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NOTES ON MEROMORPHIC DYNAMICAL SYSTEMS II

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The terminology, notation and many assumed results are those of [5]. In particular, the object studied is a 2-dimensional dynamical system (called a *meromorphic* system)

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = f(z) \tag{1}$$

defined by a function f meromorphic in an open $G \subset S^2$ (the 2-sphere). The *singular* points of (1) are the poles and zeros of f.

A problem was formulated [*l.c.*, problem 2], to obtain reasonably effective sufficient conditions for the existence of cycles of (1), containing within their interior a prescribed set Z of zeros of f with

$$\sum_{z \in Z} \operatorname{res}_z \frac{1}{f} = i\alpha\,, \quad \text{real } \alpha \neq 0 \tag{2}$$

($\sum m(z) = 1$ was also required there).

In the present paper, a necessary and sufficient condition is given (lemma 1 and theorem 1, respectively). Whether or not it is "reasonably effective" is open to doubt; however, it is effective enough to answer problem 1 (*l.c.* – is every cycle of (1) nested about a center?) by an example, in the negative. As another consequence we obtain that the limit set of a noncyclic trajectory of (1), if in a simply connected G, reduces to a single singular point. In particular, trajectories cannot spiral toward limit sets more complicated than foci.

First we shall need slightly more information about the boundary of an annulus of cycles than is given in theorems 2 and 3 of [5]. To this end we introduce the following definition.

Definition. Assume given a meromorphic system (1). A set $K \subset G$ will be called an s-curve if there exist a parametrisation of K and a component Q of $S^2 - K$ with the following properties:

(i) K is a parametrised closed elementary curve,

(ii) K is the complete boundary of Q,

(iii) $\operatorname{ind}_K z \neq 0$ implies $z \in G$,

(iv) K is the union of a finite set of poles of f and a finite set of complete trajectories of (1), all having the same or all the opposite orientation to that of K.

K will be called an interior s-curve if, in addition,

(v) no trajectory entering or leaving any point on K intersects Q, and

(vi) either $\operatorname{ind}_K z = 1$ for all $z \in S^2 - \bar{Q}$ or $\operatorname{ind}_K z = -1$ for all $z \in S^2 - \bar{Q}$.

By possibly changing the orientation of K, we may and occasionally shall assume that some fixed alternative occurs in (iv) or (vi). Requirement (i) means, in greater detail, that

(3) $$K = \{p(\theta) : \alpha \leqq \theta \leqq \beta\}$$

for some real $\alpha < \beta$ and some continuous $p : \langle \alpha, \beta \rangle \to K$ with $p(\alpha) = p(\beta)$ and such that the set

$$\{\theta : p(\theta) = p(\theta_1) \text{ for some } \theta_1 \neq \theta\}$$

is finite. (*Cf.* [2]; this latter condition implies that the parametrisation p is locally simple.)

Lemma 1. *Let C be a cycle of* (1) *with* $\operatorname{int} C \subset G$; *let A be the maximal open connected neighbourhood of C consisting of complete cycles*; *let K be that component of the boundary of A which lies in* $\operatorname{int} C$. *Then K is a center or an interior s-curve.*

Proof. *Cf.* [5], theorems 2 and 3. It remains to prove that K is elementary, property (ii) of the definition, and that K is interior.

By unicity, distinct trajectories do not intersect; thus only the finite set of singular points on K are possible self-intersections. Obviously the counter-images (in the parametrisation) of the singular points on K form an isolated closed set.

Let Q be the component of $S^2 - K$ containing A; obviously the boundary of Q is K.

Since every trajectory in Q sufficiently near K is in A, and hence a cycle, it cannot enter or exit from any point on K. Finally, for all such cycles C_j one has *e.g.* (possibly after a re-orientation)

$$\operatorname{ind}_{C_j} z = 1 \quad \text{for all } C_j \text{ and } z \in S^2 - \bar{Q}\,.$$

Since one may take $C_j \to K$ uniformly, there follows $\operatorname{ind}_K z = 1$ for $z \in S^2 - \bar{Q}$. This completes the proof of lemma 1.

