We have $\sigma(\lambda_s, x) = \sigma(\lambda, sx)$, whenever at least one side of this equality has a meaning.*

Proof. If either $\sigma(\lambda_s, x)$ or $\sigma(\lambda, sx)$ has a meaning, then there exists an a such that $\sigma(\lambda_s, x) = \gamma((\lambda_s)_x, a) + \lambda_s(x + ax) = \gamma(\lambda_{sx}, a) + \lambda(sx + asx) = \sigma(\lambda, sx)$.

14. If there is no danger of misunderstanding, we omit the symbol λ and write $C(\lambda) = C$, $\sigma(\lambda, x) = \sigma(x)$ etc.

15. *If $s \in S$, $a \in A$, $as = a$, then $a \in C$.*

Proof. Choose a $b \in H(s)$. Then $b \in C(\lambda_s)$, whence $ab \in C(\lambda_s)$. Since $abs = ab$, we have by 5 $ab \in C$ and from $s + bs \in M$ we infer that $a + ab = a(s + bs) \in \in M \cap A \subset C$; therefore $a = (a + ab) + ab \in C$.

16. *Suppose that $a \in A$. Then $\gamma(a) = \sigma(a)$, whenever at least one side of this equality has a meaning. Especially, $A \cap S = C$.*

(This follows immediately from 9 and 15.)

17. *Suppose that $x_1x_2 = x_1$, $x_3x_4 = 0$. Then $\sigma(x_1 + x_2) = \sigma(x_2) - \sigma(x_1)$, $\sigma(x_3 + x_4) = \sigma(x_3) + \sigma(x_4)$, whenever the corresponding right-hand side has a meaning.*

Proof. Put $x = x_1 + x_2$. If $t \in M(\lambda_{x_2}) \cap M(\lambda_{x_1})$, then $\lambda_{x_2}(t) - \lambda_{x_1}(t) = \lambda(x_2t) - - \lambda(x_1t) = \lambda(xt) = \lambda_x(t)$. Now, if $x_1 \in S$, $x_2 \in S$, we get, with the help of 13, 12 and 10, $\sigma(x_2) - \sigma(x_1) = \sigma(\lambda_{x_2}, x_2) + \sigma(-\lambda_{x_1}, x_2) = \sigma(\lambda_x, x_2) = \sigma(x)$. The second relation can be proved similarly.

18. *We have $\sigma(x_1 + x_2) = \sigma(x_1) + \sigma(x_2) - 2\sigma(x_1x_2)$, whenever the right-hand side has a meaning.*

Proof. Put $y_i = x_i + x_1x_2$. As $x_ix_1x_2 = x_1x_2$ and $y_1y_2 = 0$, it follows from 17 that $\sigma(x_1) - \sigma(x_1x_2) + \sigma(x_2) - \sigma(x_1x_2) = \sigma(y_1) + \sigma(y_2) = \sigma(y_1 + y_2) = \sigma(x_1 + x_2)$.

19. $C + M \subset S$.

Proof. Choose $c \in C$, $m \in M$. As $cm \in AM \subset M$, it follows from 9 that $c, m, cm \in \in S$ and by 18 we get $c + m \in S$.

20. *If $a \in A$, $b \in H(s)$, then $ab \in H(as)$.*

Proof. Since $b \in C(\lambda_s)$, we have $ab \in C(\lambda_s)$ and by 5 (where we write a, ab, λ_s instead of s, a, λ) we obtain $ab \in C((\lambda_s)_a) = C(\lambda_{as})$. From $s + bs \in M$ it follows that $as + abas = a(s + bs) \in M$, which completes the proof.

21. $AS \subset S$.

(This follows from 20.)

