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REPRESENTATION OF FINITE-LENGTH MODULAR LATTICES

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A canonic representation of modular lattices of finite length is given, in terms of simple nondistributive and finite distributive lattices. A fundamental characteristic, the defect of lattice, is defined and some of its properties exhibited.

There is a fundamental theorem stating that every distributive lattice is the subdirect product of replicas of 2, the 2-element lattice [1, ch. IX, th. 6]. This completely characterises the structure of distributive lattices, at least if we are only concerned with the algebraic (finitary) properties of the lattice operations. (On the other hand, if infinite meets and joins are studied, there remains an infinite variety of further properties.)

Naturally, the next step would be a representation theorem for modular lattices, and a reasonable first step in this direction is representation of finite-length modular lattices. For complemented finite length modular lattices, we already have an elegant representation: a direct product of a finite Boolean algebra and a finite system of projective geometries [1, ch. VIII, th. 6]. It may be then said that the results of the present paper notice the consequences of omitting the complementation hypothesis; however, interest centers more on unicity of representation (for the direct representation of complemented lattices, this is trivial).

The main results, theorems 4 and 5, may be stated as follows (all for lattices of finite length):

1. Every modular lattice $L$ is the subdirect product of a distributive $D$ and a modular $M$;

2. $M$ may be taken as the subdirect product of a finite system of simple non-distributive lattices $M_j$; with this condition, the decomposition of $L$ — and hence also $D$ and $M$ — is determined uniquely by $L$;

3. the subdirect representation of $M$ by the $M_j$'s may be taken in a certain economic fashion (determined by a relation between certain lattice-theoretic characteristics
of $M$ and $M_i$); with this condition, the representation of $M$ is determined uniquely by either of $M$ or $L$.

For assumed results and most of notation and terminology, the reader is referred to [1]. If $L$ is a lattice, then the set of all congruence relations of $L$ (ordered naturally, [1], ch. II, §§ 5, 6) forms a distributive lattice which will be denoted by $\Theta_L$ or $\Theta(L)$, and called the congruence lattice of $L$ (an unsatisfactory translation of Kongruenzverband). If $L$ is a lattice and $\theta \in \Theta_L$, then the lattice of equivalence classes of $L$ modulo $\theta$ will be denoted by $L/\theta$. Homomorphism will always mean a lattice-homomorphism. The length of a lattice $L$ is the maximum cardinality of subchains if this is finite, and $\infty$ if not; it will be denoted by $l_L$ or $l(L)$, the length of $\Theta_L$ by $\lambda_L$ or $\lambda(L)$, etc. For conciseness, m.l.f.l. will mean ‘modular lattice of finite length’.

Between lattices, $L_1 \approx L_2$ means that $L_1$ is isomorphic to $L_2$; and

$$L = P_{L_a} \quad \text{or} \quad L \leq P_{L_a}$$

(and also $L = L_1 \times \ldots \times L_n$ or $L \leq L_1 \times \ldots \times L_n$) will be used to mean that $L$ is the direct (cardinal in [1]) product, or a subdirect product, respectively, of the $L_a$’s. This notation is rather unfortunate in one respect: a subdirect decomposition is not determined by the system $\{L_a\}$, but by a system of “decomposing” homomorphisms $L \to L_a$ -- or equivalently, by a system of “decomposing” congruence relations $\{\theta_a\}$ on $L$, whereupon $L_a \approx L/\theta_a$, $\Lambda \theta_a = O$. (An instance of the difficulties this may lead to appears in [3], th. 12.) By a system $\{L_a\}$, say, we mean a mapping of the indices $a$ into the set with elements $L_a$; in a related sense, e.g. “there are $n$ distributive lattices in the system $\{L_a\}$” will mean that there are $n$ distinct indices $a$ such that $L_a$ is a distributive lattice.

1. FIRST RESULTS

In this section some relations between the lengths $l_L$ and $\lambda_L$ are considered. As working apparatus, the following two basic theorems will be used, often without further reference.

The congruence relations on a m.l.f.l. $L$ are in $I - 1$ correspondence with the sets of classes of projective prime quotients which they annul. Hence $\Theta_L$ is a finite Boolean algebra [1, ch. V, th. 10]. In particular, an atom of $\Theta_L$ corresponds to a single class of projective prime quotients in $L$, and there are precisely $\lambda_L - 1$ of these atoms.

In a modular lattice, any two finite chains between the same end-points possess refinements such that each quotient in either refinement is projective to some quotient in the other (cf. [1], ch. V, corollary to th. 5). In particular, in a m.l.f.l., each prime quotient in a given maximal chain is projective to some prime quotient in any other maximal chain. For completeness we also recall that in modular lattices all maximal chains have the same length [1, ch. V, th. 3].

There follows immediately the
**Lemma 1.** Let $C$ be a maximal chain in a m. l. f. l. $L$. Then there are $\lambda_L - 1$ not mutually projective prime quotients in $C$, and any prime quotient in $L$ is projective to one of these.

**Lemma 2.** For modular lattices $L$, $\lambda_L \leq l_L$.

(Proof. It suffices to consider only the case of $l_L$ finite, whereupon the result follows from lemma 1.)

**Lemma 3.** If $L$ is distributive, then $\lambda_L = l_L$.

(Proof. In view of lemma 2 it suffices to show that, if there is a chain $x_1 < x_2 < \ldots < x_m$ in $L$, then there exist $\theta_1 > \theta_2 > \ldots > \theta_m$ in $\Theta_L$. Now, either $L = 1$ and the assertion is trivially true; or $L$ is subdirectly decomposable into replicas of $2$ [1, ch. IX, th. 6], and hence there exist homomorphisms $h_i : L \rightarrow 2$ with

$$O = h_i(x_i) < h_i(x_{i+1}) = 1 \quad \text{for } 1 \leq i < m,$$

and $h_m(x_m) = O$. With each $h_i$ associate a $\mu_i \in \Theta_L$ in the usual manner, defining $x \equiv y(\mu_i)$ iff $h_i(x) = h_i(y)$; and set $\Theta_i = \bigwedge_{j<i} \mu_j$, $\theta_i = 1$. Then these $\theta_i$ are as asserted, since evidently $\theta_i \geq \theta_{i+1}$ and since it is readily verified that $x_j \equiv x_{j+1}(\theta_i)$ iff $i \leq j$ (for $1 \leq i \leq m$, $1 \leq j < m$).

