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EQUIVALENT SYSTEMS OF SETS AND HOMEOMORPHIC TOPOLOGIES

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Let P be a set, $X, Y \subset P$. Let us say that X *is congruent with* Y, if a permutation f of the set P (i.e. a one-to-one mapping of the set P on P) exists such that $f(X) = Y$. We write $X \sim Y$[1]). Evidently there holds: $X \sim X$; $X \sim Y \Rightarrow Y \sim X$; $(X \sim Y, Y \sim Z) \Rightarrow X \sim Z$; $X \sim Y \Rightarrow P - X \sim P - Y$; $X \sim Y \equiv$ (card X = card Y, card $P - X$ = card $P - Y$).

Let $\mathscr{S}$ and $\mathscr{T}$ be systems of subsets of P such that a permutation f of the set P exists for which $\mathscr{T} = \{Y : Y = f(X), X \in \mathscr{S}\}$. Then we say that $\mathscr{S}$ is an *equivalent system to* $\mathscr{T}$ and we write $\mathscr{S} \sim \mathscr{T}$ (or also $\mathscr{T} = f(\mathscr{S})$). Evidently

$$\mathscr{S} \sim \mathscr{S}\,; \quad \mathscr{S} \sim \mathscr{T} \Rightarrow \mathscr{T} \sim \mathscr{S}\,; \quad (\mathscr{S} \sim \mathscr{T}\,, \quad \mathscr{T} \sim \mathscr{U}) \Rightarrow \mathscr{S} \sim \mathscr{U}\,.$$

Let $\mathscr{S}' = \{Y : Y = P - X, X \in \mathscr{S}\}$. $\mathscr{S}'$ is called the system of complements to $\mathscr{S}$. We have

$$(\mathscr{S}')' = \mathscr{S}\,; \quad \mathscr{S} \sim \mathscr{T} \equiv \mathscr{S}' \sim \mathscr{T}'\,.$$

The following statement is evident, too.

Theorem 1. *Let* $\mathscr{S} \subset 2^P$. *Then the following statements are equivalent*:

1) $\mathscr{S} \sim \mathscr{T} \Rightarrow \mathscr{S} = \mathscr{T}$.

2) $(X \in \mathscr{S}, X \sim Y) \Rightarrow Y \in \mathscr{S}$.

Theorem 2. *Let* card $P = p \geqq \aleph_0$. *Let* $c(\mathscr{S})$ *denote the system of all systems* $\mathscr{T} \subset 2^P$ *for which* $\mathscr{S} \sim \mathscr{T}$. *Let* card $c(\mathscr{S}) \neq 1$. *Then* card $c(\mathscr{S}) \geqq p$.

Proof. By Theorem 1, a pair of congruent sets X and Y exists such that $X \in \mathscr{S}$ and Y non $\in \mathscr{S}$.

1) The concept "congruent sets" has been introduced in [1] II. part, pp. 84. See [2] and [3], too.

1. Let card $X < p$. Then such a set Z exists, for which $Z \cap (X \cup Y) = \emptyset$ and $Z \sim X$. If $Z \in \mathscr{S}$, put $X_1 = Y$, $X_2 = Z$, if Z non $\in \mathscr{S}$, put $X_1 = X$, $X_2 = Z$. Let R be a decomposition on P into sets congruent with X such that $X_1, X_2 \in R$. Let R_1 be a system of those elements of the decomposition R belonging to $\mathscr{S}$, R_2 a system of elements of the decomposition R not belonging to $\mathscr{S}$. $R_1 \neq \emptyset \neq R_2$ (as it follows from the choice of X_1 and X_2) and at least one of these sets has the cardinality p. Let it be e.g. R_1 (for R_2 the procedure is analogous). Let the elements of the system R_1 be denoted by $Y_1, Y_2, \ldots, Y_\iota, \ldots$. Let Y' be an element of the system R_2. There exists such a permutation f_ι of the set P for which $f_\iota(Y') = Y_\iota$, $f_\iota(Y_\iota) = Y'$ and for $(V \in R,\ V \neq Y,\ V \neq Y') \Rightarrow f_\iota(V) = V$. Thus, Y_ι non $\in f_\iota(\mathscr{S})$, $Y' \in f_\iota(\mathscr{S})$, $(V \neq Y_\iota,$ $V \neq Y') \Rightarrow V \in \mathscr{S} \equiv V \in f_\iota(\mathscr{S})$. From it follows immediately $\iota \neq \kappa \Rightarrow f_\iota(\mathscr{S}) \neq$ $\neq f_\kappa(\mathscr{S})$. As the set of indices ι has the cardinality p, we have consequently card $c(\mathscr{S}) \geqq p$.