Lemma 2. *Let K be an interior s-curve. Then the parametrisation of the definition may be chosen such that*

$$K = \{p(\theta) : 0 \leqq \theta \leqq 2\pi|\alpha|\}$$

where real $\alpha \neq 0$ *satisfies* $\sum \operatorname{res}_z 1/f \,.\, \operatorname{ind}_K z = i\alpha$ *and* $p(0)$ *is any given point on* K. *Furthermore, to each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that, for any solution* $z(.)$ *of* (1) *with* $|z(0) - p(0)| < \delta$, $z(0) \in Q$, *there is*

$$|z(\theta) - p(\theta)| < \varepsilon \quad for \quad 0 \leqq \theta \leqq 2\pi|\alpha| \,.$$

Proof. In the parametrisation given in the definition, take consecutive arcs of trajectories; on each choose a point z_j ($1 \leqq j \leqq n$, $n \geqq 1$). Let $z_j(.)$ be the solution of (1) with $z_j(0) = z_j$. If there are no singular points on K, then K is a cycle, $z_1(.)$ is periodic and its period satisfies the condition to be proved [5, lemma 4]. If there exist singular points on K, then there exist finite $\theta_j > 0 > \theta_j^*$ such that both $\lim\limits_{\theta \to \theta_j^{*}+} z_j(\theta)$, $\lim\limits_{\theta \to \theta_j -} z_j(\theta)$ are singular points. Then set $p(\theta) = z_j(\theta + \theta_j^* - \theta_1^* - \sum\limits_{k<j} (\theta_k - \theta_k^*))$ for $\sum\limits_{k<j} (\theta_k - \theta_k^*) \leqq \theta \leqq \sum\limits_{k \leqq j} (\theta_k - \theta_k^*)$. Obviously this parametrises K, $0 \leqq \theta \leqq \sum_1^n (\theta_k - \theta_k^*)$. From [5, theorem 4] it follows that $\sum_1^n (\theta_k - \theta_k^*) = 2\pi|\alpha|$ where α is as described in the formulation. By shifting the interval (and periodic continuation of p), any point on K may be made to coincide with $p(0)$.

Finally, consider the stability property. It suffices to show that if $z_n(.)$ are solutions of (1) with $z_n(0) \to p(0)$, $z_n(0) \in Q$, then

$$z_n(\theta) \to p(\theta) \quad \text{uniformly} \tag{4}$$

for $0 \leqq \theta \leqq 2\pi|\alpha|$. Obviously (4) for $0 \leqq \theta \leqq \theta_1 - \theta_1^*$ (using the previous θ_j, θ_j^*). Then $z_n(\theta_1 - \theta_1^*)$ converges to $p(\theta_1 - \theta_1^*)$ and lies in Q; we may then conclude, similarly, that (4) for $\theta_1 - \theta_1^* \leqq \theta \leqq (\theta_1 - \theta_1^*) + (\theta_2 - \theta_2^*)$, *etc.* (At each step, the point is that every point of K is a boundary point of Q.) Thus we may continue (4) over the whole interval $2\pi|\alpha|$.

Lemma 3. *Let* U *be an open neighbourhood of a closed connected set* F; *let* f *be meromorphic in* U *with all poles in* F *and with*

$$\sum \operatorname{res}_z f = 1 \,.$$

Then in $U - F$ *there exists a (multivalued) primitive function* g *to* f *such that* $\exp g$ *is holomorphic.*

Proof. If $P \subset F$ is the set of all poles of f, then every point of $U - P$ is the center of an element of a primitive function to f, arbitrarily prolongable in $U - P$. *A fortiori*, there is an analytic function g on $U - F$, arbitrarily prolongable in $U - F$, and such that $g_0' = f$ for every element g_0 of g.

Now consider two elements g_1, g_2 of g, with the same center z_0. Then there is a closed parametric curve C in $U - F$ such that g_2 is a prolongation of g_1 along C. We may assume C is simple; otherwise one would have a finite sequence $g_1 = g_1^*, g_2^*, g_3^*, \ldots, g_n^* = g_2$ of elements with center z_0 and such that consecutive elements are indeed prolongations along simple closed curves, and then one could treat each pair in turn.

P is in a unique component of $S^2 - C$ (since $P \subset F$ connected, $C \subset S^2 - F$); thus all points of P have the same index with respect to C; denote it by k. In the course of prolongation along C, the element g_1 is increased by

$$\sum \operatorname{res}_z f \,.\, 2\pi i k = 2\pi i k \,.$$

Hence exp g_1 is unchanged, *i.e.* exp g is single-valued. Obviously exp g has no poles in $U - F$ and thus it is holomorphic.

Lemma 3 concludes the preliminaries necessary for the first main result.

