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PROLONGATION OF SECTIONS IN LOCAL DYNAMICAL SYSTEMS

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This paper is closely connected with [1], and aims to extend some of the results obtained there. The generalisation is as follows:

(i) From the global dynamical systems of [1] to local dynamical systems (cf. [2]). Formally, this is almost trivial – one need only take a little more care in the proofs – but quite useful as far as applications are concerned.

(ii) It is shown that every compact section S_0 may be embedded in another section which then generates a neighbourhood of S_0 (theorem 5). The motivation for this was the special case described in theorem 7. Obviously, if a single noncritical point is taken for S_0, one obtains the Whitney-Bebutov theorem.

(iii) Finally it is proved that in theorem 1 of [1], local connectedness may be omitted from the assumptions (theorem 8).

Let P be a completely regular topological space. A *local dynamical system* on P is a mapping T with the properties $1^\circ - 3^\circ$ (cf. [2]):

1° T is a continous map of an open subset of $P \times \mathsf{E}^1$ into P (taking the usual product topology of $P \times \mathsf{E}^1$); for each $x \in P$ there are $-\infty \leqq \alpha_x < 0 < \beta_x \leqq +\infty$ such that T is defined at (x, θ) iff $\alpha_x < \theta < \beta_x$ (the value of T at (x, θ) will be denoted by $x\mathrm{T}\theta$);

2° $x\mathrm{T}0 = x$;

3° $(x\mathrm{T}\theta_1)\mathrm{T}\theta_2 = x\,\mathrm{T}\,(\theta_1 + \theta_2)$ whenever both $x\mathrm{T}\theta_1$ and either the left or right side are defined.

If domain T is $P \times \mathsf{E}^1$ itself, T may be called a *global* dynamical system. These form the subject of [1]; see also [3, chap. V]. The difference between local and global dynamical systems may be illustrated by the fundamental application: In vector notation, let

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = f(x)$$

denote an autonomous system of differential equations in E^n. Let $f : \mathsf{E}^n \to \mathsf{E}^n$ be continous, and assume some local unicity condition. For $x \in \mathsf{E}^n$, $\theta \in \mathsf{E}^1$ let $x\mathrm{T}\theta$ be the value at θ of that solution which has initial value x at $\theta = 0$. By classical theorems, this defines a local dynamical system; it is global iff each solution can be prolonged over the entire θ-axis.

Henceforth we assume that there is given a local dynamical system T on a separated uniformisable space P.

In the usual manner, if $X \subset P$ and $A \subset \mathsf{E}^1$, and if $x\mathsf{T}\theta$ is defined for all $x \in X$, $\theta \in A$, then $X\mathsf{T}A$ will denote the set of all these elements. A point $x \in P$ is called *critical* iff $x = x\,\mathsf{T}\,\theta$ for all θ, $\alpha_x < \theta < \beta_x$.

Lemma 1. *Let $X \subset P$, $A \subset \mathsf{E}^1$, $X\mathsf{T}A$ defined. If $\overline{A}$ is compact, then* $\overline{X\mathsf{T}A} = \overline{X}\,\mathsf{T}\,\overline{A}$

For proof, see [1, lemma 2]. The following are easily proved: If both X, A are compact or both connected then the same holds for $X\mathsf{T}A$. If X is open then $X\mathsf{T}A$ is open if either T is global or P is locally euclidean.

Next we modify a definition from global dynamical system theory [3, p. 352], [1]:

Definition 2. A subset $S \subset P$ is a *section* if there exists a $\lambda > 0$ such that $x\mathsf{T}\theta$ is defined for $(x, \theta) \in S \times \langle -\lambda, \lambda \rangle$ and that

$$S \cap (S\mathsf{T}\theta) = \emptyset \quad \text{for} \quad 0 < |\theta| \leqq \lambda .$$

Any such λ may then be called a *length* of S. Given S and λ, the set $S\mathsf{T}\langle -\lambda, \lambda \rangle$ is said to be *generated* by S.

The following are immediate: $S \subset P$ is a section of length $\lambda > 0$ iff the sets $S\mathsf{T}\theta$, $S\mathsf{T}\theta'$ are disjoint for $-\lambda/2 \leqq \theta < \theta' \leqq \lambda/2$. Any subset of a section is a section. A singleton is a section iff it is noncritical. A compact $S \subset P$ is a section iff it is a section locally at each $x \in S$ (or equivalently, at each $x \in P$, since $\emptyset$ is a section).

Construction 3. Let there be given a compact nonvoid section S_0. We shall first construct a mapping φ, then a neighbourhood U of S_0, and finally a set S whose properties will be examined.

Let S_0 have length $2\lambda_0 > 0$. Since sets $S_0\mathsf{T}\theta$ with distinct θ's are disjoint, we may define a map ψ_0: $S_0\,\mathsf{T}\,\langle -\lambda_0, \lambda_0 \rangle \to \mathsf{E}^1$ by $\psi_0(x\,\mathsf{T}\,\theta) = \theta$ for $x \in S_0$, $|\theta| \leqq 2_0$. Obviously ψ_0 is continuous on a compact domain (lemma 1), so that there is a continuous extension ψ, $\psi_0 \subset \psi : P \to \mathsf{E}^1$. Now define, wherever possible, $\varphi(x) = \int_{-\lambda_0}^{\lambda_0} \psi(x\mathsf{T}\theta)\,\mathrm{d}\theta$. Obviously $\varphi(x)$ is defined at least for $x \in S_0$, and then

$$\varphi(x) = \int_{-\lambda_0}^{\lambda_0} \psi_0(x\mathsf{T}\theta)\,\mathrm{d}\theta = \int_{-\lambda_0}^{\lambda_0} \theta\,\mathrm{d}\theta = 0\,. \tag{1}$$

From this point on, the construction parallells that of [4].

