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Otomar Hájek

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NOTES ON MEROMORPHIC DYNAMICAL SYSTEMS, III

OTOMAR HÁJEK, Praha

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This note is connected with a preceding paper [2]; the notation and terminology is preserved as far as possible. In particular, the object studied is a *meromorphic* system

$$\frac{dz}{d\theta} = f(z) \tag{1}$$

with f meromorphic in an open $G \subset S^2$ (the 2-sphere); the *singular* points of (1) are the critical points (zeros of f) and the poles of f. In the present paper, however, we will occasionally relax these assumptions, requiring (i) f to be a continuous maps $G \to E^2$ (thus "poles" are proscribed), and (ii) the system (1) to have unicity of solutions; in this case (1) will be called a *continuous* system.

We begin with an example.

Example 1. Consider the orthogonal pair of meromorphic systems

$$z' = z^2\,, \quad z' = iz^2$$

in $G = E^2$. There is a unique singular point, a zero of multiplicity 2 at the origin. The characteristic solutions are, respectively,

$$\frac{1}{1/z - \theta}\,, \quad \frac{1}{1/z - i\theta}$$

(for $z \neq 0$, in some neighbourhood of $\theta = 0$).

Now, start at a point $z \neq 0$, move along the trajectory to the first (second) system to a point with parameter θ (or τ), and thence along the trajectory to the second (first) system to a point with parameter τ (or θ). One obtains

$$z \begin{array}{l} \nearrow \dfrac{1}{1/z - \theta} \to \dfrac{1}{(1/z - \theta) - i\tau} \\ \searrow \dfrac{1}{1/z - i\tau} \to \dfrac{1}{(1/z - i\tau) - \theta} \end{array}$$

Two consequences may be pointed out here. First, the result is a holomorphic function of $\theta + i\tau$ near 0; and second, that the same result is obtained in both procedures, for small $|\theta + i\tau|$ ("commutativity"). Such phenomena are studied in the present paper; the results are that (i) two meromorphic systems commute iff they are isogonal, and (ii) if continuous isogonal systems commute, they are meromorphic (under further assumptions – theorem 2).

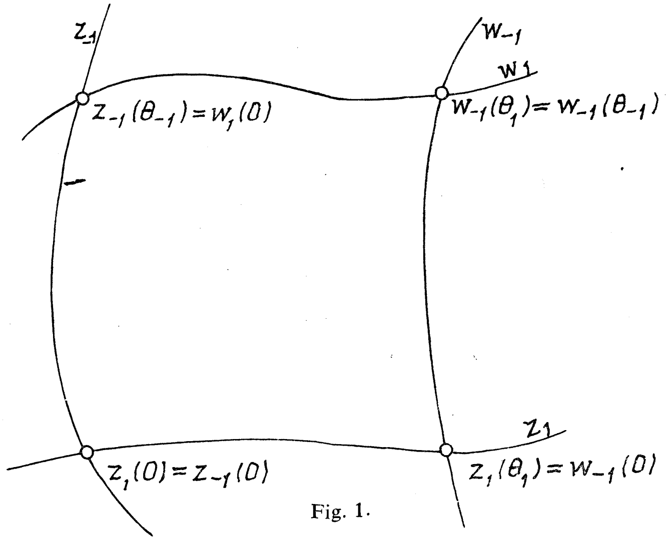

Fig. 1.

For definiteness, dynamical systems $z' = f_j(z)$ $(j = 1, 2, z \in G)$ are called *isogonal* if $f_1 = af_2$ for some complex constant $a \neq 0$; for $a = \pm i$ the term is *orthogonal* (sometimes also for $a = i\alpha$, $\alpha \neq 0$ real). If f_1/f_2 is non-real whenever $f_1 f_2 \neq 0$, they are called *transversal*; isogonality with non-real a is a very special case of transversality.