22. *Suppose that a convergence on Z and a convergence on 𝔖 with the same support are given (in the sense of [1], 1). Let the convergence on Z fulfil the conditions 1), 2) of [1], 3 and let the convergence on 𝔖 fulfil the condition 3) of [1], 5. Suppose that λ is continuous and that $\gamma(\lambda_s)$ is continuous for each $s \in S$. Then σ is continuous as well.*

Proof. Let $s_n \to s$,

(3) $$\sigma(s) = \gamma(\lambda_s, a) + \lambda(s + as) .$$

Since $s + s_n = s(s + s_n) \in SA \subset S$ (see 21), $s_n = s + (s + s_n)$, we get by 17 $s_n \in S$. Put $a_n = a + as + as_n$. From the relations $as_n \to as$, $as(a + as) = 0$ it follows that $a_n \to a + as + as = a$. As $as_n \in AS \subset S$, we get by 17

(4) $$\sigma(s_n) = \sigma(as_n) + \sigma(s_n + as_n) .$$

The equalities $sa_n = as_n$ imply, by 13 and 16, that $\sigma(as_n) = \sigma(sa_n) = \sigma(\lambda_s, a_n) = \gamma(\lambda_s, a_n) \to \gamma(\lambda_s, a)$. Since $s_n + as_n + s + as = (s_n + s)(s + as) \in AM \subset M$, we have also $s_n + as_n \in M$, so that, by 9, $\sigma(s_n + as_n) = \lambda(s_n + as_n) \to \lambda(s + as)$. Hence it follows from (4) that $\sigma(s_n) \to \gamma(\lambda_s, a) + \lambda(s + as) = \sigma(s)$.

23. Remark. For each $\lambda \in \Lambda$ put $T(\lambda) = C(\lambda) + M(\lambda)$. As $AM(\lambda) \subset M(\lambda)$, $T(\lambda)$ is a ring; it is evidently the smallest ring containing both $C(\lambda)$ and $M(\lambda)$. By 19 we have $T(\lambda) \subset S(\lambda)$. In the following example (Theorem C) we show that $S(\lambda)$ is not necessarily an additive group; then, of course, $T(\lambda) \neq S(\lambda)$. If $\lambda, \mu, \nu \in \Lambda$ and if $\lambda(x) + \mu(x) = \nu(x)$ for each $x \in M(\lambda) \cap M(\mu)$, then, according to 10, $S(\lambda) \cap S(\mu) \subset S(\nu)$; we shall see, however, that the inclusion $T(\lambda) \cap T(\mu) \subset T(\nu)$ may be false (Theorem D).

24. Example. Let K, N be two copies of the set of all natural numbers and let Z be the set of all functions x on K such that for each $k \in K$ either $x(k) = 0$ or $x(k) = 1$. If $x_1, x_2 \in Z$, put $x_1 + x_2 = x$, $x_1 x_2 = y$, where $x(k) = |x_1(k) - x_2(k)|$, $y(k) = x_1(k)\, x_2(k)$ $(k \in K)$. Evidently $x, y \in Z$, $x(k) \equiv x_1(k) + x_2(k) \pmod 2$. If we put $j(k) = 1$ $(k \in K)$, then j is the unit of Z. For each $x \in Z$ put

$$\|x\| = \sum_{k=1}^{\infty} |x(k) - x(k+1)| , \quad \eta(x) = \inf \{k; x(k) = 1\}$$

(so that $\eta(0) = \infty$). It is easy to see that

(5) $$\|x + y\| \leqq \|x\| + \|y\| , \quad \|xy\| \leqq \|x\| + \|y\| , \quad \eta(x) \leqq \eta(xy)$$

for arbitrary $x, y \in Z$.

Put, further, $A = \{x; \|x\| < \infty\}$ and let $\mathfrak{P}$ be the system of all sequences $\{x_n\}$ $(n \in N,\ x_n \in Z)$ such that $\sup_n \|x_n\| < \infty$, $\eta(x_n) \to \infty$. It follows from (5) that A is a ring and that $\{ax_n\} \in \mathfrak{P}$, $\{x_n + ax_n\} = \{(j + a)\, x_n\} \in \mathfrak{P}$ for each $a \in A$ and each $\{x_n\} \in \mathfrak{P}$. Now we define a convergence $x_n \to x$ on Z by the relations $xx_n = x_n$, $\{x_n + x\} \in \mathfrak{P}$. By [1], 4 this convergence fulfils the conditions 1) and 2) of [1], 3. Further let $\mathfrak{G}$ be the additive group of real numbers with the usual convergence.

