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REMARKS ON THE KERNEL OF A MATRIX SEMIGROUP

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The intersection of all the (two-sided) ideals of a semigroup S is, if not empty, an ideal K of S contained in every ideal of S . K is a simple subsemigroup of S , called the kernel of S ([1], p. 67). In 1928, SUSCHKEWITSCH [9] showed that every finite semigroup has a kernel and determined the structure thereof. Ever since then, the questions of the existence and structure of the kernel of a semigroup have been in the forefront of the development of the theory of semigroups.

The purpose of the present paper is to investigate these questions for a semigroup of matrices of finite degree over a field.

If S is a matrix semigroup, we shall let $m(S)$ denote the set of elements of S of minimal rank. Since the rank of the product of two matrices does not exceed that of either factor, $m(S)$ is clearly an ideal of S . If S contains a kernel K , then clearly $K \subseteq m(S)$. We shall show that if S is pseudo-invertible, i.e., some power of every element of S lies in a subgroup of S , then $m(S)$ is a minimal ideal. It follows then from a theorem of MUNN [7] that $m(S)$ is completely simple in the sense of Rees (see [1] § 2.7). Moreover, if S is any matrix semigroup having a completely simple kernel K , then we must have $K = m(S)$.

An example is given in § 2 of a matrix semigroup S having a kernel K (necessarily not completely simple) such that $K \neq m(S)$. This example also answers negatively a question raised by CLIFFORD and PRESTON [1, p. 70], namely: *if K is the kernel of a semigroup S , is every left ideal of K also a left ideal of S ?*

The question of the structure of the kernel is tantamount to that of the structure of a simple semigroup. Consequently, for us this question takes the form: *what is the structure of a simple semigroup of matrices (of finite degree)?* As we shall see the possibilities are rather limited.

First we note, from considerations of rank, that any idempotent element of a simple matrix semigroup K is primitive relative to K ; hence if K contains an idempotent, it is completely simple (by definition). This excludes, for example, a wide class of non-completely simple (inverse) semigroups studied by Clifford [2]. In particular, for an example like that in § 2, K cannot contain an idempotent.

We prove that a simple matrix semigroup K is always contained in a completely simple semigroup S of matrices of the same degree as those of K . It follows easily from this, that if K contains a minimal left (or right) ideal, then it must be completely simple. This excludes those simple semigroups containing minimal left (but not minimal right) ideals, studied by MARIANNE TEISSER [11], R. CROISOT [3], and T. SAITÔ and S. HORI [8].

The author wishes to express his deep gratitude to Professor A. H. Clifford for his encouragement and generous help in the preparation of this paper. The proof of Theorem 2.2 (via Lemma 2.1) was suggested by Professor Clifford and is a considerable improvement over the original proof.

Preliminaries. By M_n we shall denote the multiplicative semigroup of the algebra of all matrices of degree n over an arbitrary field. We shall also regard M_n as the semigroup of all linear transformations on an n -dimensional vector space. The product ab of two elements of M_n will be their composition, first a and then b . By the rank $\varrho(a)$ of a is meant the dimension of the range of a .

If $S \subseteq M_n$ is a subsemigroup, we let $m(S) = \{x \in S : \varrho(x) = \min_{y \in S} \varrho(y)\}$. If e is an idempotent of S , we shall let $H(e) = H(e, S)$ denote the maximal subgroup of S containing e .

Throughout this paper we shall assume a knowledge of the results and notation of [1, chapters, 1, 2 and 3] concerning completely o -simple semigroups.

1. We state first without proof the following fundamental lemma due to Suschkewitsch; a proof of which may be found in [6, p. 273].

1.1. Lemma. *If $a \in M_n$, then a lies in a subgroup of M_n if and only if $\varrho(a) = \varrho(a^2)$.*

Moreover, some power of every element of M_n has this property.

1.2. We shall say that a semigroup S is *pseudo-invertible* if some power of every element of S lies in a subgroup of S . (The notion of pseudo-invertibility is due to DRAZIN [4], and the above characterization is that of MUNN [7].) Thus the last sentence of 1.1 states that M_n is pseudo-invertible.