2. Let card $X = p$, card $P - X = p$. Let $X' = P - X$, $Y' = P - Y$. We have $X' \sim X$, $Y' \sim Y$. First, we shall define certain sets X'_1, X'_2 as follows:

a) If X' non $\in \mathscr{S}$, put $X'_1 = X$, $X'_2 = X'$.

b) If a) does not occur and if $Y' \in \mathscr{S}$, put $X'_1 = Y'$, $X'_2 = Y$.

c) Don't let occur either a) or b). Let Z be such a set from sets X and X', for which card $(Z \cap Y') = p$. Let such a subset Z' exist, $Z' \subset Z \cap Y'$, $Z' \sim X$ and $Z' \in \mathscr{S}$. Then put $X'_1 = Z'$, $X'_2 = Y$. Let Z' non $\in \mathscr{S}$ for any $Z' \subset Z \cap Y'$, $Z' \sim X$. Then, denote Z_1 one such subset and put $X'_1 = P - Z$, $X'_2 = Z_1$. (Thus, in all cases we have defined two sets X'_1 and X'_2 such that $X'_1 \sim X'_2 \sim X$, $X'_1 \cap X'_2 = \emptyset$, $X'_1 \in \mathscr{S}$, X'_2 non $\in \mathscr{S}$.)

α) Let there exist $Z^* \subset X'_1$, $Z^* \sim X$, card $X'_1 - Z^* = p$, $Z^* \in \mathscr{S}$. Then put $X_1 = Z^*$, $X_2 = X'_2$.

β) Don't let α) occur. Let there exist $Z^* \subset X'_2$, $Z^* \sim X$, card $X'_2 - Z^* = p$, Z^* non $\in \mathscr{S}$. Then, we put $X_1 = X'_1$, $X_2 = Z^*$.

γ) Let neither α) nor β) occur. Let Z_1 be a subset in X', $Z_1 \sim X$, card $X'_1 - Z_1 = p$, let Z_2 be a subset in X'_2, $Z_2 \sim X$, card $(X'_2 - Z_2) = p$. Then, Z_1 non $\in \mathscr{S}$, $Z_2 \in \mathscr{S}$. Put $X_1 = Z_2$, $X_2 = Z_1$.

Thus, there always exist sets $X_1 \in \mathscr{S}$, X_2 non $\in \mathscr{S}$, $X_1 \sim X_2 \sim X$, card $P - X_1 \cup$ $\cup X_2 = p$. There exists a decomposition R on P such that it contains p elements, that X_1 and X_2 are elements of this decomposition and all elements are congruent with X. The proof is to be continued as in the preceeding case.

3. Let card $X = p$, card $P - X < p$. Then, $P - X \in \mathscr{S}'$, $P - Y$ non $\in \mathscr{S}'$. Thus, according to 1. card $c(\mathscr{S}') \geqq p$, and consequently card $c(\mathscr{S}) =$ card $c(\mathscr{S}') \geqq p$.

Thus, the proof of the theorem is finished.

Theorem 2 does not hold for finite sets, as it can easily be seen from the following example: $P = \{1, 2, 3, 4\}$, $\mathscr{S} = \{\{1, 2\}, \{3, 4\}\}$. It is clear that just the systems $\mathscr{S}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$ are equivalent to $\mathscr{S}$.

As 2^p permutations of P exist, there exist at most 2^p systems equivalent to a given system $\mathscr{S}$. Next, let us prove the following theorem.

Theorem 3. *Let $p \geqq \aleph_0$. Then 2^{2^p} non-equivalent systems $\mathscr{S} \subset 2^P$ exist such that the number of systems equivalent to them is exactly 2^p.*

Proof. Let $P = P_1 \cup P_2$, card P_1 = card $P_2 = p$, $P_1 \cap P_2 = \emptyset$. Let $\mathscr{S}$ be a subsystem in 2^{P_1} containing P_1. We shall show that card $c(\mathscr{S}) = 2^p$. Let f be a one-to-one mapping of P_2 on P_1. Let $X \subset P_2$. Let us define the permutation f_X of the set P in the following way:

$x \in X \Rightarrow f_X(x) = f(x)$,

$x \in f(X) \Rightarrow f_X(x) = f^{-1}(x)$,

else $f_X(x) = x$.