An element $z \in Z$ belongs to A if and only if there exists the limit $\lim_{k \to \infty} z(k)$; we denote it by $z(\infty)$. Let A_0 be the set of all $z \in A$ such that $z(\infty) = 0$. Now define

$$j_n(k) = 1 \quad \text{for} \quad k \leqq n\,, \quad j_n(k) = 0 \quad \text{for} \quad k > n\,. \tag{6}$$

It is easy to see that $aj_n \in A_0$ and $aj_n \to a$ for each $a \in A$. Thus we get $\boldsymbol{u}A_0 = A$; A_0 is clearly an ideal in Z.

Let $\{a_k\}_{k \in K}$ be an arbitrary sequence of finite real numbers. Let M be the set of all $z \in Z$ such that $\sum_{k=1}^{\infty} |a_k z(k)| < \infty$. To each $z \in M$ we attach the number $\lambda(z) = \sum_{k=1}^{\infty} a_k z(k)$. Thus we have defined a mapping λ of M into $\mathfrak{G}$. It is obvious that M is an ideal in Z and that λ is additive. If $z \in M$ and $\{h_n\} \in \mathfrak{P}$, then $|\lambda(h_n z)| \leqq \sum_{k=1}^{\infty} |a_k h_n(k) z(k)| \leqq \sum_{k=\eta(h_n)}^{\infty} |a_k z(k)|$, so that $\lambda(h_n z) \to 0$. According to [1], 6, λ is continuous.

We say that λ is determined by the sequence $\{a_k\}$. Let Λ be the set of all mappings determined by a sequence of real numbers. If λ is determined by $\{a_k\}$ and if $z \in Z$, then λ_z is determined by $\{a_k z(k)\}$ so that $\lambda_z \in \Lambda$ as well. Evidently $\lambda'_z = \lambda_v$, where $v = j + z$.

With each $\lambda \in \Lambda$ we can associate, according to [1], 24, a set $B(\lambda)$ and a mapping $\beta(\lambda)$. If we put, e.g., $\omega t = t$ for each $t \in \mathfrak{G}$, then, by 3, the conditions R1)–R5) of 2 are fulfilled (we have, of course, $\gamma = \beta$, $C(\lambda) = B(\lambda)$). Now, by 6 and 8, a set $S(\lambda)$ and a mapping $\sigma(\lambda, .)$ can be attached to each $\lambda \in \Lambda$.

Lemma a). *Let $\sum_{k=1}^{\infty} a_k$ be a convergent series. For $k = 1, 2, \ldots$ put $r(k) = \max_{j \geqq k} \left|\sum_{i=j}^{\infty} a_i\right|$; further put $r(\infty) = 0$. Then, for each $x \in A$, the series $\sum_{k=1}^{\infty} a_k x(k)$ is convergent and*

$$\left|\sum_{k=1}^{\infty} a_k x(k)\right| \leqq (1 + 2\|x\|)\, r(\eta(x))\,. \tag{7}$$

Proof. The convergence of $\sum_{k=1}^{\infty} a_k x(k)$ is obvious. We may suppose that $\eta = \eta(x) < \infty$. Put $s_k = a_1 + \ldots + a_k$, $s = a_1 + a_2 + \ldots$ For each $p > \eta$, $\sum_{k=1}^{p} a_k x(k) = \sum_{k=\eta}^{p-1} (s_k - s_{\eta-1})(x(k) - x(k+1)) + (s_p - s_{\eta-1})\, x(p)$; hence

$$\sum_{k=1}^{\infty} a_k x(k) = \sum_{k=\eta}^{\infty} (s_k - s_{\eta-1})(x(k) - x(k+1)) + (s - s_{\eta-1})\, x(\infty)\,. \tag{8}$$

As $|x(\infty)| \leqq 1$ and $|s_k - s_{\eta-1}| \leqq |s - s_k| + |s - s_{\eta-1}| \leqq 2r(\eta)$ for each $k \geqq \eta$, (7) is an easy consequence of (8).