**Lemma 4.** If $L$ is a m. l. f. l. and $\lambda_L = l_L$, then $L$ is distributive.

(Proof. Assume $L$ is not distributive; then $L$ must have the 4-element modular nondistributive lattice of fig. 1 as sublattice (cf. [1], ch. IX, th. 2 and ch. V, th. 2). Construct a maximal refinement $C$ of $\{0, a\}$ with the same end-points. Now use [1, ch. V, th. 6] to obtain mappings

$$x \rightarrow x \vee b, \quad x \rightarrow x \wedge c, \quad x \rightarrow x \vee a$$

Fig. 1.

which map $C$ into, in turn, $[b, i]$, $[0, c]$, $[a, i]$; the result is a maximal chain $C'$ between end points $a, i$, and prime quotients of $C$ are mapped into projective prime quotients of $C'$. Finally take a maximal chain in $L$ prolonging $C \cup C'$. The resulting
chain has at least one pair of distinct projective prime quotients. From lemma 1 we then conclude that $\lambda_L - 1 < l_L - 1$, i.e. $\lambda_L < l_L$ as asserted.

Some examples may be illustrative at this stage.

Examples. The modularity assumption is rather essential in lemma 2: if $n > m \geq 4$ then there is a (nonmodular) lattice $L$ with prescribed $l_L = m$, $\lambda_L = n$. Thus for $m = 4$ such a lattice is easily described by its graph, fig. 2; its congruence lattice is a Boolean algebra whose atoms $x_j$ annul only $[x_j, y_j]$; there are $n - 1$ atoms, and hence $\lambda_L = n$.

For $m \geq 4$ we may take, analogously, the lattice of fig. 3. (There are obvious modifications for $m$ or $n$ infinite.)

If $2 \leq n \leq m$ then there is a modular lattice $L$ with prescribed $l_L = m$, $\lambda_L = n$ (cf. fig. 4).

From these examples it may be concluded that the estimate $2 \leq \lambda_L \leq l_L$ cannot be improved for modular lattices $L \neq 1$, and also that infinite lengths are not allowed in lemma 4.

2. THE DEFECT
OF A M. L. F. L.

The results of lemmas 2 to 4 may be conveniently formulated using the

Definition 1. If $L$ is a m. l. f. l., then the defect $\delta_L$ (or $\delta(L)$, etc.) of $L$ is defined by

$$\delta_L = l_L - \lambda_L.$$  

We then have $0 \leq \delta_L \leq l_L - 2$ (except for the singular case $L = 1$, where $l = \lambda = 1$, 

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\( \delta = 0 \); and \( \delta_L = 0 \) if and only if \( L \) is distributive. Thus the defect of a lattice may be thought of as a measure of its departure from distributivity. The "most defective" lattices have \( \delta_L = l_L - 2 \), i.e. \( \lambda_L = 2 \); these are precisely the simple m. l. f. l. This section is devoted to the study of elementary properties of \( \delta \).

**Theorem 1.** Let \( L_1 \) be a sublattice of a m. l. f. l. \( L_2 \); then \( \delta(L_1) \leq \delta(L_2) \).

**Proof.** Denote concepts pertaining to \( L_1 \) or \( L_2 \) by the corresponding indices. In \( L_1 \) take a maximal chain \( C_1 \), and in each class of projective prime quotients select one belonging to \( C_1 \) (cf. lemma 1); thus there are precisely \( \lambda_1 - 1 \) distinguished prime quotients in \( C_1 \). Now in \( L_2 \), prolong \( C_1 \) to a maximal chain \( C_2 \). In \( C_2 \), at least \( (l_1 - 1) - (\lambda_1 - 1) = \delta_1 \) of its prime quotients are contained within quotients of \( C_1 \) distinct from all the distinguished ones; and each of these prime quotients of \( C_2 \) is carried by the projectivities obtaining in \( L_1 \) into a prime quotient of \( C_2 \) contained within some distinguished quotient. Thus there remain, in \( C_2 \), \( (l_2 - 1) - \delta_1 \) prime quotients with the property that any other prime quotient in \( L_2 \) is projective to one of these. Hence

\[ \lambda_2 - 1 \leq (l_2 - 1) - \delta_1 \]

or \( \delta_1 \leq (l_2 - 1) - (\lambda_2 - 1) \leq \delta_2 \) as it was required to prove.

We have shown that for sublattices \( L \) of a fixed m. l. f. l. \( L_0 \), the characteristic \( \delta_L \) is nondecreasing as a function of \( L \). Obviously \( l_L \) also has this property; also obviously, \( \lambda_L \) is neither nonincreasing nor nondecreasing. However, for sublattices \( L \) with \( l_L = = l_{L_0} \), \( \lambda_L \) is nonincreasing (this follows from theorem 1 and (2), or from lemma 1 directly); and we have the

**Corollary.** If a m. l. f. l. \( L \) has a simple sublattice of the same length, then \( L \) itself is simple.

Theorem 1 may be interpreted as treating changes of decrement when the lattice is subjected to a \( 1-1 \) homomorphism (into). Naturally, one then inquires about changes of \( \delta \) when \( L \) is homomorphically mapped onto.

**Theorem 2.** If \( L \) is a m. l. f. l. \( \theta \) is a nonpositive valuation on \( \Theta_L \).

For the proof it will be useful to recall or introduce, more or less provisionally, some further notions. Let \( d[. .] \) be the dimension function on \( \Theta_L \) [1, ch. 1, § 9].

Given a m. l. f. l. \( L \) and a congruence relation \( \theta \) on \( L \), define \( \delta(\theta) \) as follows. Take a maximal chain \( C \) in \( L \); then let \( \delta(\theta) \) be the number of prime quotients of \( C \) which \( \theta \) annihil (It may be shown directly that \( \delta(\theta) \) is independent of the choice of \( C \); this also follows from (5).)

For a m. l. f. l. \( L \), its congruence lattice \( \Theta_L \) is a Boolean algebra [1, ch. V, th 10]. By our lemma 2, \( \Theta_L \) has a finite length \( \lambda_L \); hence \( \Theta_L \) itself is finite and has precisely \( \lambda_L - 1 \) atoms \( \alpha_j \) and \( \lambda_L - 1 \) dual atoms \( \beta_j \).

Next we shall need two lemmas.
Lemma 5. Given a m. l. f. l. L. Then, on \( \Theta_L \),

(i) \( \delta(\cdot) \) is a positive valuation with \( \delta(O) = 0 \), \( \delta(I) = l_L - 1 \);

(ii) \( \delta(\cdot) - d[\cdot] \) is a nonnegative valuation with terminal values 0 and \( \delta_L \).