Theorem 1. *Let there be given a meromorphic system* (1). *Then every interior s-curve K is a component of the boundary of an annulus of cycles of* (1). (*Furthermore, this annulus is in the component Q from the definition.*)

Proof. From [5, theorem 4] again, there is a real $\alpha \neq 0$ with

$$\sum \operatorname{res}_z \frac{1}{f} \,.\, \operatorname{ind}_K z = i\alpha \,.$$

Now recall that $\operatorname{ind}_K z = 1$ for all $z \in S^2 - \bar{Q}$, and apply lemma 3. As the function-take $1/(i\alpha f)$. For the set F one may take $S^2 - Q$, or any connected closed subset containing all zeros of f in $S^2 - Q$. For the set U take an arbitrary open neighbourhood of $S^2 - Q$ which again contains only those zeros of f which are in $S^2 - Q$ (recall there are only poles and no zeros on K). There results a holomorphic function $h = \exp g$ on $U - F = U \cap Q$ with $h' = h/(i\alpha f)$.

Take any $z_0 \in U \cap Q$, and consider the solution $z(.)$ of (1) with $z(0) = z_0$. From lemma 2 it follows that if z_0 is taken sufficiently near K, then $z(\theta) \in U \cap Q$ for $0 \leqq \theta \leqq 2\pi|\alpha|$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\theta} h(z(\theta)) = h'(z(\theta))\, z'(\theta) = h(z(\theta)) \frac{1}{i\alpha f(z(\theta))}\, z'(\theta) = h(z(\theta)) \frac{1}{i\alpha}$$

so that

$$h(z(\theta)) = h(z(0)) \exp \frac{\theta}{i\alpha}$$

and

$$h(z(2\pi|\alpha|)) = h(z_0) \,. \tag{5}$$

We emphasise that (5) holds for any $z_0 \in U \cap Q$ sufficiently near K.

Next, take any $z_1 \in K$, not a pole of f. Obviously h may be prolonged over a neighbourhood of z_1, and then at z_1, $h' = h/(i\alpha f) \neq 0$. Thus in some neighbourhood of z_1, the function h is $1 - 1$. This with (5) proves that if z_0 is taken in $U \cap Q$ sufficiently near K, then the solution $z(.)$ through z_0 has $z(2\pi|\alpha|) = z(0) = z_0$, *i.e.* has period $2\pi|\alpha|$.

Thus there are cycles of (1) arbitrarily (uniformly) near K in Q; this and theorem 1 of [5] conclude the proof of the theorem.

Remarks. 1. If K contains at least one pole, then K is a component of the boundary of a maximal open connected annulus of cycles of (1).

2. Theorem 1 was stated to be a solution of problem 2; however condition (2) does not appear explicitly in the formulation (but also see [5], theorem 4).

3. It may be noticed that the proof of theorem 1 is a distant development of Gregor's proof [3] of a theorem stating that a simple zero z_0 of f with pure imaginary $f'(z_0)$ (or, what comes to the same thing, with pure imaginary $\text{res}_{z_0} 1/f$) is a center. Individual steps of this latter proof are parallelled by lemmas 2 and 3.

As in [5], let $m(z)$ denote the signed multiplicity function. Then we have the following

Corollary. *If K is an interior s-cirve of* (1) *and Q a component of* $S^2 - K$ *as in the definition, then*

$$\sum_{z \notin Q} m(z) = 1 .$$

(Proof.) Again, we may assume $\infty \in Q$. Then the singular points not in Q are precisely the singular points in the interior of any cycle C of the annulus which K bounds (theorem 1). Using [5, theorem 1],

$$\sum_{z \notin Q} m(z) = \sum m(z) \left|\text{ind}_C z\right| = 1 .$$

For the special case that K is a simple closed curve this reduces to [5], theorem 5 ($|k_j| = \delta_j = 1$).

The *limit set* (or ω-limit set) of a trajectory of (1) through a non-singular point z_0 is defined thus (*e.g.* [1], ch. XVI, § 1): Let $z(.)$ be the solution of (1) with $z(0) = z_0$ and with a maximal open domain of definition (α, β) (possibly $\alpha = -\infty$ or $\beta = +\infty$); then w is in the limit set iff $w = \lim z(\theta_n)$ for some $\theta_n \nearrow \beta$.

Since S^2 is compact, each limit set is always non-empty connected compact; if $w \in G$ is in a limit set, then the complete trajectory though w is in the limit set [*l.c.*, theorem 1.1 and 1.2].

