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THE INTERVAL TOPOLOGY OF A CERTAIN L -GROUP

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TO ALEXANDER DONIPHAN WALLACE
on the occasion of his 60th birthday.

It is the purpose of this note to present a counterexample to a long-standing conjecture about lattice-ordered groups (l -groups). It will be shown that there exists an l -group which is not totally ordered, but which is a topological group and a topological lattice in its interval topology.

An l -group is a partially ordered group which is a lattice in its partial ordering. If G is an l -group, the *interval topology* of G is that topology obtained by taking as a subbasis for the closed sets, the sets (called *cones*) of the form $\{f \in G \mid f \geq f_0\}$ and $\{g \in G \mid g \leq g_0\}$. It is easily verified that any totally ordered l -group is a topological group and a topological lattice in its interval topology. In [1], BIRKHOFF raised the question whether every l -group is a topological group and a topological lattice in its interval topology. NORTHAM [6] showed that there is an l -group which is not a Hausdorff space in its interval topology, and hence is not a topological group. CHOE [2], WOLK [7], CONRAD [3], and JAKUBÍK [5] found larger and larger classes of l -groups such that if an l -group belongs to the class and is a Hausdorff space in its interval topology, then it is totally ordered. In particular, the class of all l -groups which are subdirect sums of totally ordered groups is such a class, and hence so are the commutative l -groups [5]. In all of this work, no example of a non-totally ordered l -group which is a Hausdorff space in its interval topology was presented. In fact, the nature of the theorems proved, especially in [7], [3], and [5], would lead one to conjecture that no such l -group exists. Nevertheless, we now give an example of a non-totally ordered l -group which is not only a Hausdorff space, but even a topological group and a topological lattice in its interval topology.

Let G denote the set of all order-preserving permutations f of the ordered set R of real numbers, which satisfy the condition

$$(*) \quad (x - 1)f = xf - 1 \quad \text{for all } x \in R.$$

It was observed in [4] that G is an l -group under the operation of composition of

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functions, and the order: $f \leq g$ if and only if $xf \leq xg$ for all $x \in R$. Clearly, G is not totally ordered.

Lemma. *If $a, b, a', b' \in R$, if $a < b < a + 1$, and if $a' < b' < a' + 1$, then there exists $f \in G$ such that $af = a'$ and $bf = b'$.*

Proof. Obviously, there is an order-preserving permutation f' of R such that $af' = a'$, $(a + 1)f' = a' + 1$, and $bf' = b'$. Let f'' be the restriction of f' to the interval $[a, a + 1]$. Then f'' can be uniquely extended, via the equation (*), to an element f of G .

As it is somewhat more natural to deal with open sets, we take the equivalent definition of the interval topology as that topology whose open sets are subgenerated by sets of the form $\{f \in G \mid f \not\leq f_0\}$ and $\{g \in G \mid g \not\geq g_0\}$. It is easy to see that if U is a subbasic open set, then so are U^{-1} , Uf , and fU for any $f \in G$. It is also obvious that the interval topology is always T_1 . Thus, in order to show that G is a topological group, it suffices to show that if U is a subbasic open set containing the identity e of G , then there exists an open set V containing e such that $V^2 \subseteq U$. This we proceed to do.

Let $U = \{g \in G \mid g \not\leq g_0\}$ be a subbasic open set containing $e \in G$. (The case of a U of the other form is omitted, but can be handled similarly.) As $e \in U$, $e \not\leq g_0$, so there is some $\gamma \in R$ such that $\gamma g_0 < \gamma = \gamma e$. Choose real numbers u, v, w, x, y , and z , such that

$$\begin{aligned} \gamma g_0 < u < v < w < x < y < z < \gamma, \quad x < w + 1, \quad v < u + 1, \\ \gamma < z + 1 \quad \text{and} \quad y < x + 1. \end{aligned}$$

Then since $w < x < w + 1$ and $v < u + 1 < v + 1$, by the lemma there is an $f_1 \in G$ such that $wf_1 = v$ and $xf_1 = u + 1$. Also, since $z < \gamma < z + 1$ and $y < x + 1 < y + 1$, there is an $f_2 \in G$ such that $zf_2 = y$ and $\gamma f_2 = x + 1$.

Let $V = \{f \in G \mid f \not\leq f_1 \text{ and } f \not\leq f_2\}$. Then V is an open set (being the intersection of two subbasic open sets). Also, $e \in V$, because $zf_2 = y < z = ze$ and $wf_1 = v < w = we$.

Suppose $g, g' \in V$. Then since $g' \not\leq f_1$, for some $\alpha \in R$, $\alpha f_1 < \alpha g'$, and since g' and f_1 satisfy (*), we may assume $x - 1 < \alpha < x$. Likewise, for some $\beta \in R$, $\gamma - 1 < \beta < \gamma$, $\beta f_2 < \beta g$. Therefore,

$$\begin{aligned} \gamma g_0 < u &= xf_1 - 1 = (x - 1)f_1 < \alpha f_1 < \alpha g' < xg' = \\ &= (\gamma f_2 - 1)g' = (\gamma - 1)f_2 g' < \beta f_2 g' < \beta g g' < \gamma g g'. \end{aligned}$$

Thus, $gg' \not\leq g_0$, and so $gg' \in U$, or $V^2 \subseteq U$. Hence G is a topological group in its interval topology.