Proof. We know that \( d[\cdot] \) is a positive valuation with terminal values 0 and \( \lambda_L - 1 \). From the definition of \( \delta(\cdot) \) it is immediate that it is a valuation, obviously nonnegative with terminal values 0, \( l_L - 1 \). Hence \( \delta(\cdot) - d[\cdot] \) is a valuation with \( \delta(O) - d[O] = 0 \). For every atom \( x \) of \( \Theta_L \) we have \( d[x] = 1 \) \( \leq \delta(x) \), and hence

\[
\delta(\theta) - d[\theta] = \sum_{x \leq \theta} (\delta(x) - d[x]) \geq 0.
\]

Since \( d[\cdot] \) is positive, so is \( \delta(\cdot) \). Finally,

\[
\delta(I) - d[I] = (l_L - 1) - (\lambda_L - 1) = \delta_L.
\]

This completes the proof.

Lemma 6. If \( \theta \) is a congruence relation on a m. l. f. l. L, then

\[
\delta(L/\theta) = \delta_L - (\delta(\theta) - d[\theta]) \tag{3}
\]

Proof. From lemma 5 it follows that, for any m. l. f. l. L,

\[
\delta_L = \delta(I) - d[I] = \sum_x (\delta(x) - d(x)) = \sum_x (\delta(x) - 1),
\]

summing over all atoms \( x \) of \( \Theta_L \). We may then apply this to the lattice \( L/\theta \),

\[
\delta(L/\theta) = \sum_y (\delta(y) - 1)
\]

summing over all atoms \( y \) in \( \Theta(L/\theta) \) \( (\delta(y) \) is of course the corresponding function on \( \Theta(L/\theta) \), not on \( \Theta_L \). Now, these atoms \( y \) correspond, via a 1-1 homomorphism, to elements \( \theta \vee x \) of \( \Theta_L \) with \( x \) an atom, \( \theta \wedge x = O \); the next step is to prove that \( \delta(y) = \delta(x) \). Recall the definition of \( \delta(\cdot) \), take a maximal chain \( C \) in \( L \), and the corresponding maximal chain \( C' \) in \( L/\theta \); consider any prime quotient \( [x, y] \) in \( C \). If \( \theta \) annihilates \( [x, y] \), then by assumption \( \theta \wedge x = O \) it contributes 0 towards \( \delta(x) \) and does not occur in \( C' \). If \( x \neq y(\theta) \), then \( [x, y] \) is in \( C' \), and is annulled by \( y \) iff it is annulled by \( x \) in \( C \); thus its contributions towards \( \delta(y) \) and \( \delta(x) \) coincide.

Observing there are precisely \( \lambda_L - 1 - d[\theta] \) such \( x \)'s, one may continue,

\[
\delta(L/\theta) = \sum_{x \wedge \theta = 0} \delta(x) - \sum_{x \wedge \theta = 0} 1 = (l_L - 1 - \sum_{x \leq \theta} \delta(x)) - (\lambda_L - 1 - d[\theta]),
\]

having used

\[
l_L - 1 = \sum_{x \wedge \theta = 0} \delta(x) = \sum_{x \wedge \theta = 0} \delta(x) + \sum_{x \leq \theta} \delta(x);
\]

and thus

\[
\delta(L/\theta) = (l_L - 1 - \delta(\theta)) - (\lambda_L - 1 - d[\theta]) = \delta_L - (\delta(\theta) - d[\theta]).
\]

This proves (3).

As for the proof of theorem 2, it suffices to apply Lemma 5 to (3).
Corollary 1. We have the following formulae (all for m. l. f. l.):

\[(4) \quad \delta(L|\theta) \leq \delta_L.\]

\[(5) \quad \lambda(L|\theta) = \lambda_L - d[\theta], \quad l(L|\theta) = l_L - \delta(\theta),\]

\[(6) \quad \delta(L_1 \times \ldots \times L_n) = \sum_{j=1}^{n} \delta(L_j).\]

Proof. (4) follows directly from theorem 2: \(O \leq \theta\) implies \(\delta_L = \delta(L/O) \geq \delta(L|\theta)\). Since \(\Theta_L\) is a Boolean algebra, and \(\Theta(L|\theta)\) is isomorphic to the dual ideal of all \(\eta \leq \theta\) in \(\Theta_L\), we have \(\lambda(L|\theta) = \lambda_L - d[\theta]\). Hence and from (2), (3) there follows

\(l(L|\theta) = l_L - \delta(\theta).\)

For (6), consider \(L = L_1 \times L_2\); there exist \(\theta_k \in \Theta_L\) such that

\[L_k \approx L|\theta_k, \quad \theta_1 \wedge \theta_2 = O, \quad \theta_1 \vee \theta_2 = I.\]

From theorem 2 we then have that

\[\delta(L|\theta_1 \vee \theta_2) + \delta(L|(\theta_1 \wedge \theta_2)) = \delta(L|\theta_1) + \delta(L|\theta_2),\]

i.e.

\[0 + \delta_L = \delta(L_1) + \delta(L_2).\]

Since direct products are associative, we immediately obtain (6).

Corollary 2. If \(L = \bigoplus_{a_i} \) is a m. l. f. l., then there are at most \(\delta_L\) nondistributive factors in the system.

Proof. Let the nondistributive factors be \(L(a_i), 1 \leq i \leq n\), and set \(M = \bigoplus_{1 \leq i \leq n} L(a_i)\).

Then, obviously, \(M\) is isomorphic to a sublattice of \(L\), so that according to theorem 1 and (6),

\[\delta_L \geq \delta_M = \sum_{i=1}^{n} \delta(L(a_i)).\]

By assumption, each \(L(a_i)\) is nondistributive, so that \(\delta(L(a_i)) \geq 1\), and the displayed inequality then yields \(\delta_L \geq n\) as asserted.

Corollary 3. If \(L\) is a subdirect product of \(L_j (1 \leq j \leq n, \text{all m. l. f. l.})\), then

\[(7) \quad \max \delta(L_j) \leq \delta_L \leq \sum_{j=1}^{n} \delta(L_j).\]

Proof. The first inequality follows from (4). The second from theorem 1, \(\delta_L \leq \sum_{j=1}^{n} \delta(L_j)\), and (6).

3. Plain M. L. F. L.

For conciseness of formulation in the representation theorems, we will introduce two new notions, that of a plain lattice and of an exact decomposition. (In a sense, these generalise, respectively, simple lattices and direct decompositions.) The aim of this and the following sections is the study of these notions.
Definition 2. A lattice $L$ is called plain if every homomorphism of $L$ into a distributive lattice is constant.

Equivalent formulations: every homomorphism of $L$ into a given distributive lattice not $\mathbf{1}$ (e.g., into $\mathbf{2}$) is constant; or, $I$ is the only congruence relation $\theta$ on $L$ for which $L/\theta$ is distributive. Obviously the only plain distributive lattice is $\mathbf{1}$. Some elementary properties of plain lattices are formulated in the following lemma; the proofs are trivial.