Theorem 2. *Given, a meromorphic dynamical system* (1) *in an open set G with* $S^2 - G$ *connected. If the limit set of a non-cyclic trajectory is in G, then it reduces to a single singular point.*

Proof. Let T be a trajectory of (1), ωT its limit set, $\omega T \subset G$, and assume that T is not a cycle and ωT is not a singular point; to obtain a contradiction we shall first show that then ωT is an s-curve.

If ωT contains no singular points, it is a cycle [*l.c.*, theorem 2.1]; but from [5, theorem 2] it follows that a cycle of (1) is the limit set of no trajectory except itself. If ωT contains a zero z_0 of f, then from [5, lemma 2] there follows $\omega T = z_0$, contradiction. Since ωT is a compact subset of G, the set of poles of f on ωT is finite. Only a finite set of trajectories enters or exits from a pole of f, according to [3, th. 4]. Thus ωT is the union of a finite set of trajectories and poles of f, with at least one pole on ωT.

As in the proof of lemma 2 it can be shown that ωT is a closed parametric curve. As in the proof of lemma 1, this parametrisation is elementary.

That $z \in G$ whenever $\operatorname{ind}_{\omega T} z \neq 0$ is a consequence of the assumption that $S^2 - G$ is connected, if (after inversion, if necessary), $\infty \in S^2 - G$.

It remains to prove property (ii) of the definition. We shall show that that component Q of $S^2 - \omega T$ which contains the trajectory T has ωT as its boundary (by a classical theorem, T intersects ωT iff T is a cycle; *e.g.* [1], ch. XVI, lemma 2.3). This is a general property of plane dynamical systems: [6, section 4]. Since ωT is compact and Q a component of $S^2 - \omega T$, the boundary of Q lies within ωT. Secondly, $T \subset Q$ implies

$$\omega T \subset \bar{T} \subset \bar{Q}\,;$$

from these two inclusions we conclude that the boundary of Q is $\omega T - Q$, which is ωT since Q does not intersect ωT.

Now, finally, from theorem 1 we have that there exist cycles in Q arbitrarily uniformly near ωT; each of these then separates T from its limit set ωT, a contradiction. This proves theorem 2.

Example. Consider the meromorphic system $z' = f(z)$ with

$$f(z) = i\,\frac{(z^2-4)^2}{z^2-1}\,, \quad G = S^2\,.$$

This has the special property that f is even, and also "formally pure imaginary" in the sense that $\overline{f(z)} = -f(\bar{z})$. Thus, if $z(\theta)$ is a solution, then so are $-z(-\theta)$, $\bar{z}(-\theta)$, $-\bar{z}(\theta)$; therefore the configuration of trajectories is symmetrical about both the real and imaginary axes. The singular points z, their signed multiplicities $m(z)$, residues of $1/f$ and the corresponding entrant and exitant exceptional directions w_i, w_o are collected in the following table

z	$m(z)$	$\operatorname{res}_z 1/f$	w_i	w_o
-2	2	$\frac{5}{32}i$	i	$-i$
-1	-1	0	$\pm e^{i\pi/4}$	$\pm e^{-i\pi/4}$
1	-1	0	$\pm e^{-i\pi/4}$	$\pm e^{i\pi/4}$
2	2	$-\frac{5}{32}i$	i	$-i$

(∞ is a nonsingular point).

From the column of residues and [5, theorem 1] we conclude that any cycle – if such exists – must contain in its interior precisely one of the zeros 2, −2; neither of these is a center.

Now assume there is a trajectory T with limit points 1, −1, the poles of f. By symmetry, there is a simple closed separatrix S passing through both poles but neither zero. From [5, theorem 4], int S contains a zero of f: contradiction. Since the imaginary axis is an axis of symmetry, no trajectory with a pole of f as limit point can intersect the imaginary axis.

