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NOTES ON MEROMORPHIC DYNAMICAL SYSTEMS, I

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In the theory of dynamical systems in the plane, one naturally needs examples; the most elementary are linear systems (and, possibly, "polar" systems). However, it seems a giant step from linear systems, with an entirely trivial theory in the large, to say polynomial systems, where even the local theory is rather involved (and the theory in the large is quite formidable; *e.g.*, the van der Pol equation). A possible candidate for a class intermediate in complexity are the systems

$$\frac{d\xi_1}{d\theta} = \varphi_1(\xi_1, \xi_2), \quad \frac{d\xi_2}{d\theta} = \varphi_2(\xi_1, \xi_2)$$

(θ, ξ_j, φ_j real) with $f = \varphi_1 + i\varphi_2$ a polynomial in $z = \xi_1 + i\xi_2$; and the immediate generalisations to f holomorphic, or rational, meromorphic. (The linear systems are not a subclass.)

The restriction of the vector-field function f to these classes naturally has as consequence special properties of the dynamical system, and some of these are the subject of the present paper. Specifically, this paper is devoted to the qualitative theory of cycles of these systems. It appears that 'f holomorphic' is a rather too strict restriction (there are then no saddle points, *etc.*). Now, poles of f are "saddle-points"; but it may not be immediately apparent whether these have any connection with the concept of saddle point customary in differential equation theory. However, this is simple: if f has a pole of order k at 0, then

$$z' = f(z), \quad z' = |z|^\alpha f(z) \quad (\alpha > k)$$

have the same trajectories (with distinct parametrisations), and the second of these has a critical point at 0, namely a saddle point.

Let there be given a nonvoid open subset G of the 2-sphere S^2; a *meromorphic system* in G is determined by a meromorphic function f on G, or by the differential equation (in a local complex coordinate z)

$$\frac{dz}{d\theta} = f(z) \tag{1}$$

with θ real.

A *solution* of (1) is a mapping $z : I \to G$ of a nonvoid open interval $I \subset \mathsf{E}^1$, which has $f(z(\theta)) \neq \infty$ for $\theta \in I$, and which satisfies equation (1) whenever $z(\theta) \neq \infty$; if $z(\theta) = \infty$ we of course require, instead of (1), that $1/z(.)$ be a solution of

$$\frac{\mathrm{d}w}{\mathrm{d}\theta} = -w^2 f\left(\frac{1}{w}\right)$$

at θ. It seems convenient not to define solutions through poles of f, since then unicity of solution would not be preserved.

A *trajectory* of (1) is the image of a nonconstant solution of (1) with a maximal open interval as domain of definition. A *cycle* of (1) is the image of a solution with a (finite positive primitive) period. From unicity it follows that we may speak of the period (e.g., primitive) of a cycle. A singular point of (1) is a pole or zero of f in $G \subset \mathsf{S}^2$; zeros of f are, in a more general sense than ours, often called the *critical points* of f.

As an example, consider the system in S^2, defined on $\mathsf{E}^2 = \mathsf{S}^2 - \infty$ by $z' = iz^2/(z-1)$, and described near ∞ by $1/z = w$, $w' = iw/(w-1)$. Thus there are three singular points in S^2, zeros at 0 and ∞, and a pole at 1. We do not introduce the concept of "singular points at infinity", since this would lead to difficulties (thus the former system would then have three, and the latter two singular points).

The local theory of these systems was established by Gregor [2]. Several of his results may be summarised as follows.

Lemma 1. *Let* (1) *be a meromorphic system. Then*

1. *Every pole of f is a saddle point*;

2. *Multiple zeros of f are nodes if* $\operatorname{res} 1/f = 0$;

3. *A zero of multiplicity one is either a dicritical node (iff f' is real), a center (iff f' is pure imaginary) or a focus*; *moreover, then,* $\operatorname{Re} f' < 0$ *iff the critical point is asymptotically stable*;

4. *If z_0 is a center, i.e. if* $f(z_0) = \operatorname{Re} f'(z_0) = 0 \neq \operatorname{Im} f'(z_0)$, *then all cycles C near z_0 have the same primitive period T,*

$$T = \frac{2\pi i}{f'(z_0)} \operatorname{ind}_C z_0 \,. \tag{2}$$

In particular, since $T > 0$, $\operatorname{sgn} \operatorname{Im} f'(z_0)$ determines $\operatorname{ind}_C z_0$, the orientation of C.

The basic idea in [2] is that the trajectories of (1) are similar (in some respect at least) locally at a singular point *e.g.* $z_0 = 0$, to those of "canonic" systems of the form

$$z' = az^m \tag{3}$$

where $|m|$ is the multiplicity of the zero ($m > 0$) or pole ($m < 0$) of f, and $a = \lim f(z)\, z^{-m}$. For $m \neq 1$ there are then $2|m-1|$ *exceptional directions*, defined as (unit vectors) $w \in \mathsf{E}^2$, solutions of $w^{1-m} = \mp(a/|a|)$ (entrant or exitant according as ∓ 1 is the sign taken); these have the property that if a solution $z(\theta)$ tends to $z_0 = 0$, then $z(\theta)/|z(\theta)|$ tends to an exceptional direction, entrant for increasing θ, exitant for decreasing θ.