Definition. Given, two continuous dynamical systems

$$(2) \qquad z' = f_j(z) \quad (j = -1, 1)$$

in an open $G \subset S^2$. We will say that they are *integrally commutative* if the following property obtains: For $j = \pm 1$, let $z_j(.)$ be a solution of the j-th system, and $z_1(0) = z_{-1}(0)$; let θ_j be within the domain of definition of $z_j(.)$, and let $w_j(.)$ be the solution of the j-th system with $w_j(0) = z_{-j}(\theta_{-j})$. Then (*cf. fig.* 1) $w_1(\theta_1) = w_{-1}(\theta_{-1})$. If this holds only for sufficiently small $|\theta_j|$ (say for $|\theta_j| < \varepsilon$ with $\varepsilon > 0$ possibly depending on $z_j(0)$), we will say that (2) are (*locally*) *commutative.*

Remarks. Since we will mostly be concerned with local rather than integral commutativity, the qualifier "local" will usually be omitted. In the preceding definition, it is required that w_j be defined at θ_j.

The definition needs a trivial modification for meromorphic systems: two meromorphic systems (2) are said to commute if they satisfy the definition in the open set $G - \{\text{poles of } f_j\}$.

Lemma. *Assume given a meromorphic system* (1), *and a point p at which f is holomorphic. From p more along the trajectory to* (1) *to a point with parameter* θ, *and then along the trajectory to*

$$z' = if(z) \tag{3}$$

to a point with parameter τ. *The point so obtained is a meromorphic function of* $\theta + i\tau$ *near* 0.

Proof. By a classical theorem (*e.g.* [1], chapter I, theorem 8.1), there exists a meromorphic function $z(.)$ defined near 0 and such that

$$\frac{dz(w)}{dw} = f(z(w)), \quad z(0) = p$$

Now consider $z(\theta + i\tau)$; obviously, for fixed τ, it is a solution of (1) (by unicity, it is the only solution of (1) with $z(0) = p$); for fixed θ it is a solution of (3). This proves the lemma.

Theorem 1. *Two meromorphic systems in a region G, neither vanishing identically, commute if and only if they are isogonal in G.*

Proof. First take isogonal meromorphic systems, say (1) and ($a \neq 0$)

$$z' = a\, f(z) \tag{4}$$

Take $p \in G$ with $f(p) \neq \infty$, and let $z(\theta + i\tau)$ be the function constructed in lemma 1 (thus $z(0) = p$). Obviously, for small $|c|$, $z(c + \theta)$ is a solution of (1), and $z(c + a\theta)$ a solution of (4). Now, for small $|\theta + i\tau|$,

$$p \begin{matrix} \nearrow z(\theta) & \to z(\theta + a\tau) \\ \searrow z(a\tau) & \to z(a\tau + \theta) \end{matrix}$$

with coinciding end-points; this is commutativity.

For the converse assertion, consider two meromorphic systems

$$z' = f_j(z), \quad (j = 1, 2, \; z \in G),$$

and assume they commute. Take a point $p \in G$ non-singular for both systems. In the now familiar manner, denote by $z(\theta, \tau)$ the point obtained by first moving from p along the trajectory of the first system to a point with parameter θ, and then along the trajectory of the second system to a point with parameter τ (θ, τ real, $|\theta + i\tau|$ small, $z(0, 0) = p$). In any case

$$z(\theta, \tau) - z(\theta_0, \tau) - z(\theta, \tau_0) + z(\theta_0, \tau_0) = z(\theta, \tau) - z(\theta, \tau_0) - z(\theta_0, \tau) + z(\theta_0, \tau_0)$$

for small $|\theta + i\tau|$, $|\theta_0 + i\tau_0|$, so that

$$\int_{\theta_0}^{\theta} \frac{\partial}{\partial\theta}(z(\theta,\tau) - z(\theta,\tau_0))\,\mathrm{d}\theta = \int_{\tau_0}^{\tau} \frac{\partial}{\partial\tau}(z(\theta,\tau) - z(\theta_0,\tau))\,\mathrm{d}\tau$$

By construction, $z(\theta, \tau)$ with fixed θ is a solution of the second system; by commutativity, $z(\theta, \tau)$ with fixed τ is a solution of the first system; thus

$$\int_{\theta_0}^{\theta} [f_1(z(\theta,\tau)) - f_1(z(\theta,\tau_0))]\,\mathrm{d}\theta = \int_{\tau_0}^{\tau} [f_2(z(\theta,\tau)) - f_2(z(\theta_0,\tau))]\,\mathrm{d}\tau$$

Differentiating $\partial^2/\partial\theta\,\partial\tau$ (f_j are meromophic, $z(.)$ is C^∞), one obtains $f_1'f_2 = f_2'f_1$ at $z = z(\theta, \tau)$; thus f_1/f_2 is constant on the trajectories through p. Since p is non-critical, these trajectories have accumulation points in G, and thus f_1/f_2 is constant throughout G. This completes the proof of theorem 1.