Theorem A. *Let λ be determined by $\{a_k\}$. Then $B(\lambda)$ is the set of all $b \in A$ such that the series $\sum_{k=1}^{\infty} a_k b(k)$ converges; its sum is $\beta(\lambda, b)$ for each $b \in B(\lambda)$.*

Proof. Let B_1 be the set of all $b \in A$ such that $\sum_{k=1}^{\infty} a_k\, b(k)$ converges; we denote this sum by $\varphi(b)$. It is easy to see that B_1 is an ideal in A. If $b \in B_1$, $\{h_n\} \in \mathfrak{P}$, $bh_n = h_n$, then, by lemma a), $\varphi(h_n) \to 0$. According to [1], 6, φ is continuous. Evidently $\varphi(x) = \lambda(x)$ for each $x \in B_1 \cap M(\lambda)$. Since $A_0 \subset M(\lambda)$, we have $A = \mathbf{u}(A_0) \subset \subset \mathbf{u}(M(\lambda))$. It follows from [1], 19 that $B_1 \subset B(\lambda)$.

Choose, conversely, a $b \in B(\lambda)$ and define j_n by means of (6). Then $bj_n \to b$, $\sum_{k=1}^{n} a_k\, b(k) = \sum_{k=1}^{\infty} a_k\, b(k)\, j_n(k) = \lambda(bj_n) = \beta(\lambda, bj_n) \to \beta(\lambda, b)$, whence $b \in B_1$, $\sum_{k=1}^{\infty} a_k\, b(k) = \beta(\lambda, b)$.

Theorem B. *Let λ be determined by $\{a_k\}$. Then $S(\lambda)$ is the set of all $z \in Z$ such that $\sum_{k=1}^{\infty} a_k\, z(k)$ converges; its sum is $\sigma(\lambda, z)$ for each $z \in S(\lambda)$.*

Proof. If $z \in S(\lambda)$, then $\sigma(\lambda, z) = \sigma(\lambda_z, j) = \beta(\lambda_z, j) = \sum_{k=1}^{\infty} a_k\, z(k)$ by 13, 16 and by Theorem A. The same is true, if $\sum_{k=1}^{\infty} a_k\, z(k)$ converges.

Lemma b). *If $a_k \geqq 0$, $\sum_{k=1}^{\infty} a_k = \infty$ and if $\lim_{k\to\infty} a_k = 0$, then there exist $x, y \in Z$ such that $x + y = j$, $\sum_{k=1}^{\infty} a_k(x(k) - y(k)) = 0$.*

Proof. We find easily numbers $b_k = \pm 1$ such that $\sum_{k=1}^{\infty} a_k b_k = 0$. Now we put $x(k) = \frac{1}{2}(1 + b_k)$, $y(k) = \frac{1}{2}(1 - b_k)$.

Theorem C. *Let $\sum_{k=1}^{\infty} a_k$ be a non-absolutely convergent series of real numbers and let λ be determined by $\{a_k\}$. Then there exist $x, y \in S(\lambda)$ such that*

$$\sum_{k=1}^{\infty} a_k\, x(k)\, y(k) = \infty\,, \quad \sum_{k=1}^{\infty} a_k |x(k) - y(k)| = -\infty\,;$$

hence xy, $x + y \in Z - S(\lambda)$.

Proof. Put $z^+(k) = 1$ for $a_k > 0$, $z^+(k) = 0$ for $a_k \leqq 0$, $z^-(k) = 1 - z^+(k)$ $(k = 1, 2, \ldots)$. Clearly $\sum_{k=1}^{\infty} a_k\, z^+(k) = \sum_{k=1}^{\infty} (-a_k)\, z^-(k) = \infty$, $z^+z^- = 0$, $z^+ + z^- = j$.

For each $z \in Z$ and each $n \in N$ put $\lambda_n(z) = \sum_{k=1}^{n} a_k\, z(k)$. Then

$$\lambda_n(z^+) \to \infty\,, \quad \lambda_n(z^-) \to -\infty \tag{9}$$

and by lemma b) there exist $t^+, v^+, t^-, v^- \in Z$ such that $t^+ + v^+ = t^- + v^- = j$ (hence $t^+v^+ = t^-v^- = 0$) and that

$$\lambda_n(z^+t^+) - \lambda_n(z^+v^+) \to 0\,, \quad \lambda_n(z^-t^-) - \lambda_n(z^-v^-) \to 0\,. \tag{10}$$

We now define

$$x = t^+z^+ + t^-z^- , \quad x' = v^+z^+ + v^-z^- ,$$

$$y = t^+z^+ + v^-z^- , \quad y' = v^+z^+ + t^-z^- .$$

It follows from (10) that

$$\lambda_n(x) - \lambda_n(x') \to 0 , \quad \lambda_n(y) - \lambda_n(y') \to 0 .$$

Since $x + x' = y + y' = j$, we have

$$\lambda_n(x) + \lambda_n(x') = \lambda_n(y) + \lambda_n(y') = \lambda_n(j) \to \beta(\lambda, j) ,$$

whence $\lambda_n(x) \to \frac{1}{2}\beta(\lambda, j)$, $\lambda_n(y) \to \frac{1}{2}\beta(\lambda, j)$, so that, by Theorem B, $x, y \in S(\lambda)$.