Example 1. Consider again the system (1) with

$$f(z) = \frac{iz^2}{z-1}, \quad G = \mathsf{S}^2 .$$

Then we have the following information about the singular points of this system

singular point	m	type	entrant ex. dir.	exitant ex. dir.	res $1/f$
0	2	node	$-i$	i	$-i$
1	-1	saddle	$\pm e^{-i\pi/4}$	$\pm e^{i\pi/4}$	0
∞	1	center	none	none	i

This example will be examined further later.

Shortly later, the present author announced [3] that there exists a homeomorphism mapping the field of trajectories of (1) into that of (3), locally at the singular point 0 (in fact, the homeomorphism is piecewise conformal), thus removing the restriction res $1/f = 0$ assertion in 2 of lemma 1. It is intended to give detailed proofs in a subsequent paper.

We will need two further results; both are trivial consequences of this local homeomorphism of (1) and (3).

Lemma 2. *Let* 0 *be a singular point of* (1). *Then there exist arbitrarily small neighbourhoods* U *of* 0 *such that*:

1. *If* 0 *is a zero of* f, *then separately for each solution* $z(.)$ *of* (1) *with* $z(0) \in U$, $z(\theta) \in U$ *either for all* $\theta \geqq 0$ *or for all* $\theta \leqq 0$;

2. *If* 0 *is a pole of* f, *then in* U, *the set of trajectories which have* 0 *as accumulation point is finite nonempty.*

Two "indices" will be used. The first is the notion familiar from complex variable theory: the index of a point $z_0 \in \mathsf{E}^2$ with respect to a closed rectifiable parametric curve $C \subset \mathsf{E}^2 - z_0$ is

$$\operatorname{ind}_C z_0 = \frac{1}{2\pi i}\int_C \frac{dz}{z - z_0} .$$

If, furthermore, C is simple closed, then let int C be the bounded component of $\mathsf{E}^2 - C$, and set ind $C = \operatorname{ind}_C z_0$ for any $z_0 \in$ int C.

The second is a generalisation of this notion, the Kronecker index of a point in a vector field. For our purposes it may be defined as follows. Given a meromorphic function f in G, define $m(z)$ or $m_f(z)$ for $z \in G$ thus: if z is a zero of f of multiplicity k, set $m(z) = k$; if z is a pole of f of multiplicity k, set $m(z) = -k$; in the remaining cases, set $m(z) = 0$. Thus $|m(z)|$, if nonzero, is the multiplicity of z in the usual sense. Obviously

$$m_{fg} = m_f + m_g$$

for meromorphic f, g in G; also (if $C_r = \{z + re^{i\theta} : 0 \leqq \theta \leqq 2\pi\}$)

$$\begin{aligned} m_f(z) &= \operatorname{res}_z \frac{f'}{f} = \frac{1}{2\pi i} \lim_{r \to 0+} \int_{C_r} \frac{f'}{f} \mathrm{d}z = \\ &= \frac{1}{2\pi} \lim_{r \to 0+} \operatorname{Im} \int_{C_r} \frac{f'}{f} \mathrm{d}z = \frac{1}{2\pi} \lim_{r \to 0+} \int_{C_r} \mathrm{d} \arg f \end{aligned}$$

and thus $m_f(z)$ is indeed the Kronecker index of z in f [1, XVI, § 4].

A general theorem going back to Poincaré (and with well known topological generalisations) states that the (Kronecker) index of a cycle is 1. For meromorphic systems this specialises to the

Lemma 3. *Let C be a cycle of* (1) *with* int $C \subset G$. *Then*

$$\sum_{z \notin C} m(z) \left|\operatorname{ind}_C z\right| = 1\,.$$

Proof. The Kronecker index of C in the vector field of (1) is defined [1, XVI, § 4], essentially, as the number

$$\operatorname{ind} C \,.\, \frac{1}{2\pi} \int_C \mathrm{d} \arg f$$

where $\arg f$ is any branch of $\arg f(z)$, single-valued and piecewise continuous along C. In the usual definition (*l.c.*) there is a convention on the orientation of C. However C may be oriented otherwise, *e.g.* by the solution of (1) which parametrises C; the ind factor corrects for this effect on the $\int_C$.