It follows, of course, that G is a Hausdorff space. However, we prefer to show this directly, since, in fact, G is a Hausdorff space in a very strong sense. It was observed in [6] that a lattice L is a Hausdorff space in its interval topology if and only if for

each two elements g and g' of L , there is a covering of L by a finite number of cones such that no cone contains both g and g' . We shall show that for each $g, g' \in G$, $g \neq g'$ there are two cones whose union contains G and such that neither cone contains both g and g' .

Without loss of generality, assume $g' = e$ and $g \not\leq e$. Then for some $\gamma \in R$, $\gamma < \gamma g$. Choose $u, v, w, x, y, z \in R$ such that

$$\begin{aligned} \gamma < u < v < w < x < y < z < \gamma g, \quad v < u + 1, \quad z < y + 1, \\ w < v + 1 \quad \text{and} \quad y < x + 1. \end{aligned}$$

Then since $u < v < u + 1$ and $z < y + 1 < z + 1$, by the lemma, there exists $f_1 \in G$ such that $uf_1 = z$ and $vf_1 = y + 1$. Similarly, there exists $f_2 \in G$ such that $wf_2 = x$ and $(v + 1)f_2 = y$. Let $U = \{f \in G \mid f \leq f_1\}$ and $V = \{f \in G \mid f \geq f_2\}$. Then since $uf_1 = z < \gamma g < ug$, $g \notin U$. Likewise, since $wf_2 = x > w = we$, $e \notin V$, and hence neither cone contains both e and g .

Let $f \in G$. If $f \notin V$ then for some $\alpha \in R$, $v \leq \alpha \leq v + 1$, $\alpha f < \alpha f_2$. Let $\beta \in R$, $v \leq \beta \leq v + 1$. Then

$$\begin{aligned} \beta f &\leq (v + 1)f = vf + 1 \leq \alpha f + 1 < \alpha f_2 + 1 \leq (v + 1)f_2 + 1 = \\ &= y + 1 = vf_1 \leq \beta f_1. \end{aligned}$$

Hence $f < f_1$ and $f \in U$. Therefore the union of U and V contains G .

Finally, we wish to show that G is a topological lattice. It will suffice to show that if $f_1, f_2 \in G$ such that $f_1 \cap f_2 = e$ (\cap and \cup indicate the lattice operations in G), and if W is a subbasic open set containing e , then there exist open sets U and V such that $f_1 \in U$, $f_2 \in V$, and for every $f \in U$, $g \in V$, $f \cap g \in W$. The continuity of \cap everywhere follows by homogeneity, and the continuity of \cup follows by duality.

As \cap does not play the same role with respect to the two different kinds of subbasic open sets, we must consider two cases.

Case 1. $W = \{f \in G \mid f \leq g_0\}$. Since $e \in W$, $e \leq g_0$, and there exists $\gamma \in R$ with $\gamma g_0 < \gamma$. Choose $x, y \in R$ such that

$$\gamma g_0 < x < y < \gamma < y + 1 \quad \text{and} \quad x < \gamma g_0 + 1.$$

Then by the lemma, there exists $h \in G$ such that $yh = x$ and $\gamma h = \gamma g_0 + 1$.

Let $U = V = \{f \in G \mid f \leq h\}$. Since $yh = x < y = ye$, $e \leq h$, and therefore $f_1 \leq h$ and $f_2 \leq h$. Hence $f_1 \in U$ and $f_2 \in V$. Now if $f \in U$, then for some $\alpha \in R$, $\gamma - 1 < \alpha < \gamma$, $\alpha f > \alpha h$. Hence

$$\gamma g_0 = \gamma h - 1 = (\gamma - 1)h < \alpha h < \alpha f < \gamma f.$$

Thus if $f \in U$ and $g \in V = U$, $\gamma(f \cap g) = \min(\gamma f, \gamma g) > \gamma g_0$, so $f \cap g \not\leq g_0$ and $f \cap g \notin W$.

Case 2. $W = \{f \in G \mid f \not\leq g_0\}$. By duality with case 1, we can choose $\gamma < x < y < \gamma g_0$ and $h \in G$ such that $xh = y$ and $(\gamma + 1)h = \gamma g_0$. Let $U = \{f \in G \mid f \not\leq h\}$. By duality, if $f \in U$, $\gamma f < \gamma g_0$. Now, as $\min(xf_1, xf_2) = x(f_1 \cap f_2) = xe = x < y = xh$, then either $xf_1 < xh$ or $xf_2 < xh$. That is, either $f_1 \in U$ or $f_2 \in U$. Say $f_1 \in U$. Let $V = G$. Then $f_2 \in V$. Now if $f \in U$ and $g \in V$, $\gamma(f \cap g) \leq \gamma f < \gamma g_0$, so $f \cap g \not\leq g_0$ and $f \cap g \in W$.

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Резюме

ИНТЕРВАЛЬНАЯ ТОПОЛОГИЯ В НЕКОТОРОЙ l -ГРУППЕ

ЧАРЛЕС ГОЛАНД, (Charles Holland), Медисон, Висконсин

В статье строится пример структурно упорядоченной группы, которая не является линейно упорядоченной, но является топологической группой и топологической структурой относительно интервальной топологии. Этим решается вопрос, поставленный в работе [3].