1.3. Let J_i ($i = 0, 1, 2, \dots, n$) denote the elements of rank not exceeding i in M_n . It is well known (see e.g. [5], p. 162) that these are the only ideals of M_n . From [1, § 2.6] one readily deduces that the Rees factor semigroup J_i/J_{i-1} ($i = 1, 2, \dots, n$) are o -simple. Clearly each factor is pseudo-invertible since M_n is. Thus by a result of MUNN (see [1] p. 82 or [7], Theorem 2) each factor J_i/J_{i-1} is completely o -simple.

1.4. Lemma. *If e, f and g are idempotents in M_n such that*

- (i) $\varrho(e) = \varrho(f) = \varrho(g)$
- (ii) $ef \in H(g, M_n)$

then $\{g, f\}$ is a left zero semigroup and $\{g, e\}$ is a right zero semigroup.

Proof. By 1.3 it suffices to prove that if e, f and g are idempotents in a completely o -simple semigroup S such that $ef \in H(g, S)$ then the conclusions of the lemma hold. Now applying the Rees representation theorem for completely o -simple semigroups [1, chapter 3] one may verify in a straightforward manner that $\{g, f\}$ and $\{g, e\}$ are, respectively, left and right zero subsemigroups of S .

1.5. Theorem. *If S is a pseudo-invertible semigroup of matrices, then $m(S)$ is a completely simple minimal ideal of S .*

Proof. Let $K = m(S)$. We first establish that K is a union of groups. Let $x \in K$, then $\varrho(x) = \varrho(x^2)$; hence by 1.1 x lies in a subgroup of M_n , say $H(e, M_n)$ for some idempotent e in M_n . Since S is pseudo-invertible, some power of x , say x^q , lies in a subgroup $H(f, S)$ of S , where $f = f^2 \in S$. Since $x^q \in H(e, M_n) \cap H(f, S)$, it is well known [1, p. 22] that we must have $f = e$ and $H(f, S) \subseteq H(e, M_n)$. Hence $exe = x$, and if y is the inverse of x^q in $H(e, S)$ then $y' = x^{q-1}y$ is a left inverse and $y'' = yx^{q-1}$ is a right inverse for x , both of which are in S . It follows that $y' = y''$ and that $x \in H(e, S)$. Since $H(e, S)$ meets K and K is an ideal, $H(e, S) \subseteq K$. Therefore K is a union of groups.

Now let I be any ideal of S . We wish to show that $K \subseteq I$. Let x be in K . x lies in some subgroup $H(e, S)$ of K . It is clear that $I \cap K$ is an ideal and is a union of groups since K is. Let f be an idempotent in $I \cap K$. Let $H(g, S)$ be the subgroup of $I \cap K$ which contains fe . Since e, f , and g are in K we have $\varrho(e) = \varrho(f) = \varrho(g)$. It now follows from 1.4 that $\{e, g\}$ is a left zero semigroup. In particular $e = eg$. Since $g \in I$, e must be also. Therefore $x = ex \in I$, implying $K \subseteq I$. Now by a theorem of Munn (Theorem 2.55, p. 82 of [1]) it follows that since K is simple and pseudo-invertible, K is completely simple, q.e.d.

1.6. Corollary. *If S is a finite semigroup of matrices, then $m(S)$ is the kernel of S .*

1.7. Theorem. *If S is a semigroup of matrices with a completely simple kernel K , then $K = m(S)$.*

Proof. Since $m(S)$ is an ideal, we need only show that $m(S) \subseteq K$. Let $x \in m(S)$. Then $\varrho(x) = \varrho(x^2)$ and by 1.1 $x \in H(e, M_n)$ for some $e = e^2 \in M_n$. Let y be an element of K . Then $xyx \in K$ and $xyx = exyxe \in eKe$. Since $K \subseteq m(S)$ we know that $\varrho(e) = \varrho(xyx)$. These facts imply that e and xyx have the same range and null space; hence by [1, 6(b), p. 57] e and xyx are in the same \mathcal{H} -class of M_n . Since this \mathcal{H} -class contains an idempotent, it is a group [1, p. 59]. Now since K is a union of groups and $xyx \in K$ we must have $e \in K$. Whence $x = ex \in K$, q.e.d.

1.8. Corollary. *If S is a compact semigroup of real or complex matrices, then $m(S)$ is the kernel of S .*

Proof. This follows from the well-known fact that a compact topological semigroup always has a completely simple kernel (see e.g. [12]).