Theorem 2. *Given, two isogonal transversal continuous dynamical systems in an open $G \subset \mathsf{S}^2$, with critical points isolated. Then, if the systems commute, they are meromorphic.*

Proof. Let the systems be (1) and (4) again, with non-real constant a. Take any $p \in G$ with $f(p) \neq 0$. Construct a mapping $z(.)$ of a neighbourhood of 0 in S^2 into G in the usual manner as follows. From p move along the trajectory to (1) to a point with parameter θ; and then along the trajectory to (4) to a point with parameter τ; denote the resulting point by $z(\theta + a\tau)$ (recalling $a \neq \bar{a}$).

Since (1), (4) are isogonal and p non-critical, $z(.)$ is $1 - 1$ in some disc-neighbourhood D of 0 in E^2; obviously $z(.)$ is continuous, so that $U = z(D)$ is a neighbourhood of p (the Invariance of Domain Theorem).

Now take any $p_0 \in U$, $p_0 = z(\theta_0 + a\tau_0)$. By construction $z(\theta_0 + a\tau)$ with variable τ is a solution of (4), so that

$$\frac{\partial}{\partial\tau} z(\theta_0 + a\tau)|_{\tau=\tau_0} = a\,f(p_0)\,.$$

By commutativity, $z(\theta + a\tau_0)$ (variable: θ) is a solution of (1) and thus

$$\frac{\partial}{\partial\theta} z(\theta + a\tau_0)|_{\theta=\theta_0} = f(p_0)\,.$$

Hence

$$a\frac{\partial}{\partial\theta} z(\theta + a\tau) = \frac{\partial}{\partial\tau} z(\theta + a\tau) \quad \text{for} \quad \theta + a\tau \in D\,,$$

the "oblique" Cauchy-Riemann equation. We conclude that $z(.)$ is holomorphic (and $1 - 1$) in D, and then

$$\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{\mathrm{d}z}{\mathrm{d}\theta} = f(z)$$

is also holomorphic; and thus so is f, the composition of $f(z(.))$ and $z^{-1}(.)$. Summarising, f is holomorphic at any $p \in G$ with $f(p) \neq 0$. Since f is continuous and the critical points are isolated by assumption, it follows from a familiar theorem that f is holomorphic throughout G. This concludes the proof.

Example 2. There do exist non-meromorphic commutative continuous dynamical systems: a diffeomorphic but non-conformal map of isogonal meromorphic systems will usually have this property (however, by theorem 2, these cannot be isogonal).

However, there exist quite simple dynamical systems which do not commute with any transversal dynamical system (except the trivial $z' = 0$). Thus, consider the continuous dynamical system

$$z' = i|z|\, z\,.$$

Its characteristic function is $z \exp i|z|\,\theta$; thus each $z \in S^2$, $0 \neq z \neq \infty$, is on a cycle with period $2\pi/|z|$.

Now, consider any non-trivial transversal system; each trajectory then intersects an infinity of cycles of the former system; if the systems were commutative, then all these cycles would have the same period, a contradiction.

Problems.

1. Prove that a meromorphic system $z' = f(z)\,(z \in G)$ is integrally commutative with some transversal isogonal iff f has no poles in G.

2. For continuous dynamical systems in S^2, is commutativity transitive? (From theorem 1 it follows that it is transitive for meromorphic systems.)

References

[1] *Coddington E. A., Levinson N.:* Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[2] *Hájek O.:* Notes on Meromorphic Dynamical Systems I, Czech. Math. J. *16* (1966), 14—27.

Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK)

Резюме

О МЕРОМОРФНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ, III

ОТОМАР ГАЕК (Otomar Hájek), Прага

Пара динамических систем называется перестановочной если — грубо говоря — при переключительном режиме концевая точка перемещения не зависит от порядка переключений. Показано, что между мероморфными системами перестановочность эквивалентна изогональности — теорема 1; и что изогональные перестановочные системы (непрерывные с изолированными особенностями, в плоскости) обязательно мероморфны.