Evidently, $\bigcup f_X(\mathscr{S}) \cap P_2 = X$. Thus, $(X \neq Y;\ X, Y \subset P_2) \Rightarrow f_X(\mathscr{S}) \neq f_Y(\mathscr{S})$. Thus card $c(\mathscr{S}) = 2^p$.

Let $\mathfrak{S}$ be the class of all systems $\mathscr{S} \subset 2^{P_1}$, containing P_1. Evidently card $\mathfrak{S} = 2^{2^p}$. Let us decompose $\mathfrak{S}$ in classes of mutually equivalent systems. As every of this classes has cardinality at most 2^p, there exist 2^{2^p} these classes. Let $\mathfrak{S}_1$ be the class containing one element of each class of the mentioned decomposition. In accordance with what was said, we have card $\mathfrak{S}_1 = 2^{2^p}$; $(\mathscr{S}_1, \mathscr{S}_2 \in \mathfrak{S}_1;\ \mathscr{S}_1 \neq \mathscr{S}_2) \Rightarrow \mathscr{S}_1$ non $\sim \mathscr{S}_2$; $\mathscr{S} \in \mathfrak{S}_1 \Rightarrow$ card $c(\mathscr{S}) = 2^p$.

Let F and G be mappings of the system 2^P into 2^P (thus $X \subset P \Rightarrow F(X) \subset P$; $X \subset P \Rightarrow G(X) \subset P$). We say that the mapping F is *equivalent to* G if a permutation f of the set P exists such that $X \subset P \Rightarrow f(F(X)) = G(f(X))$. We write $F \sim G$ or also $G = f \circ F$. The relation $\sim$ is evidently an equivalence. Assign a mapping F' to the mapping F as follows: $F'(X) = P - F(P - X)$. Call the mapping F' the complementary mapping to F. It holds $F \sim G \Rightarrow F' \sim G'$. If, namely, $G = f \circ F$, then $f(F'(X)) = P - f(F(P - X)) = P - G(f(P - X)) = G'(f(X))$. Thus $G' = f \circ F'$. Furthermore $(F')' = F$.

Theorem 4. *Let* card $P = p \geqq \aleph_0$. *Let $F \in (2^P)^{2^P}$. Then the cardinality of the set mappings G equivalent with F is 1 or at least p. The first case occurs exactly when F has these two properties*:

1) $X \subset P \Rightarrow F(X) \in \{P, X, P - X, \emptyset\}$.

2) *If $X \sim Y$ then*

$F(X) = P \Rightarrow F(Y) = P.$

$F(X) = X \Rightarrow F(Y) = Y.$

$F(X) = P - X \Rightarrow F(Y) = P - Y.$

$F(X) = \emptyset \Rightarrow F(Y) = \emptyset.$

Proof. It can be readily seen that a mapping F fulfilling the relations 1) and 2) is equivalent to itself only.

Let $F \in (2^P)^{2^P}$, $X \subset P$. Put

$$n_F(X) = \text{card}\,(F(X) - X)\,, \quad d_F(X) = \text{card}\,(X - F(X))\,,$$

$$o_F(X) = \text{card}\,([P - F(X)] - X)\,, \quad m_F(X) = \text{card}\,(F(X) \cap X)\,.$$

The ordered quadruple of cardinal numbers $(n_F(X), d_F(X), o_F(X), m_F(X))$ is called the *type* of the set X in the mapping F and we denote it by $T_F(X)$. Put $S_F(X) = \{Y: Y \sim X \text{ and } T_F(Y) = T_F(X)\}$. Let $S_X = \{Y: Y \sim X\}$.

First it is evident that $T_F(X) = T_{fFf^{-1}}(f(X))$ for every permutation f. Further, it is evident that systems $S_F(Y)$ for $Y \in S_X$ form a decomposition on S_X.