Now for meromorphic functions f,

$$\int_C \mathrm{d} \arg f = \int_C \mathrm{d} \operatorname{Im} \log f = \operatorname{Im} \int_C \frac{f'}{f} \mathrm{d}z = 2\pi \sum m(z) \operatorname{ind}_C z\,.$$

When multiplying through by the ind factor, notice that $\operatorname{ind} C \,.\, \operatorname{ind}_C z = |\operatorname{ind}_C z|$, since $\operatorname{ind}_C z = 0$ outside C, and in int C both factors coincide.

Our second result states that the primitive period of a cycle is completely determined by the behaviour of f at its zeros inside the cycle.

Lemma 4. *If C is a cycle of (1) with* int $C \subset G$, *then its primitive period T satisfies*

$$T = 2\pi i \sum_{z \notin C} \operatorname{res}_z (1/f) \operatorname{ind}_C z = 2\pi i \,.\, \operatorname{ind} C \,.\, \sum_{z \in \operatorname{int} C} \operatorname{res}_z 1/f \,.$$

Proof. This is almost trivial: from $z' = f(z)$ it follows that

$$\frac{1}{f(z(\theta))} z'(\theta) = 1\,, \quad \int_C \frac{dz}{f(z)} = T\,,$$

and the residue theorem yields our formula immediately.

Notice that formula (2) is a special case. The results of lemmas 3 and 4 may be put in another – possibly more convenient – form.

Theorem 1. *Let C be a cycle of the meromorphic system $z' = f(z)$,* int $C \subset G$. *Then*

$$\sum\nolimits^{Z} |m(z)| - \sum\nolimits^{P} |m(z)| = 1\,,$$

$$\sum\nolimits^{Z} \operatorname{Re} \operatorname{res}_z 1/f = 0\,, \quad \sum\nolimits^{P} \operatorname{Im} \operatorname{res}_z 1/f \neq 0\,;$$

here $\sum^{Z}$ and $\sum^{P}$ *denote summation over all zeros and poles, respectively, of f in* int C.

Obviously sgn $\sum^{Z} \operatorname{Im} \operatorname{res}_z 1/f$ determines the orientation of C.

Corollary. *Under the same assumptions,*

1. int C *contains at least one zero; if it contains more than one zero, it must also contain a pole;*

2. *If* int C *contains at most one singular point, then it must be a center;*

3. *If f is holomorphic in* int C *(e.g. a polynomial), then the case described in 2 obtains;*

Example 1 (contd.). We may apply theorem 1 to the case $f(z) = iz^2/(z-1)$ considered previously. Since ∞ is a center, there do exist cycles C in E^2; for every such cycle, int C must contain the unique zero 0 (corollary, 1); since $m(0) = 2$, int C must also contain a pole, *i.e.* the pole 1. (Since ∞ is a center, this is obvious for cycles sufficiently near ∞; however, we have proved it for all cycles of the system.) The formula of lemma 4 then yields, for the period T of every cycle, $T = 2\pi i \,.\, 1 \,.\, (-i + 0) = 2\pi$.

Theorem 1 then suggests, as most theorems do, several further questions.

In the situation described in 2 above, is int C completely filled by cycles encircling the singular point? (An affirmative answer follows from theorem 2 or 3.)

Problem 1. Do there exist other types of cycles except these?

If the statement of theorem 1 is interpreted as a necessary condition, is it sufficient? This is vaguely put, and the answer is negative; however, we may formulate a less ambitions problem:

Problem 2. Given a meromorphic system (1), and some zeros and poles z_k of f with $\sum m(z_k) = 1$, $\sum \operatorname{res}_{z_k} 1/f = i\alpha$, $\alpha \neq 0$ real. Find effective conditions for existence of cycles C with $\operatorname{ind}_C z_k \neq 0$.

Now we will notice the neighbourhood of a cycle. Given the meromorphic system (1), take a point $z \in G$ (not a pole) and consider the solution $z(.)$ of (1) with $z(0) = z$. This solution may be prolonged, either indefinitely, or until it meets a pole of f or approaches the boundary of G. In any case we may define a function X on a subset of $\mathsf{E}^1 \times G$ such that

$$\text{(4)} \qquad \frac{\partial}{\partial \theta} X(\theta, z) = f(X(\theta, z)), \quad X(0, z) = z, \quad X(\theta, z) \in G;$$

X is defined for all $z \in G$ not poles of f, and for all θ in some open interval $I_z \subset \mathsf{E}^1$ containing 0. (Then $\varphi(\theta, \theta_0; \xi, \eta) = X(\theta - \theta_0, \xi + i\eta)$ is the familiar "characteristic function").

Lemma 5. *If $X(\theta_0, z_0)$ is defined for given θ_0, z_0, then $X(\theta_0, z)$ is holomorphic in z near z_0.*

The proof may be carried out directly; however, the assertion is a special case of a theorem on the analytic dependence on initial data, *e.g.* [1, chap. I, th. 8.2].