Lemma 7. Every simple lattice not $\mathbf{2}$ is plain. The homomorphic image of a plain lattice is plain. If $L$ is a lattice of finite length and has a plain sublattice of the same length, then $L$ itself is plain.

The last statement is a counterpart to the corollary to theorem 1. Now let us return to our main interest, m. l. f. l.

Lemma 8. A m. l. f. l. $L$ (not $\mathbf{1}$) is plain if and only if it is the subdirect product of nondistributive simple m. l. f. l. In the positive case the set of factors is determined uniquely.

Part of the proof may be conveniently separated out:

Lemma 9. Every m. l. f. l. $L$ is the subdirect product of $\lambda_L - 1$ simple lattices; the set of factors is determined uniquely.

Definition 3. The decomposition of a m. l. f. l. $L$ to be constructed will be termed the canonical decomposition of $L$.

Proof of lemma 9. Every lattice $L$ is the subdirect product of subdirectly irreducible lattices [1, ch. VI, th. 10]. Interpreting this in the congruence lattice, $L/\theta$ is subdirectly irreducible iff

\[ \wedge \theta_a = \theta \quad \text{implies some} \quad \theta_a = \theta \]

[1, l. c.]. Now, if $L$ is a m. l. f. l., then $\Theta_L$ is a finite Boolean algebra and (8) holds iff $\theta$ is a dual atom, i.e. iff $L/\theta$ is simple. This proves the first part of the lemma.

Now let a m. l. f. l. $L$ be the subdirect product of simple $L_a$; let $\{\theta_a\}$ be the congruence relations defining the product, $L_a = L/\theta_a$ and $\wedge \theta_a = 0$. Since $L/\theta_a$ is simple, $\phi_a$ is a dual atom of $\Theta_L$; since $\wedge \phi_a = 0$, the set of all $\phi_a$ must consist of all dual atoms of $\Theta_L$. This completes lemma 9.

Proof of lemma 8. First let $L$ be a plain m. l. f. l. not $\mathbf{1}$. There is a subdirect decomposition into simple factors (lemma 9); since $L$ is plain, so are the factors (lemma 8). Omit trivial factors $\mathbf{1}$, if necessary. Then each factor is nondistributive (lemma 8 again); thus the factors are nondistributive simple m. l. f. l.

Now, conversely, let a m. l. f. l. $L \leq PL_a$ with the $L_a$ simple nondistributive. To prove $L$ is plain, consider a homomorphism $k : L \to 2$. Assume $k$ is nonconstant. Then the congruence relation $\theta_k$ defined on $L$ by $x \equiv y (\theta_k)$ iff $kx = ky$ has $\theta_k < I$, so that there is a dual atom $\beta \geq \theta_k$. By unicity (lemma 9), $L_a = L/\beta$ for some $a$.

On $L_a$, define $gt = kx$ if $t$ is the $a$-th coordinate of $x$ (i.e. if $x \in t$, $t$ an equivalence
class of $L$ modulo $\beta$). From $\beta \geq \theta_k$ it follows that $g$ is defined unequivocally, and obviously it is a nonconstant homomorphism $g : L_a \to 2$. But $L_a$ is simple nondistributive, i.e. plain; this contradiction completes the proof.

It may be said that lemma 8 characterises plain m. l. f. l. constructively; however it also makes possible a simple descriptive characterisation: The plain m. l. f. l. form the least set (of lattices) which contains all nondistributive simple m. l. f. l. and is closed under the operation of taking finite subdirect products (or of taking subdirect products of finite length). This follows from the

**Corollary.** For a m. l. f. l. let $L \leq PL_a$ with the $L_a$ plain; then $L$ itself is plain.

**Proof.** According to lemma 8, $L_a \leq PL_{a,b}$ with the $L_{a,b}$ simple nondistributive; hence $L \leq PL_{a,b}$; again from lemma 8, $L$ is plain.

Plain m. l. f. l. may also be characterised in terms of their congruence lattices:

**Lemma 10.** Let $L$ be a m. l. f. l.; $L$ is plain if and only if $\delta(L/\theta)$ is a negative valuation on $\Theta_L$.

**Proof.** In any case, $\delta(L/\theta)$ is a nonpositive valuation (theorem 2). It is a negative valuation iff $\delta(\alpha) - d[\alpha] > 0$ on every atom $\alpha$ of $\Theta_L$ (lemma 6), or, symmetrically, if

$$\delta(\beta) - d[\beta] < \delta_L \quad (= \delta(I) - d[I])$$

for every dual atom $\beta$.

Now, if $L$ is plain, then $L/\theta$ is nondistributive for every $\theta < I$, and thus

$$0 < \delta(L/\theta) = \delta_L - (\delta(\theta) - d[\theta]) \quad \text{i.e.} \quad \delta(\theta) - d[\theta] < \delta_L$$

for $\theta < I$, and in particular for all dual atoms $\beta$.

Conversely if $\delta(L/\theta)$ is a negative valuation, then in a decomposition $L \leq PL/\theta_a$ into simple factors every $\theta_a$ is a dual atom; hence $\delta(\theta_a) - d[\theta_a] < \delta_L$ by assumption, and therefore $\delta(L/\theta_a) > 0$ (lemma 6), i.e. $L/\theta_a$ is nondistributive, $L$ is plain (lemma 8). This proves lemma 10 in entirety.

4. EXACT DECOMPOSITIONS

**Definition 4.** A subdirect decomposition $L \leq PL/\theta_a$ of a m. l. f. l. will be termed exact if

$$\theta'_a = \Lambda_{b+a} \theta_b \quad \text{for all} \quad a.$$  

($\theta'_a$ is, of course, the complement of $\theta_a$ in $\Theta_L$; also see remarks at the end of the paper.)

**Lemma 11.** Each of the following subdirect decompositions of a m. l. f. l. $L$ is exact: any direct decomposition, the canonic decomposition into simple factors, the subdirect decomposition corresponding to a neutral element.
Proof. Let \( L = PL/\theta_a \) be a direct decomposition; then also

\[ L \cong L/\theta_a \times P_{b+a} L/\theta_b = L/\theta_a \times L/(\wedge_{b+a} \theta_b) = L_1 \times L_2 \]

is a direct decomposition. Set \( \theta = \theta_a, \tau = \wedge_{b+a} \theta_b \); we are to prove that \( \theta \vee \tau = I \).