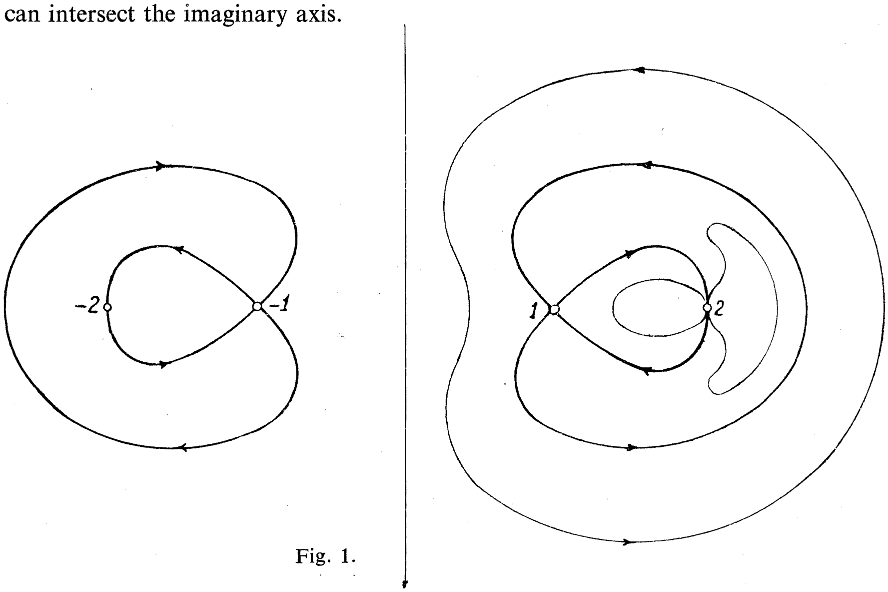

Fig. 1.

Next, consider a pole (*e.g.* 1) and the two trajectories T_1, T_2 exiting from it. By theorem 2, they tend either to 1 or to 2. In the former case we have a simple closed separatrix without zeros, which must then contain 2 in its interior.

If both T_j tend to 1, then the so obtained elementary closed curve K may be oriented in such a manner that $\text{ind}_K 2 = 0$, in contradiction with [5, theorem 4].

It is not possible for both T_j to tend to the zero 2, since by symmetry (to the real axis) the trajectory exiting from 1 with direction $w_o = -e^{i\pi/4}$ cannot enter 2 with the prescribed direction $w_i = i$.

We conclude that both the pole-zero pairs generate configurations similar to those of [5], example 1 and fig. 1. From theorem 2 it then follows that every trajectory in the remaining part of the plane is a cycle. (This also follows from theorems 3 and 4 of [5], since the imaginary axis is a cycle, in S^2.) In particular, *there exist meromorphic systems with cycles not nested about a center.*

It may be in place here to list some open questions which might prove interesting.

1. Given a meromorphic system (1) in G, and an open invariant set $H \subset G$ (a set H is invariant if $T \subset H$ whenever T is a trajectory incident with H). Prove that

$$\sum_{z \notin H} \text{Re}\,\text{res}_z \frac{1}{f} = 0.$$

(*Cf.* theorems 1 and 4 of [5].)

2. Given a finite sequence of non-zero integers $\{m_j\}_1^n$, describe the qualitative theory of all non-homeomorphic dynamical systems (1) where $G = S^2$, f is rational, and the singular points have multiplicities $\{m_j\}$. (For $n \leqq 2$ it suffices to consider the canonic systems $z' = az^m$ of [5].)

3. Treat the perturbation theory of meromorphic systems, with (i) fixed singular points, (ii) variable singular points, including the bifurcation of singular points.

4. Describe the boundary behaviour of meromorphic systems, with special reference to essential singularities and critical points of f. (For algebraic critical points, *cf.* [4].)

5. Obtain "reasonable" *a priori* bounds for regions of attraction of singular points (*i.e.* conformal maps of neighbourhoods of 0 in the canonic systems).

References

[1] *Coddington E. A., Levinson N.:* Theory of Ordinary Differential Equations, McGraw-Hill, New York 1955.

[2] *Černý I.:* Decomposition of an elementary curve into Jordan curves (in Czech), Čas. pěst. mat. *89* (1964), 173—204.

[3] *Gregor J.:* Dynamical systems with regular right-hand side (in Czech), Pokroky MFA *2*, III (1958), 153—160.

[4] *Gregor J.:* Dynamical systems with regular right-hand side (in Czech), Pokroky MFA *3*, III (1958), 266—270.

[5] *Hájek O.:* Notes on meromorphic dynamical systems I, Czech. Math. J. *16* (1966), 14—27.

[6] *Hájek O.:* Plane Dynamical Systems. Thesis (unpublished) 1960.

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Резюме

О МЕРОМОРФНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, II

ОТОМАР ГАЕК (Otomar Hájek), Прага

Для мероморфных динамических систем (см. [5]) доказаны следующие результаты:

1. Необходимое (лемма 1) и достаточное (теорема 1) условие для того чтобы замкнутая кривая была частью границы полосы циклов данной системы. Одним следствием является отрицательное решение гипотезы [5, проблема 1], что всякий цикл мероморфной системы окружает сингулярную точку типа центра.

2. Всякое непустое множество нециклической траектории, целиком содержащееся в односвязной области определения мероморфной системы, является единственной точкой (теорема 2).