Theorem 2. *Let C be a cycle of the meromorphic system (1). Then there is an annular neighbourhood U of C consisting of complete cycles of (1), with the same primitive period.*

Proof. By definition, $C \subset G$ and there is no pole on C. Let T be the primitive period of C; then

$$\text{(5)} \qquad X(T, z) = z \quad \text{for} \quad z \in C.$$

Both sides of this equation are holomorphic in z near C (lemma 5), so that (5) must hold in a neighbourhood U of C. Thus each $z \in U$ is on a cycle of (1) with period T.

It remains to prove that T is the primitive period, at least for small $U \supset C$. Assume the contrary; then there are z arbitrarily near C and cycles $C(z)$ passing through z with primitive periods $T(z) \neq T$; however, since T is at least some period of $C(z)$, we have $T = n(z)\, T(z)$, $2 \leqq n(z)$ integral. There are then two possibilities (and both lead to contradictions).

Either the $n(z)$ are bounded; then we may take convergent subsequences $n(z) \to n_0$; $T(z) \to T_0$ (since $0 < T(z) < n(z)\, T(z) = T$), whereupon $n_0 \geqq 2$, $T = n_0 T_0$, hence

$T > T_0 > 0$; also, by continuity, T_0 is a period of C; however T is the primitive period, a contradiction.

The second possibility is that some subsequence $n(z) \to +\infty$. Now, for any two points $z(\theta_1)$, $z(\theta_2)$ on $C(z)$,

$$|z(\theta_2) - z(\theta_1)| = \left| \int_{\theta_1}^{\theta_2} f(z(\theta))\, d\theta \right| \leqq MT(z) = MT/n(z) \to 0\,,$$

where $M = \sup_U |f| < +\infty$ for small $U \supset C$. Therefore there is a point z_0 on C such that any disc neighbourhood of z_0 contains complete trajectories (the cycles $C(z)$); in particular, there are zeros of f arbitrarily near z_0, a contradiction.

This completes the proof of theorem 2.

More information may be had concerning the neighbourhood U:

Theorem 3. *Let C be a cycle of the meromorphic system* (1). *Then there is a* (*maximal*) *neighbourhood U of C consisting of complete cycles of* (1), *such that U is a region, the boundary U consists of two components K_1, K_2 separated by C; furthermore, each K_j is a closed parametric curve consisting of complete trajectories, singular points and boundary points of G; and either*

1. *K_j is a single point, a center; or*
2. *K_j consists of a finite set of complete trajectories and poles of f, at least one of each; or*
3. *K_j contains no zeros of f and intersects the boundary of G.*

Sketch of proof. Assume $z_0 \in K_j \cap G$ is a zero of f. Since every neighbourhood of z_0 intersects U, from 1 of lemma 2 it follows that arbitrarily small neighbourhood of z_0 contain cycles. Hence z_0 is a center (*cf.* lemma 1), and as K_j is connected, $K_j = z_0$. Thus the only zeros on K_j are centers, whereupon K_j degenerates.

If a nondegenerate $K_j \subset G$ were to contain no poles, then the function $X(\theta, z)$ would be holomorphic on K_j, and thus K_j would itself be a cycle (theorem 2); application of theorem 2 to K_j then yields a contradiction.

If a $K_j \subset G$ were to contain infinitely many poles or trajectories, then it would also contain an essential singularity of f (K_j closed in compact S^2).

Example 1 (contd.) We now have that cycles fill out region H in S^2 with ∞ as the "outer" boundary, and with a closed parametric curve S through the saddle point 1 as "inner" boundary. Since 0 is a zero of f, it is not in H, nor on S. Finally S can enter 1 only with direction $e^{-i\pi/4}$ or $-e^{-i\pi/4}$, and exit from 1 only with directions $\pm e^{i\pi/4}$.

The last group of results concerns *separatrices*, by which we shall understand parametric curves consisting of a finite set of complete trajectories and singular points of (1), containing at least one of each and oriented in agreement with the constituent trajectories. The components K_j of theorem 3 are an instance of these, at least if $K_j \subset G$ and K_j contains at least two points. The following lemma is immediate.

Lemma 6. *Assume that $z(.)$ is a solution of (1) and*

$$z(\theta) \to z_0 \quad as \quad \theta \to \theta_0 ,$$

with z_0 a singular point and $|\theta_0| = +\infty$ not excluded. Then

1. *z_0 is not a center;*
2. *If z_0 is not a focus, then*

$$\frac{z(\theta) - z_0}{i|z(\theta) - z_0|}$$

tends to an exceptional direction as $\theta \to \theta_0$;

3. *If z_0 is a zero of f, then $|\theta_0| = +\infty$; however, if z_0 is not a focus, then the trajectory determined by $z(.)$ at least has finite arc-length near z_0,*

$$\int^{\theta_0} \left|\frac{dz(\theta)}{d\theta}\right| d\theta < +\infty ;$$

4. *If z_0 is a pole of f, then $|\theta_0| < +\infty$.*

Lemma 7. *A closed separatrix cannot contain simple zeros of f.*

Proof. A simple zero z_0 of f, not a center, is either stable or unstable, according as $\operatorname{Re} f'(z_0) < 0$ or >0. Thus a separatrix cannot both enter and exit from z_0.