Clearly $xy = t^+z^+$, $x + y = z^-$. According to (9), $\lambda_n(x + y) \to -\infty$. Since $t^+z^+ + v^+z^+ = z^+$, $t^+v^+ = 0$, we have $\lambda_n(t^+z^+) + \lambda_n(v^+z^+) = \lambda_n(z^+) \to \infty$ and by (10) we get $\lambda_n(xy) = \lambda_n(t^+z^+) \to \infty$, which completes the proof.

Theorem D. *Suppose that $z, z' \in Z - A$, $z + z' = j$. Let $\sum_{k=1}^{\infty} a_k$ be such a non-absolutely convergent series that $a_k z(k) = a_k$ for all k. Let the sequences $\{a_k\}$, $\{z'(k)\}$, $\{a_k + z'(k)\}$ determine mappings λ, μ, ν respectively. Then $\lambda(x) + \mu(x) = \nu(x)$ for each $x \in M(\lambda) \cap M(\mu)$, but the relation $T(\lambda) \cap T(\mu) \subset T(\nu)$ does not hold.*

Proof. Since $j \in B(\lambda)$, $z' \in M(\lambda)$, we have $z = j + z' \in T(\lambda)$; evidently $z \in M(\mu)$, whence $z \in T(\lambda) \cap T(\mu)$. Suppose that $z \in T(\nu)$. Then

$$(11) \qquad z = b + m , \quad b \in B(\nu) , \quad m \in M(\nu) .$$

As $\sum_{k=1}^{\infty} (a_k + z'(k))\, b(k)$ converges by Theorem A, there exists a k_0 such that $b(k)\, z'(k) = 0$ for each $k > k_0$. Since the set $\{k;\, z'(k) = 1\}$ is infinite and since $b \in A$, there exists a k_1 such that $b(k) = 0$ for all $k \geqq k_1$. By (11), $z(k) = m(k)$ for these k; it follows that

$$\sum_{k=k_1}^{\infty} |a_k| = \sum_{k=k_1}^{\infty} |a_k|\, z(k) = \sum_{k=k_1}^{\infty} |a_k + z'(k)|\, m(k) .$$

As $m \in M(\nu)$, we obtain $\sum_{k=k_1}^{\infty} |a_k| < \infty$, in contradiction to our hypothesis. Thus we get $z \in (T(\lambda) \cap T(\mu)) - T(\nu)$.

References

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Резюме

ПРОДОЛЖЕНИЯ АДДИТИВНЫХ ОТОБРАЖЕНИЙ

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Пусть Z — кольцо Буля и пусть $\mathfrak{G}$ — абелева группа. Пусть Λ — определенное семейство, элементы которого суть аддитивные отображения λ некоторого множества $M(\lambda) \subset Z$ в группу $\mathfrak{G}$. Всякому $\lambda \in \Lambda$ поставим в соответствие его аддитивное продолжение $\sigma(\lambda)$, отображающее множество $S(\lambda) \subset Z$ в группу $\mathfrak{G}$, и для $x \in S(\lambda)$ положим $(\sigma(\lambda))(x) = \sigma(\lambda, x)$. Если $\lambda, \lambda_1, \lambda_2 \in \Lambda$ и если $\lambda_1(x) + \lambda_2(x) = \lambda(x)$ для $x \in M(\lambda_1) \cap M(\lambda_2)$, то $\sigma(\lambda_1, x) + \sigma(\lambda_2, x) = \sigma(\lambda, x)$ для $x \in S(\lambda_1) \cap S(\lambda_2)$. Эти результаты используются в дальнейшей работе для обобщения интеграла Лебега.