2. Example. Define a product on Euclidean 4-space as follows:

$$(a, b, c, d)(x, y, z, w) = (a + bx + cw, by, bz, d).$$

This semigroup may be faithfully represented as a semigroup of real matrices by letting (x, y, z, w) correspond to

$$\begin{pmatrix} 1 & 0 & 0 \\ x & y & z \\ w & 0 & 0 \end{pmatrix}.$$

Associativity is then automatic. We now take K to be the subsemigroup consisting of all (x, y, z, w) with

$$x > 0, y > 0, z \geq 0 \text{ and } w \geq 0.$$

Let $e = (0, 1, 0, 0)$ and set $S = K \cup \{e\}$. It is routine to verify that S is a semigroup and that K is the kernel of S . Clearly $m(S) = S$ and hence $m(S) \neq K$.

To see that S settles the question of Clifford and Preston mentioned in the introduction, let L be the principal left ideal of K generated by any element $p = (a, b, c, d)$ of K with $d > 0$. One may easily show that $ep \notin L$ and hence L is not a left ideal of S .

The following lemma is straightforward and we therefore omit its proof.

2.1. Lemma. *Let Q be a completely o -simple semigroup, and let S be a subsemigroup of $Q \setminus \{0\}$ such that every \mathcal{H} -class of Q meeting S is a group. Then the union T of all \mathcal{H} -classes of Q meeting S is a completely simple semigroup contained in $Q \setminus \{0\}$.*

We observe that if S is a simple subsemigroup of M_n , then $m(S) = S$ since $m(S)$ is an ideal of S . Hence $\varrho(x) = \varrho(x^2)$ for all x in S , and by 1.1, x lies in a subgroup $H(x, M_n)$ of M_n .

2.2. Theorem. *If S is a simple subsemigroup of M_n , then*

$$T = \bigcup \{H(x, M_n) : x \in S\}$$

is a completely simple subsemigroup of M_n .

Proof. As remarked above $m(S) = S$, and therefore every element of S has the same rank, say k . In the notation of 1.3, let $Q = J_k/J_{k-1}$. Identifying the elements of T with those of Q which correspond under the quotient mapping, one easily sees that T is the union of all \mathcal{H} -classes of Q meeting S . The theorem now follows from the lemma of 2.1.

2.3. Corollary. *If S is a simple semigroup of matrices, the following are equivalent:*

- (i) S is completely simple.
- (ii) S contains a minimal left (right) ideal.
- (iii) $a \in Sa \cup aS$ for some a in S .
- (iv) S contains an idempotent.

Proof. Clearly (i) \Rightarrow (ii) \Rightarrow (iii). To show that (iii) \Rightarrow (iv), we first note that by 2.2 we may assume that S is contained in a completely simple semigroup of matrices. Using the Rees representation it is easily seen that $a = ba$ in a completely simple semigroup implies that $b^2 = b$. Similarly $a = ba$ implies $b^2 = b$. In either case S contains an idempotent. Now any idempotent in S must be primitive since every element of S has the same rank; so clearly (iv) implies (i).

Remark. SCHWARZ [10, p. 229] shows that if S is a simple subsemigroup of a completely o -simple semigroup T then $P_{\alpha\beta} = S \cap H_{\alpha\beta}$, where $H_{\alpha\beta}$ is an \mathcal{H} -class of T , is also simple and contains an idempotent if and only if S does. If S is completely simple then $P_{\alpha\beta}$ are isomorphic. Is this true if S is simple but not completely simple?

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Резюме

ЗАМЕТКИ О ЯДРЕ МАТРИЧНЫХ ПОЛУГРУПП

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Пусть S — полугруппа матриц конечного порядка. Пусть $m(S)$ -множество всех элементов $\in S$, имеющих минимальный ранг. Если S содержит минимальный двусторонний идеал K (так называемое ядро S), то $K \subseteq m(S)$. На примере показано, что K необязательно равно $m(S)$.

Назовем S псевдо-инверсной полугруппой, если для всякого $a \in S$ существует натуральное число $n = n(a)$ так, что a^n содержится в некоторой подгруппе, лежащей в S . В работе доказывается, что для псевдо-инверсной полугруппы матриц S имеет место $m(S) = K$ и K -вполне простая полугруппа. Наоборот, если S -любая полугруппа матриц, для которой ядро K вполне простое, то $K = m(S)$.

Цитированный пример дает отрицательный ответ на один открытый вопрос из теории полугрупп. А именно, доказывается: Если K -ядро полугрупп S , то левый идеал из K не является необходимо левым идеалом из S .