Now we shall attempt to extend the formulas of theorem 1 to closed separatrices. First, we have as trivial generalisation of lemma 4,

Theorem 4. *Let S be a closed parametric curve in G (not necessarily simple), consisting of a finite set of complete trajectories and poles of f, and assume that all points z with $\operatorname{ind}_S z \neq 0$ belong to G. Then there is a real $\alpha > 0$ with $2\pi i \sum_{z \notin S} \operatorname{res}(1/f) \,.\, \operatorname{ind}_S z = \alpha$.*

Proof. From 4 of lemma 6 it follows that C may be parametrised using solutions of (1), with the parameter varying from 0 to a finite $\alpha > 0$. Then, except at a finite number of points on S,

$$2\pi i \sum_{z \notin S} \operatorname{res}_z \frac{1}{f} \,.\, \operatorname{ind}_S z = \int_S \frac{dz}{f(z)} = \int_0^\alpha \frac{z'(\theta)}{f(z(\theta))} d\theta = \int_0^\alpha d\theta = \alpha .$$

Problem 3. Let S be a simple closed curve consisting of complete trajectories and singular points of (1) (and containing at least one zero of f). Prove that $\operatorname{Re} \sum_{z \notin S} \operatorname{res}_z (1/f) \,.\, \operatorname{ind}_S z = 0$.

In this connection, it is not true that $\operatorname{Im} \sum \operatorname{res}_z (1/f) \,.\, \operatorname{ind}_S z \neq 0$; a counter-example is provided by the system with $f(z) = iz^2/(z-1)$ treated in example 1 (cf. fig. 1).

Corollary. *With the assumptions of theorem 4, in each component of* $\mathsf{E}^2 - S$ *relatively to which* S *has non-zero index, there is at least one zero of* f.

We cannot conclude, in analogy with the corollary to theorem 1, that if there is precisely one singular point in a component of $\mathsf{E}^2 - S$, then it must be a center. The example mentioned again affords a counter-example. The reason why the analogy fails may be traced to that closed separatrices need not (though cycles must) have unity Kronecker index.

Nevertheless, it is natural to inquire about the sum of Kronecker indices of singular points in the bounded component of a simple closed separatrix. The remaining part of the paper is devoted to this question.

Assume there is given a system (1), and a simple closed curve S consisting of a finite set of complete trajectories and singular points of (1); in particular, then, we have the results of lemma 7 and 6.

Lemma 8. *Let* $\{z_j\}_1^n$ *be the singular points on* S. *Then*

$$\sum_{z \notin S} m(z) \left|\operatorname{ind}_S z\right| = 1 + \operatorname{ind} S \sum_j \left((m_j - 1)\, v_j + (\tfrac{1}{2} - m_j \delta_j) \operatorname{sgn} v_j\right), \tag{6}$$

with the following notation:

$$m_j = m(z_j)\,; \quad v_j = \frac{1}{2\pi} \operatorname{Arg} \frac{w_0}{w_i}\,;$$

w_0, w_i *are the exceptional directions (entrant, resp. exitant) of* S *at* z_j; δ_j *is defined as follows*: S *separates sufficiently small disc neighbourhoods of* z_j *into two curvilinear sectors*; *then* $\delta_j = 0$ *or* 1 *according as* $\operatorname{int} S$ *is or not within the sector with convex angle at vertex (within* $(-\pi, \pi\rangle$*), locally at* z_j.

Proof. Please refer to Hopf's proof [1, chap. XVI, th. 4,3] of the "cycle index is 1" theorem.

Given a simple closed positively oriented parametric curve $C = \{z(\sigma) : 0 \leqq \sigma \leqq 1\}$; omit the hypothesis that C have a smoothly varying tangent, and assume only that $z'(0) \neq 0$ exists. Then the proof yields, at least, that the variation of argument of the vector $u(\sigma, \tau) = z(\tau) - z(\sigma)$ is 2π along any simple curve Q in the $\sigma - \tau$ plane leading from $(0, 0)$ to $(1, 1)$, and except for these end-points, entirely within the triangle $0 < \sigma < \tau < 1$. However, if a continuous $z'(\sigma)$ exists for $\sigma_1 \leqq \sigma \leqq \sigma_2$ (some $0 \leqq \sigma_1 < \sigma_2 \leqq 1$), then we may also admit simple curves Q which touch the diagonal $\sigma = \tau$ along $\sigma_1 \leqq \sigma \leqq \sigma_2$. (The original proof in [1] consists in taking for Q the whole diagonal $0 \leqq \sigma = \tau \leqq 1$.)