We have $fS_F(X) = S_{fFf^{-1}}(f(X))$ for every permutation f. It holds, namely, $Z \in fS_F(X) \Rightarrow Z = f(Z_1)$ for a suitable $Z_1 \in S_F(X) \Rightarrow T_{fFf^{-1}}(Z) = T_F(Z_1) = T_F(X) = T_{fFf^{-1}}(f(X)) \Rightarrow Z \in S_{fFf^{-1}}(f(X))$.

$$Z \in S_{fFf^{-1}}(f(X)) \Rightarrow T_{fFf^{-1}}(Z) = T_{fFf^{-1}}(f(X)) = T_F(f^{-1}Z) = T_F(X) \Rightarrow f^{-1}Z \in S_F(X) \Rightarrow Z \in fS_F(X)\,.$$

Now, let F be such a mapping 2^P into 2^P that the cardinality of the set of mappings equivalent to F (denote it by M) is less than p. We shall show that

$$\text{(A)} \qquad S_F(X) = S_X\,.$$

Suppose that this is not true. Then the system of all $fS_F(X)$, where f runs through all possible permutations of the set P, contains at least p different sets according to Theorem 2. Thus, two different permutations f and g exist such that $fS_F(X) \neq gS_F(X)$ and $fFf^{-1} = gFg^{-1} = G$. Then $fS_F(X) = S_G(f(X))$, $gS_F(X) = S_G(g(X))$. As the sets of the form $S_G(Y)$ for $Y \sim X$ constitute a decomposition on S_X, we have $S_G(f((X)) \cap S_G(g(X)) = \emptyset$. Simultaneously, $T_G(f(X)) = T_F(X) = T_G(g(X))$. Thus, $g(X) \in S_G(f(X))$ which is a contradiction. Hence, (A) is valid.

Let us choose $X \subset P$ arbitrarily but fixed. Suppose that 1) does not hold.

α) Let card $(P - X) = p$. α_1) Let $\emptyset \neq F(X) - X \subsetneqq P - X$. There exist at least p sets Z in $P - X$ congruent with the set $F(X) - X$. Thus, card $M \geqq p$, which is a contradiction.

α_2) Let $\emptyset \neq F(X) \subsetneqq X$. Let R be a decomposition on P into sets congruent with X and let card $R = p$. From (A) it follows $Y \in R \Rightarrow \emptyset \neq F(Y) \subsetneqq Y$. For every $Y \in R$ choose $a(Y) \in Y - F(Y)$, $b(Y) \in F(Y)$. Let f_Y be such a permutation of the set P that

$$f_Y(a(Y)) = b(Y),\, f_Y(b(Y)) = a(Y)\,, \quad \text{otherwise} \quad f_Y(x) = x\,.$$

For $Z \in R$, $Z \neq Y$ we have $f_Y(Z) = Z$, $f_Y(F(Z)) = f(Z)$. For Y it holds $f_Y(Y) = Y$, $f_Y(F(Y)) \neq F(Y)$. Thus $(Y, Z \in R;\ Y \neq Z) \Rightarrow f_Y \circ F \neq f_Z \circ F$, whence card $M \geqq p$, which is a contradiction.

α_3) Let neither α_1) nor α_2) occur, i.e. $P - X \subsetneqq F(X) \neq P$. Then, put $G(Y) = P - F(Y)$ for all $Y \subset P$. The number of equivalent mappings to G is also less than p and at the same time $G(X) \subset X$. We get a contradiction just as in α_2).

β) Let card $P - X < p$. Then instead of F we consider the complementary mapping F'. For $P - X$ 1) does not occur. In accordance with α) at least p mappings equivalent to F' exist. Constructing the complementary mappings to them, we get at least p mappings equivalent to F. Thus we get a contradiction.

Let for every set $X \subset P$ 1) be fulfilled. Then, according to (A) 2), holds, too.

From Theorem 4 the ensuing result follows immediately. *Let (P, u) be the Čech's topological space*[2]) *with* card $P = p \geqq \aleph_0$. *Then the cardinality of the set of topological spaces (P, v) homeomorphic with (P, u) is* 1 *or at least p. The cardinality of the set is* 1 *exactly if*

1. $X \subset P \Rightarrow uX = X$ or P.
2. $(X, Y \subset P;\ X \sim Y;\ uX = X) \Rightarrow uY = Y$.

This consequence follows also immediately from Theorem 1 and 2 for topologies defined by means of the system of open or closed sets (see e.g. [4]). In the case of the general Čech's topologies such a definition is impossible.