Take any \( x = [x_1, x_2], y = [y_1, y_2] \in L \). Since the decomposition into \( L_1 \times L_2 \) is direct, the element \( z = [x_1, y_2] \in L \), and thus

\[ x \equiv z(\theta), \quad z \equiv y(\tau), \quad \text{i.e.} \quad x \equiv y(\theta \tau). \]

This proves \( \theta \tau = I \) in \( \Theta_L \); thus

\[ \theta \vee \tau = \theta \cup \tau \cup \theta \tau \cup \theta \tau \theta \cup \ldots \geq \theta \tau = I \]

and hence \( \theta \vee \tau = I \), as was to be proved.

As for the canonic decomposition into simple factors, the decomposing congruence relations are all the dual atoms of \( \Theta_L \); the exactness condition is then immediate, since \( \Theta_L \) is a finite Boolean algebra.

(The final part of lemma 11 is only included for completeness and is not needed in the sequel.) Consider the subdirect decomposition \( L \leq L_1 \times L_2 \) corresponding to a neutral element \( e \in L ([3, \S 2], [1, \ch II, \S 10]) \); the decomposing congruence relations \( \theta_1, \theta_2 \) are defined by \( x \equiv y(\theta_1) \) iff \( x \wedge e = y \wedge e, x \equiv y(\theta_2) \) iff \( x \vee e = y \vee e \). Then \( e = [I, O] \). We shall show that, more generally, if \([I, O] \) or \([O, I] \) \( \in \in L \leq L_1 \times L_2 \), then \( \theta_1 \theta_2 \theta_1 = I = \theta_2 \theta_1 \theta_2 \) and thus \( \theta_1 \vee \theta_2 = I \), the decomposition is exact. Assume \( e = [I, O] \in L \).

Take any \( x = [x_1, x_2], y = [y_1, y_2] \in L \). The both \( [x_1, O] = x \wedge e, [y_1, O] = y \wedge e \) are in \( L \) and hence

\[ x \equiv x \wedge e(\theta_1), \quad x \wedge e \equiv y \wedge e(\theta_2), \quad y \wedge e \equiv y(\theta_1) \]

so that \( x \equiv y(\theta_1 \theta_2 \theta_1) \). Thus we have shown that \( \theta_1 \theta_2 \theta_1 = I \). (Similarly \( \theta_2 \theta_1 \theta_2 = I \); for the proof use \( x \vee e \) and \( y \vee e \).) The final step is as in the first part of the proof:

\[ \theta_1 \vee \theta_2 = \theta_1 \cup \theta_2 \cup \theta_1 \theta_2 \cup \theta_2 \theta_1 \cup \theta_1 \theta_2 \theta_1 \cup \ldots \geq \theta_1 \theta_2 \theta_1 = I. \]

This completes the proof of Lemma 11.

Since \( \Theta_L \) is finite Boolean we have immediately the

**Lemma 12.** Exact decompositions are associative in the following sense: if \( L = PL/\theta_a \) is exact, \( L, m. l. f. l. \), and if \( A = \cup_{b}A(b) \) is a decomposition into disjoint summands, then \( L \leq PL/\langle \bigwedge_{a \in A(b)} \theta_a \rangle \) is exact.

**Lemma 13.** If \( L \leq PL_a \) is an exact decomposition of a \( m. l. f. l. \), then the number of nontrivial factors in the system is finite,

\[
\delta_L = \sum \delta(L_a),
\]

\[
\Theta_L \approx P \Theta(L_a),
\]

\[
l_L = 1 + \sum (l(L_a) - 1), \quad \lambda_L = 1 + \sum (\lambda(L_a) - 1).
\]

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Proof. Let $L_a = L/\theta_a$. The congruence lattice $\Theta_L$ is finite, so that the number of distinct $\theta_a$ is also finite. Now assume some $\theta_a$ is repeated, i.e. that $\theta_a = \theta_b$ for some $a \neq b$. Then $\theta_a' = \bigwedge_{c \in a} \theta_c \leq \theta_b = \theta_a$, $\theta_a' \leq \theta_a$, $\theta_a = I$. Thus the only repeated $\theta_a$ is $I$ and $L/\theta_1 = 1$ is trivial.

As for (10) we may then repeat the proof (but not the statement!) of (6), corollary 1 to theorem 2, for a finite system of factors, e.g. for all non-trivial factors, and the trivial factors contribute $\delta(L_a) = \delta(1) = 0$.

Formula (11) is simple for two factors: we have $L \leq L_1 \times L_2$ with $L_1 = L/\theta$, $L_2 = L/\theta'$ (complement in $\Theta_L$). Every element of a Boolean algebra is central; hence $\Theta_L = \theta \times \theta' = \Theta(L_1) \times \Theta(L_2)$ (cf. [1, ch. II, § 8], [3, th. 3d]). $\theta$ denotes the set of all $x \leq \theta$, and dually $\theta$ the set of all $x \geq \theta$). The extension to several factors is trivial.

From (11) we conclude immediately the second formula in (12); the first follows hence and from (10). This completes the proof of lemma 13.

**Theorem 3.** There is a unique exact decomposition of a m. l. f. l. into simple factors, namely the canonical decomposition.

**Proof.** Let $\theta_a$ be the congruence relations associated with an exact decomposition $L = PL_a$ into simple factors. Then each $\theta_a$ is a dual atom in $\Theta_L$, and since $\bigwedge \theta_a = O$, the system $\{\theta_a\}$ contains all the dual atoms; easily from exactness, each dual atom occurs precisely once in $\{\theta_a\}$. Thus $L \leq PL_a/\theta_a$ is precisely the canonical decomposition, and $L \leq PL_a$ coincides with it up to factor isomorphism $L_a \approx L/\theta_a$.

**Corollary 1.** Each exact decomposition of a complemented m. l. f. l. is direct.

**Proof.** Let $L \leq PL_a$ be exact, $L$ complemented m. l. f. l. Let $L_a \leq P_aL_{ab}$ be the canonical decomposition of $L_a$ into simple factors $L_{ab}$, an exact decomposition according to lemma 11. It follows easily that

$$L \leq P_aL_{ab}$$

is also exact. Now, there also exists a direct decomposition $L$ of into simple factors (Dilworth's theorem [1, ch. II, th. 8]). This is exact according to lemma 11, and coincides with (13) according to theorem 3. Hence (13) is direct, and thus so are all the decompositions from which (13) is composed. Thus, indeed, $L \leq PL_a$ is direct.