Returning to our case, we have that the variation along Q is 2π ind S (since S may well be negatively oriented). Also, a continuous $z'(\sigma)$ exists except at a finite set of σ_j's with $z(\sigma_j) = z_j$, the singular points on S. Thus, for sufficiently small positive α_j, β_j,

$$2\pi \text{ ind } S = \sum_j \{\text{Var arg } z'(\sigma) : \sigma_{j-1} + \alpha_{j-1} \leqq \sigma \leqq \sigma_j - \beta_j\} + \\ + \sum_j \{\text{Var arg } (z(\tau) - z(\sigma)) : \sigma_j - \beta_j \leqq \sigma \leqq \sigma_j + \alpha_j, (\sigma, \tau) \in Q\}$$

(with obvious changes in the first and last summands near $\sigma = 0$ or $\sigma = 1$.) The first sum then constitutes a curvilinear integral over a finite set-join S' of disjoint subarcs of S missing the singular z_j; thus

$$(7) \qquad 2\pi \text{ ind } S = \text{Im} \int_{S'} \frac{f'}{f} \, dz + \sum_j \text{Var}_j$$

where Var_j has the obvious meaning.

Next, describe a circle K_j around each z_j with radius sufficiently small to have K_j intersect S at exactly two points (*cf.* assertion 2 in lemma 6); these then separate K_j into two arcs, of which precisely one, say A_j, has $A_j \subset \text{int } S$. Obviously $S - \bigcup \text{int } K_j$ is a curve S' of the type described above, and $S'' = S' \cup \bigcup A_j$ is a piecewise smooth closed curve; if the radii of K_j are taken sufficiently small, int S'' contains all the singular points in int S and none other. Thus by the residue theorem,

$$\int_{S''} \frac{f'}{f} \, dz = 2\pi i \sum m(z) \text{ ind}_S z = 2\pi i \sum m(z) \text{ ind}_S z$$

and therefore

$$(8) \qquad 2\pi \sum_{z \notin S} m(z) \text{ ind}_S z = \text{Im} \int_{S'} \frac{f'}{f} \, dz + \sum \text{Im} \int_{A_j} \frac{f'}{f} \, dz .$$

From (7) and (8) (on multiplying by $1/2\pi$ ind S),

$$(9) \qquad \sum_{z \notin S} m(z) \left|\text{ind}_S z\right| = 1 + \text{ind } S \sum_j \left(\frac{1}{2\pi} \text{Im} \int_{A_j} \frac{f'}{f} \, dz - \frac{1}{2\pi} \text{Var}_j\right).$$

Here the left side is independent of choice of the α_j, β_j, A_j; and a j-th summand on the right depends only on α_j, β_j, A_j. In particular, we may take $\alpha_j \to 0$, $\beta_j \to 0$, radius $A_j \to 0$ for each j separately.

Now consider any j; for simplicity assume $z_j = 0$, and set $m(z_j) = m$. Then, near $z_j = 0$,

$$\frac{f'(z)}{f(z)} = \frac{m}{z}(1 + o(|z|)),$$

so that, as radius $A \to 0$,

$$\int_A \frac{f'}{f} \, dz \to \operatorname{Im}(\varphi_o - \varphi_i) ;$$

here obviously

$$\varphi_o - \varphi_i = \operatorname{Arg} \frac{w_o}{w_i} - 2\pi\delta \operatorname{sgn} \operatorname{Arg} \frac{w_o}{w_i}$$

where w_i, w_o are the exceptional directions under which S enters (exits from) 0, and $\delta = 0$ or 1 according as int S is or not within the convex sector (with apex angle within $(-\pi, \pi\rangle$). Thus, for each j,

$$(10) \qquad \frac{1}{2\pi} \operatorname{Im} \int_{A_j} \frac{f'}{f} \, dz \to m_j(v_j - \delta_j \operatorname{sgn} v_j)$$

where m_j, v_j, δ_j are as in the statement of the lemma.

Finally we are to consider Var_j. Quite obviously, as $\alpha_j \to 0$, $\beta_j \to 0$ with radius $A_j \to 0$,

$$\mathrm{Var}_j \to \operatorname{Arg} \frac{w_o}{w_i},$$

where w_o, w_i have the previous meaning, so that (since $w_o \neq w_i$)

$$\mathrm{Var}_j \to 2\pi v_j - \pi \operatorname{sgn} v_j .$$

This and (10) in (9) yield (6), which was to be proved.