In this connection the following problem arises.

Is it possible to assign to any Čech's space (P, u) the system $\mathcal{S}(u) \subset 2^P$ so that $u \neq v \Rightarrow \mathcal{S}(u) \neq \mathcal{S}(v)$ and (u being homeomorphic with v) $\equiv \mathcal{S}(u) \sim \mathcal{S}(v)$?

References

[1] *Whitehead-Russell:* Principia mathematika II. part, Cambridge 1927.

[2] *E. Szpilrajn:* Sur l'équivalence des suites d'ensembles et l'équivalence des fonctions. Fund. Math. *XXVI* (1936), 302–326.

[3] *R. Sikorski:* On the inducing of homomorphismus by mappings. Fund. Math. *XXXVI* (1949), pp. 7–22.

[4] *N. Bourbaki:* Topologie générale, p. 1., Paris 1940.

[2]) Here $u \in (2^P)^{2^P}$ and we have $uX \supset X$; $u\emptyset = \emptyset$; $X \subset Y \subset P \Rightarrow uX \subset uY$.

Резюме

ЭКВИВАЛЕНТНЫЕ СИСТЕМЫ МНОЖЕСТВ И ГОМЕОМОРФНЫЕ ТОПОЛОГИИ

ФРАНТИШЕК НЕЙМАН И МИЛАН СЕКАНИНА (F. Neuman a M. Sekanina), Брно

Пусть P — множество; $X, Y \subset P$. Мы говорим, что X конгруэнтно Y, если существует такая перестановка f множества P, что $f(X) = Y$, и записываем $X \sim Y$. (Перестановка f — это взаимно однозначное отображение P на P.) Пусть $\mathscr{S}$ и $\mathscr{T}$ — системы подмножеств P. Если существует перестановка f множества P такая, что $\mathscr{T} = \{Y : Y = f(X), X \in \mathscr{S}\}$, то $\mathscr{S}$ эквивалентно $\mathscr{T}$ и мы записываем $\mathscr{S} \sim \mathscr{T}$. Пусть F и G — любые отображения 2^P в 2^P (т. е. $X \subset P \Rightarrow F(X) \subset P$ и $G(X) \subset P$). Мы говорим, что они эквивалентны, если существует f такое, что $X \subset P$ всегда влечет за собой $f(F(X)) = G(f(X))$. Основные результаты:

Теорема 2. *Пусть* $\operatorname{card} P = p \geqq \aleph_0$. *Пусть* $c(\mathscr{S})$ *— система всех тех систем* $\mathscr{T} \subset 2^P$, *что* $\mathscr{S} \sim \mathscr{T}$. *Пусть* $\operatorname{card} c(\mathscr{S}) \neq 1$. *Тогда* $\operatorname{card} c(\mathscr{S}) \geqq p$.

Теорема 4. *Пусть* $\operatorname{card} P = p \geqq \aleph_0$. *Пусть* $F \in (2^P)^{2^P}$. *Тогда мощность множества всех отображений* G, *эквивалентных* F, *равна* 1 *или* $\geqq p$. *Первый случай имеет место только тогда, когда* F *выполняет одновременно и* 1) $X \subset P \Rightarrow F(X) \in \{P, X, P - X, \emptyset\}$ *и* 2) *если* $X \sim Y$ *то*

$$F(X) = P \Rightarrow F(Y) = P$$
$$F(X) = X \Rightarrow F(Y) = Y$$
$$F(X) = P - X \Rightarrow F(Y) = P - Y$$
$$F(X) = \emptyset \Rightarrow F(Y) = \emptyset .$$

Непосредственным следствием теоремы 4 для топологий Чеха (P, u) (т. е. $u \in (2^P)^{2^P}$, $uX \supset X$, $u\emptyset = \emptyset$, $X \subset Y \subset P \Rightarrow uX \subset uY$) является утверждение:

Пусть (P, u) — топологическое пространство Чеха, $\operatorname{card} P = p \geqq \aleph_0$. Тогда мощность множества топологических пространств (P, v), гомеоморфных (P, u), равна 1 только в случае, если выполнено

и 1. $X \subset P \Rightarrow uX \in \{X, P\}$

и 2. $(X, Y \subset P;\ X \sim Y;\ uX = X) \Rightarrow uY = Y$.

Иначе, она больше или равна p.