**Corollary 2.** Let $L \leq PL_j$ be the canonical decomposition of a m. l. f. l. $L$ into nontrivial simple factors. Then

$$\sum l(L_j) = 2l_L - \delta_L - 2$$

**Proof.** The canonical decomposition is exact, hence $\delta_L = \sum \delta(L_j)$; the factors are simple, so that $\sum \delta(L_j) = l(L_j) - 2$, and there are $\lambda_L - 1$ terms; $\lambda_L = l_L - \delta_L$. All this implies our formula.

It is almost obvious that (11) in lemma 13 is a necessary and sufficient condition for exactness. Now notice condition (10); it is unaffected by zero terms, i.e. by the
presence of distributive subdirect factors. On the other hand, for plain, m. l. f. l.
in the condition is quite stringent:

**Lemma 14.** Let \( M \) be a plain m. l. f. l. Then a subdirect decomposition \( M \leq PM_a \)
is exact if and only if \( \delta_M = \sum \delta(M_a) \).

**Proof.** Necessity was proved in lemma 13. Let \( M \) be a plain m. l. f. l., \( M \leq PM_a \).
We may omit trivial factors \( 1 \); denote the resulting set of indices by \( A \), its cardinality
by \( n \). Let \( \theta_a \) be the congruence relations defining the decomposition, \( M_a = M/\theta_a \), \( \wedge \theta_a = O \). The proof seems to need a rather elaborate formal apparatus. Let \( \alpha_j \), 
\( 1 \leq j \leq \lambda M - 1 \), be all the atoms of \( \Theta_M \); thus

either \( \alpha_j \leq \theta_a \) or \( \alpha_j \wedge \theta_a = O \).

Define (analogues of Kronecker symbols)

\[
\delta^a_j = I \quad \text{and} \quad d^a_j = 1 \text{ if } \alpha_j \leq \theta_a ,
\]

\[
\delta^a_j = O \quad \text{and} \quad d^a_j = 0 \text{ if } \alpha_j \wedge \theta_a = O .
\]

Thus we have \( \theta_a = \bigvee (\alpha_j \wedge \delta^a_j) \). Also, to every \( j \) there is an \( a \) with \( \delta^a_j = O \), since
otherwise \( O < \alpha_j \leq \wedge \theta_a = O \). Hence we conclude

(15)
\[
\sum_a d^a_j \leq n - 1 \text{ for every } j .
\]

Next, consider \( \delta(M_a) = \delta(M/\theta_a) \) (cf. lemmas 5 and 6):

\[
\delta(M_a) = \delta_M - (\delta(\theta_a) - d[\theta_a]) = \delta_M - \sum_j(\delta(\alpha_j \wedge \delta^a_j) - d[\alpha_j \wedge \delta^a_j])
\]

\[
= \delta_M - \sum_j(\delta(\alpha_j) - d[\alpha_j]) d^a_j .
\]

The assumption \( \delta_M = \sum \delta(M_a) \) then implies

\[
\delta_M = n \delta_M - \sum_a \sum_j(\delta(\alpha_j) - d[\alpha_j]) d^a_j = n \delta_M - \sum_j(\delta(\alpha_j) - d[\alpha_j])(\sum_a d^a_j) ,
\]

and from (15) and lemma 5 we conclude

\[
(n - 1) \delta_M = \sum_j(\delta(\alpha_j) - d[\alpha_j])(\sum_a d^a_j) \leq (n - 1) \sum_j(\delta(\alpha_j) - d[\alpha_j]) =
\]

\[
= (n - 1)(\delta(I) - d[I]) = (n - 1) \delta_M .
\]

Since (from lemmas 6 and 10) all the coefficients \( \delta(\alpha_j) - d[\alpha_j] \) are positive, it follows
hence that all \( \sum_a d^a_j = n - 1 \). Thus to every \( j \) there is precisely one \( a \in A \) with \( \alpha_j \wedge \theta_a = O \); in other words, the relation \( \alpha_j \wedge \theta_a = O \) defines a mapping \( j \rightarrow a(j) \) of the \( j \)'s
into \( A \).

Next we shall show that this is a mapping onto \( A \). We have \( \alpha'_j \geq \theta_{a(j)} \), and thus
\( \wedge a(j) = O \). Now, suppress repetitions in \( \{\theta_{a(j)}\}_j \), obtaining say \( \{\theta_b\}_{b \in B} \). Obviously
\[ \Lambda_b \theta_b = O, \text{ the } \theta_b \text{ define a subdirect decomposition, and thus (corollary 3 to theorem 2)} \]
\[ \delta_M \leq \sum \delta(M_b) \leq \sum \delta(M_a) = \delta_M. \]
Since all \( \delta(M_a) > 0 \), we conclude \( B = A \), and the mapping \( j \to a(j) \) is indeed onto \( A \).

Now return to the statement of our lemma. Since the decomposition is subdirect, we have \( \theta_a \land \Lambda_{b+a} \theta_b = O \), i.e. \( \theta'_a \geq \Lambda_{b+a} \theta_b \). Now assume the decomposition is not exact; then in the last relation we must have \( > \) for some \( a \). Thus there is an atom \( x_j \) with
\[ \theta'_a \geq x_j, \quad x_j \land \Lambda_{b+a} \theta_b = O. \]
But then \( \theta_a \land x_j = O \), i.e. \( a = a(j) \); and therefore, for all \( b \neq a \), not \( \theta_b \land x_j = O \), i.e. \( x_j \leq \theta_b \). We conclude
\[ O < x_j \leq \Lambda_{b+a} \theta_b, \]
a contradiction. This proves lemma 14.

5. REPRESENTATION THEOREMS

**Theorem 4.** To a m. l. f. l. \( L \) there is a unique subdirect decomposition
\[ L \leq D \times M \]
into a distributive \( D \) and a plain \( M \). Furthermore, \( D \) is finite, the decomposition is exact and thus
\[ (16) \quad \delta_L = \delta_M, \quad l_L = l_D + l_M - 1, \quad \lambda_L = \lambda_D + \lambda_M - 1. \]

**Proof.** Let \( L \) be a m. l. f. l. For every homomorphism \( h : L \to 2 \) define a relation \( \theta_h \) on \( L \) by
\[ (17) \quad x \equiv y (\theta_h) \iff hx = hy. \]
Obviously \( \theta_h \) is a congruence relation. Set \( A = \Lambda_b \theta_b \); thus \( A \) is the least congruence relation \( \theta \) on \( L \) for which \( L/\theta \) is distributive.

Define \( D = L/\Lambda, \ M = L/\Lambda' \) (\( \Lambda' \) is the complement of \( \Lambda \) in \( \Theta_L \)). Obviously \( L \leq D \times M, D \) is distributive; since \( D \) has finite length \( l_D \leq l_L \), it is a sublattice of the Boolean algebra \( 2^{l_L-1} \), and hence \( D \) is finite. From the estimates (7) (corollary 3 to theorem 2) we conclude \( \delta_M \leq \delta_L \). Exactness is immediate; (16) then follows from lemma 13.