Formula (6) may be simplified further. By definition of exceptional directions, we have, at the singular point z_j, $w_i^{1-m} = -(a/|a|)$, $w_o^{1-m} = a/|a|$ where $a = \lim_{z \to z_j} f(z) \, . \, (z - z_j)^{-m}$ and $m = m_j = m(z_j)$. Then

$$v_j = \frac{1}{2\pi} \operatorname{Arg} \frac{w_o}{w_i} = \frac{1}{2} \frac{k_j}{|m_j - 1|}$$

with k_j an odd integer, $-|m_j - 1| < k_j \leqq |m_j - 1|$. It is easily seen that $|k_j|$ is the smaller number of sectors (bounded by consecutive exceptional directions at z_j), counted from the subarc of S entering z_j to that exiting from z_j; and $k_j > 0$ iff this order is in the positive direction. Obviously $\operatorname{sgn} v_j = \operatorname{sgn} k_j$. Since $0 \neq m_j \neq 1$ (lemma 7), $\operatorname{sgn} m_j = \operatorname{sgn}(m_j - 1)$.

Furthermore, there is a connection between δ_j and $\operatorname{sgn} v_j$. It is easily seen that if ind $S = 1$, then

$$v_j < 0 \quad \text{if} \quad \delta_j = 0 , \qquad v_j > 0 \quad \text{if} \quad \delta_j = 1 ,$$

i.e. $2\delta_j - 1 = \operatorname{sgn} v_j$. If ind $S = -1$, then the v_j's change sign, and δ_j's remain unchanged; thus in every case,

$$2\delta_j - 1 = \operatorname{sgn} v_j \operatorname{ind} S .$$

These results are formulated below.

Theorem 5. *Let S be a simple closed curve, consisting of a finite set of complete trajectories and singular points* z_j; *assume* $\operatorname{int} S \subset G$. *Then*

$$(11) \qquad \sum_{z \notin S} m(z) \left|\operatorname{ind}_S z\right| = 1 + \sum_j \tfrac{1}{2}(1 + |k_j| \operatorname{sgn} m_j - 2\delta_j m_j)(2\delta_j - 1),$$

where k_j *is an odd integer,* $-|m_j - 1| < k_j \leqq |m_j - 1|$, *and* m_j, δ_j *are as described in lemma* 8. *Furthermore,* $2\delta_j - 1 = \operatorname{sgn} k_j \operatorname{ind} S$.

Some rough estimates may be obtained from (11). For convenience, set $N_Z = \sum_{m_j > 0} 1$, the number of zeros on S; and $M_Z = \sum_{m_j > 0} m_j$, $M_P = \sum_{m_j < 0} |m_j|$, the sums of (positive) multiplicities of zeros and poles, respectively, of f on S. For poles one has $m_j < 0$ and $1 \leqq |k_j| \leqq |m_j - 1| = |m_j| + 1$; the corresponding terms in (11) are

$$\sum_{m_j < 0} = \tfrac{1}{2} \sum_{\delta = 0} (|k_j| - 1) + \tfrac{1}{2} \sum_{\delta = 1} (1 - |k_j| + 2|m_j|);$$

and one obtains the following estimates

$$0 \leqq 0 + \tfrac{1}{2} \sum_{\delta = 1} |m_j| \leqq \sum_{m_j < 0} \leqq \tfrac{1}{2} \sum_{\delta = 0} |m_j| + \sum_{\delta = 1} |m_j| \leqq M_P.$$

Similarly, for zeros one has $m_j > 0$ and $1 \leqq |k_j| \leqq |m_j - 1| = m_j - 1$ (in particular, $2 \leqq m_j$, *cf.* lemma 7); then from

$$\sum_{m_j > 0} = -\tfrac{1}{2} \sum_{\delta = 0} (|k_j| + 1) + \tfrac{1}{2} \sum_{\delta = 1} (1 + |k_j| - 2|m_j|)$$

there follow the estimates

$$-M_Z \leqq -\tfrac{1}{2} \sum_{\delta = 0} |m_j| + \sum_{\delta = 1} (1 - |m_j|) \leqq \sum_{m_j > 0} \leqq -\tfrac{1}{2} \sum_{\delta = 0} 2 - \tfrac{1}{2} \sum_{\delta = 1} |m_j| \leqq -\tfrac{1}{2} N_Z.$$

Using these, one obtains the following corollaries (the assumptions of theorem 5 and the preceding notation are preserved).

Corollary 1. $1 - M_Z \leqq \sum m(z) \left|\operatorname{ind}_S z\right| \leqq 1 + M_P - \tfrac{1}{2} N_Z$.

Corollary 2. *If there are no zeros on S, then* int *S contains at least one zero. If, furthermore, there are no poles in* int *S, then the sum M of multiplicities of zeros in* int *S satisfies* $1 \leqq M \leqq 1 + M_P$.

Corollary 3. *If there are no poles and at least three zeros on S, then* int *S contains at least one pole.*

Sharper results may be obtained on restricting the separatrices S by requiring all δ_j to be equal.