Now consider any homomorphism \( h : M \to 2 \); let \( p : L \to M = L/\Lambda' \) be the natural homomorphism. Define a congruence relation \( \theta \) on \( L \) by \( x \equiv y (\theta) \) iff \( hp x = hp y \). Obviously \( \theta \geq \Lambda' \), and \( L/\theta \) is either \( 1 \) or \( 2 \), i.e. distributive, and therefore \( \theta \geq A \). Thus \( \theta \geq A \land \Lambda' = I \); hence \( hp \) is constant on \( L \), hence \( h \) constant on \( M \). In conclusion, any homomorphism \( M \to 2 \) is constant, i.e. \( M \) is plain.

It remains to prove unicity. Let \( L \leq D_1 \times M_1 \) with \( D_1 \) distributive and \( M_1 \) plain; let \( D_1 = L/\Lambda, M_1 = L/\Lambda' \), \( \Lambda \land \Lambda' = O \). Since \( M_1 \) is plain, no dual atom \( \beta \geq \Lambda' \).
is of the form $\theta_{h}$ (as in (17)); thus $\Delta^{0} \supseteq \Delta'$. Since $D_{1} = L/\Delta$ is distributive, $\Delta \supseteq \Delta$. Then from $\Delta \land \Delta^{0} = 0$ there follows $\Delta = \Delta$, $\Delta^{0} = \Delta'$; this completes the proof.

**Corollary.** Let $L$, $L_{a}$ be m. l. f. l., let $L \leq D \times M$ and $L_{a} \leq D_{a} \times M_{a}$ be decompositions as in theorem 4. If $L$ is a subdirect (or exact, direct) product of the $L_{a}$, then both

\[ D \leq PD_{a}, \quad M \leq PM_{a} \]

are subdirect, exact, direct, respectively.

One aspect of these results may be summarized as follows. Given a m. l. f. l. $L$, several integer-valued characteristics of $L$ have been introduced:

\[ l_{L}, \lambda_{L}, \delta_{L}; \quad l_{D}, \lambda_{D}, \delta_{D}; \quad l_{M}, \lambda_{M}, \delta_{M}. \]

These are not independent ($\delta = l - \lambda$ in each group, (16), distributivity of $D$ implying $\delta_{D} = 0$, inequalities such as $0 \leq \delta \leq l - 2$, (21), etc.); and we may distinguish subsets of “independent” characteristics, e.g. $l_{L}, \delta_{L}, l_{M}$. Then for m. l. f. l. $L$, $\delta_{L} = 0$ characterises distributive lattices, $\delta_{L} = l_{L} - 2$ characterises simple lattices, $l_{L} = l_{M}$ characterises plain lattices (this last follows from theorem 4).

**Theorem 5.** To a m. l. f. l. $L$ there is a unique subdirect decomposition

\[ L \leq D \times M_{1} \times \ldots \times M_{n} \]

with $D$ distributive, $M_{j}$ simple modular nondistributive, $\delta_{L} = \sum \delta(M_{j})$. Furthermore, $D$ is finite, the decomposition is exact, and

\[ \sum l(M_{j}) = 2(l_{L} - l_{D}) - \delta_{L}. \quad (18) \]

**Proof.** Existence follows from theorem 4, the canonical exact decomposition of $M$ (lemmas 9, 11) and (16), (10).

Next, consider unicity. Let a subdirect decomposition of $L$ be given as described; let $D = L/\Delta$, $M_{j} = L/\theta_{j}$. Set $M = L/\land \theta_{j}$. Then $M \leq PM_{j}$, so that $M$ is plain (lemma 8); hence $M$ is determined uniquely (theorem 4), $\delta_{M} = \delta_{L} = \sum \delta(M_{j})$. But then $M \leq PM_{j}$ is exact (lemma 14), so that this decomposition is unique (theorem 3).

It remains to prove (18). From (14) (corollary 2 to theorem 3), we have $\sum l(M_{j}) = 2l_{M} - \delta_{M} - 2$. Now $\delta_{M} = \delta_{L}$, and from (16) $l_{M} - 1 = (l_{L} - 1) - (l_{D} - 1)$; hence $\sum l(M_{j}) = 2(l_{M} - 1) - \delta_{M} = 2(l_{L} - l_{D}) - \delta_{L}$, i.e. (18). This completes the proof.

**Corollary 1.** With the notation of theorems 4 and 5,

\[ \frac{1}{2}(l_{L} - l_{D}) \leq \delta_{L} \leq l_{L} - l_{D}; \quad (19) \]

\[ \delta_{L} \leq \sum l(M_{j}) \leq 3\delta_{L}; \quad (20) \]

\[ l_{M} \leq 1 + 2\delta_{L}. \quad (21) \]

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Proof. Consider (18); all $M_j$ are modular nondistributive, hence $l(M_j) \geq 3$, and there are precisely $\lambda_M - 1$ of these. Hence
\[ 2(l_L - l_D) - \delta_L \geq 3(\lambda_M - 1). \]
Now $\lambda_M - 1 = (\lambda_L - 1) - (\lambda_D - 1)$ since we have (12) and the decomposition $L \leq D \times M$ is exact; also $\lambda_D = l_D$, $\lambda_L = \lambda_L - \delta_L$. Thus $\lambda_M - 1 = \lambda_L - \lambda_D = l_L - l_D - \delta_L$ and therefore $2(l_L - l_D) - \delta_L \geq 3(l_L - l_D) - 3\delta_L$; this is the first inequality of (19).

Since $D = L \div A$, we have $\lambda_D \leq \lambda_L$, i.e. $l_D \leq l_L$; and then $\delta_L = l_L - \lambda_L \leq l_L - l_D$, the second inequality of (19). (It may be noticed that (19) generalizes the statement that $\delta_L = 0$ iff the m. l. f. l. $L$ is distributive.)

If we use (19) to estimate $l_L - l_D$ in (18), (20) results.

From theorem 5 we have $\sum \delta(M_j) = \delta_L$ with precisely $\lambda_M - 1$ summands all $\geq 1$. Thus $\lambda_M - 1 \leq \delta_L$; with $\lambda_M = l_M - \delta_M = l_M - \delta_L$, this implies (21).

Example. From (21) we have that for plain m. l. f. l. $L$, $l_L \leq 1 + 2\delta_L$. However, this condition does not characterise plain lattices among m. l. f. l. Indeed, there is no relation at all between $l_L$ and $\delta_L$ (or equivalently, between $l_L$ and $\lambda_L$) which characterises plain lattices among m. l. f. l.

Let $S_3$ be the lattice of fig. 1, $S_4$ that of fig. 6. Obviously both are simple m. l. f. l.,
\[ l(S_3) = 3, \quad l(S_4) = 4, \quad \text{hence} \quad \delta(S_3) = 1, \quad \delta(S_4) = 2. \]

Set $L_1 = S_3 \times S_3, \quad L_2 = S_4 \times 2$. Since direct decompositions are exact,
\[ l(L_1) = 5 = l(L_2), \quad \delta(L_1) = 2 = \delta(L_2) \]
(lema 12, (10) and (12)). By lemma 8, $L_1$ is plain and $L_2$ is not.

Fig. 5.

Fig. 6.
Corollary 2. If $L$ is complemented, then the decompositions of theorems 4 and 5 are direct.

This follows immediately from corollary 1 to theorem 3.

We may now attempt classification of m. l. f. l. according to their defects, and obtain some information about their structure. As an illustration, we will state the two following lemmas.

Lemma 15. A m. l. f. l. $L$ with defect 1 and length $l$ is the subdirect product of a simple modular nondistributive lattice of length 3 and a distributive lattice of length $l - 2$.

Proof. The number $n$ of simple nondistributive factors in the canonic decomposition has $n \geq 1$, since $L$ is nondistributive ($\delta_L = 1$), and also $n \leq \delta_M = \delta_L = 1$. Thus there is a unique simple nondistributive factor $M$; it has $\delta_M = 1$ and hence length 3. (Alternately, (21) may be used.) The length $l_D$ of the distributive factor satisfies (16), i.e. $l = l_D + 3 - 1$.

Lemma 16. A m. l. f. l. $L$ with defect 2 and length $l$ is the subdirect product of either

(i) two simple modular nondistributive lattices of length 3 and a distributive lattice of length $l - 4$, or

(ii) a simple modular nondistributive lattice of length 4 and a distributive lattice of length $l - 3$.

The proof is similar to the preceding. If $\lambda_M - 1 = \delta_L$ we have case (i), if $\lambda_M - 1 < \delta_L$ we have case (ii).

We then have the following:

Lemma 17. A modular lattice of length 3 is either

(i) a simple modular (nondistributive) lattice of length 3, or

(ii) 3, or

(iii) $2 \times 2$.

Lemma 18. A modular lattice of length 4 is either

(i) a simple lattice of length 4 (iff $\delta = 2$), or

(ii) a subdirect product of a simple lattice of length 3 and 2 (iff $\delta = 1$), or

(iii) a distributive lattice of length 4 (iff $\delta = 0$).

Lemma 19. A modular lattice of length 5 is either

(i) a simple lattice of length 5 ($\delta = 3$), or

(ii) a subdirect product of two simple lattices of length 3 ($\delta = 2$), or

(iii) ditto, of a simple lattice of length 4 and 2 ($\delta = 2$ again), or

(iv) ditto, of a simple lattice of length 4 and 3 ($\delta = 1$), or

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(v) *ditto,* of a simple lattice of length 3 and $2 \times 2$ ($\delta = 1$), or
(vi) a distributive lattice of length 5 (iff $\delta = 0$).

Several interesting questions, raised by the preceding results, remain open.
1. Given a finite distributive lattice $D$ and a plain m. l. f. l. $M$, determine all lattices
   which are subdirect products of $D, M$.
2. Given a finite system of simple nondistributive m. l. f. l. $M_j$, determine all
   (plain) lattices with canonic decomposition $M_1 \times \ldots \times M_n$.
3. Determine all simple modular lattices of given length.
4. How much of the preceding results remains true if the finiteness of length
   assumption is dropped?
5. Does the defect of a direct product of projective geometries have simple geometrical
   significance? (The defect of a projective geometry is length $- 2 = \text{dimension}$
   $-1$).

Remarks. The referee, M. Kolibiar, has been good enough to bring to the author's
attention the following references.
1. In [5, p. 100], Maeda defined canonic factorisations of general lattices $L$ as
   those subdirect decompositions $L \leq PL/\theta_a$ which have $\theta^*_a = \Lambda_{b \neq a} \theta_b$ for all $a (\theta_a \in \Theta_L$;
   $\theta^*_a$ denotes the pseudocomplement of $\theta_a$). Since for m. l. f. l., $\Theta_L$ is a finite Boolean
   algebra, one has $\theta^*_a = \theta_a'$; thus the exact decompositions of definition 4 are merely
   a special case of canonic factorisations.
2. The existence part of theorem 3 (i.e. lemma 9) may also be obtained from Dilworth's theorem
   [4, th. 3.3]; of course, the proof of this latter also proceeds via [1, ch. VI, th. 10].
3. The second assertion in lemma 11 follows from [6], lemma 4.
4. It may also be noted that, for canonic factorisations, Maeda obtained a theorem
   [5, th. 2.4] analogous to the Birkhoff factor theorem [1, ch. II, th. 7]; it seems probable that, using this, one might obtain a unicity theorem extending the present
   theorem 3 to general lattices.

References

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Резюме
ПРЕДСТАВЛЕНИЕ ДЕДЕКИНДОВЫХ СТРУКТУР
КОНЕЧНОЙ ДЛИНЫ

ОТОМАР ГАЕК (Otomar Hájek), Praha

В работе ищется каноническое представление дедекиндовых структур конечной длины (д. с. к. д.). Назовём структуру $L$ полупростой (plain) если не существует структурных гомоморфизмов, изображающих $L$ на двух-элементную структуру.

Теорема 4. Существует единственное полупрямое разложение $\delta_2 \leq \delta_n$ в дистрибутивную структуру $D$ и полупростую структуру $M$.

Определим дефект $d_2 \leq d_n$ как $d_2 = l_2 \leq \lambda_2$, где $l_2$ — длина $L$ и $\lambda_2$ — длина структуры отношений конгруэнтности на $L$. Полупрямое произведение $L$ в систему структур $L_1, \ldots, L_n$ называется точным, если $\delta_2 = \sum \delta_{l_2}$ ("точное произведение" — exact decomposition — используется в работе в другом смысле).

Теорема 5. Существует единственное точное произведение $\delta_2 \leq \delta_n$ в дистрибутивную структуру $D$ и конечное число простых недистрибутивных $\delta_2 \leq \delta_n$.

В работе имеется также более детальное исследование понятия дефекта (п. 1, 2) полупростой структуры и точного представления (п. 3, 4); приведены также некоторые соотношения между длинами $L, M, D, M_j$ (п. 4).