Example 1 (contd.). Now we may complete our examination of the case $f(z) = iz^2/(z-1)$, $G = S^2$. The existence of a closed separatrix $S \subset E^2$ with $1 \in S \not\ni 0$ has already been established. From the corollary to theorem 4, each component of $E^2 - S$ must contain a zero of f; thus there is a unique component and hence S is simple closed, and 0 is the unique singular point in int S. In particular, we may apply theorem 5; here $m_1 = -1$, so that $-2 < k_1 \leqq 2$, *i.e.* $|k_1| = 1$; and thus from (11)

$$2 = 1 + \tfrac{1}{2}(1 - 1 + 2\delta_1)(2\delta_1 - 1), \quad \delta_1 = 1 .$$

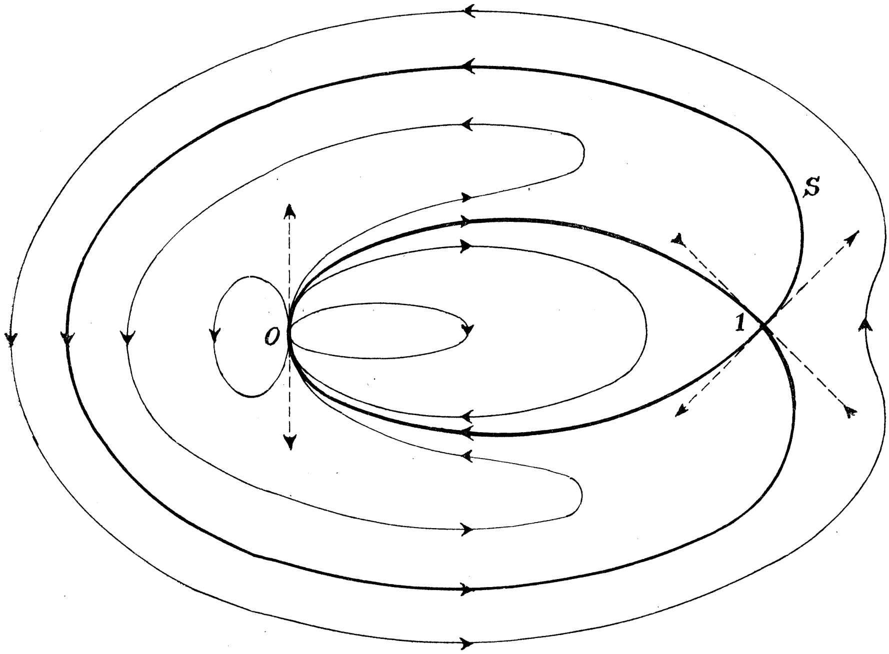

Fig. 1.

From symmetry, if $z(\theta)$ is a solution of (1), then $\overline{z(-\theta)}$ is also a solution, so that S must be symmetric about the x-axis. Thus the exceptional directions of S at 1 are either $e^{\pm i\pi/4}$ or $-e^{\pm i\pi/4}$. If it were the latter, then necessarily $\delta_1 = 0$ (cf. the definition of δ_j in lemma 8), a contradiction. This establishes how S is situated.

All trajectories in int S tend towards and from 0, with the exception of the two trajectories which have exceptional directions $e^{i\pi/4}$ and $-e^{-i\pi/4}$ at 1. Thus, finally, we have fig. 1.

Given a rational function f, we may define the *type* of the dynamical system (1) associated with f as the system of integers $\{m_f(z)\}_z$ with z varying over all the singular points of (1).

Thus the canonic systems $z' = az^m$ (m integer, $a \neq 0$) have type $(m, 2 - m)$ for $0 \neq m \neq 2$, type (2) for $m = 2$, and empty type for $m = 0$. The system of example 1 has type $(2, -1, 1)$. (Obviously the sum of multiplicities is 2 except for empty type.)

Example 2. Any rational system (1) of type $(2, -1, 1)$ is of the form

$$z' = a\,\frac{z^2}{z - 1} \quad (a \neq 0)\,, \tag{12}$$

up to a homographic mapping taking the singular points to 0, 1, ∞ respectively.

Obviously the trajectories of (12) are isogonals to those of example 1; the angle between the trajectories is Arg $-ia$.

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Резюме

О МЕРОМОРФНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, I

ОТОМАР ГАЕК (Otomar Hájek), Прага

Изучается поведение в целом траекторий динамической системы $dz/d\theta = f(z)$ где функция f мероморфная в заданной области комилексной сферы, и θ вещественная переменная. (В [2] предложена локальная теория этих систем.)

Доказано, что у мероморфных систем не существуют изолированные циклы: всякий цикл погружен в полосу циклов того хе периода (теоремы 2 и 3). Результаты о кратностях и резидуумах сингулярных точек во внутренней области цикла (теорема 1; первый из них, по существу, классический) переносятся на более общий случай замкнутой сепаратрицы: теоремы 4 и 5. Отдельные результаты иллюстрированы на качественном анализе одного примера мероморфной системы (фиг. 1).