Our next step is to obtain neighbourhoods of S_0 of a special type. Merely for the purpose of this construction, a subset of $P \times \mathsf{E}^1$ of the form $X \times \langle -\alpha, \alpha \rangle$ with $X \subset P$, $\alpha > 0$ will be termed *cartesian*; it is compact iff X is compact.

From definition 2, τ is defined on $S_0 \times \langle -2\lambda_0, 2\lambda_0 \rangle$, so that it is also defined on a cartesian neighbourhood of $S_0 \times \langle \lambda_0, \lambda_0 \rangle$. Hence φ is defined and continuous on a neighbourhood of S_0; therefore $\varphi(x\tau\theta)$ (i.e., the composition of φ with τ) is defined and continuous on a cartesian neighbourhood of $S_0 \times \{0\}$. Then

$$\varphi(x\tau\theta) = \int_{\theta-\lambda_0}^{\theta+\lambda_0} \psi(x\tau\vartheta)\, d\vartheta\,, \qquad \frac{\partial}{\partial\theta}\varphi(x\tau\theta) = \psi(x\tau\theta + \lambda_0) - \psi(x\tau\theta - \lambda_0)\,,$$

so that $(\partial/\partial\theta)\,\varphi(x\tau\theta)$ is also defined and continuous on a cartesian neighbourhood of $S_0 \times \{0\}$. Furthermore, by construction of ψ,

$$\frac{\partial}{\partial\theta}\varphi(x\tau\theta) = 2\lambda_0 \quad \text{for} \quad (x, \theta) \in S_0 \times \{0\}\,;$$

by continuity, then,

$$\frac{\partial}{\partial\theta}\varphi(x\tau\theta) > 0 \quad \text{for } (x, \theta) \in U_1 \times \langle -2\lambda, 2\lambda \rangle\,, \tag{2}$$

some cartesian neighbourhood of $S_0 \times \{0\}$ (this λ will be important later).

In particular, $\varphi(x\tau\lambda) > \varphi(x) = 0 > \varphi(x\tau - \lambda)$ for $x \in S_0$. Hence one may take a neighbourhood U_2 of S_0 with the property that

$$\varphi(x\tau\lambda) > 0 > \varphi(x\tau - \lambda) \quad \text{for} \quad x \in U_2\,. \tag{3}$$

Now take any neighbourhood U of S_0 with $\bar{U} \subset U_1 \cap U_2$ (particular choices of this U will, subsequently, determine various properties of the section to be constructed).

The final step in the construction is to set

$$S = \{x : \varphi(x) = 0\} \cap (\bar{U}\tau\langle -\lambda, \lambda \rangle)\,, \quad F = S\tau\langle -\lambda, \lambda \rangle\,.$$

Lemma 4. *Both S, F are closed, and*

$$S_0 \subset S \subset F\,, \quad S_0 \subset \operatorname{Int} U \subset \bar{U} \subset F\,.$$

The relations

$$x \in \bar{U}\,, \quad p(x) = x\,\tau\,\theta \in S\,, \quad |\theta| \leqq \lambda$$

define a continuous closed map p of $\bar{U}$ onto S.

For proof, see that of lemma 6 in [1].

Theorem 5. *To any compact section S_0 there exists a closed section $S \supset S_0$ which generates arbitrarily small neighbourhoods of S_0.*

For proof, see that of theorem 2 in [1].

Proposition 6. *In theorem* 5,

1° *if* P *is locally compact, then* S *may be chosen compact,*

2° *if* P *is locally connected and* S_0 *connected, then* S *may be chosen connected,*

3° *if* P *is metrisable with property* $\mathscr{S}$, *then* S *may be chosen locally connected;*

Furthermore, if P *has any combination of these properties, then* S *may be taken with the corresponding combination of properties.*

For proof, see that of theorem 2 in [1]; one only needs the additional easily established fact that a connected set in a locally connected space has small connected neighbourhoods.

Now we shall obtain consequences of the extension theorem in the case that the carrier space P is a 2-manifold. We recall a former result applying to this situation: every locally connected continuum section is either a simple arc or a simple closed curve [1, theorem 1]. It is easily established that the proof [1] again carries over bodily to our case of local dynamical systems.

Theorem 7. *Let* S_0 *be a simple arc section of a local dynamical system on a 2-manifold. Then there exists a second simple arc section* $S \supset S_0$ *such that neither end-point of* S_0 *is an end-point of* S.

Proof. First use proposition 6 to obtain a compact connected locally connected section $S \supset S_0$, of length say λ, which generates a neighbourhood F of S_0. Since S_0 contains at least two points, so does S; thus S is a locally connected continuum, and [1, theorem 1] applies.

Therefore there is a homeomorphism $q : Q \approx S$ (a "parametrisation" of S) where Q is either the interval $\langle 0, 1 \rangle$ in E^1 or the unit circle in E^2 (according as S is or not an arc).

Now, S is a section of length λ; it is then easily verified that the map h,

$$h(\theta, \sigma) = q(\sigma) \mathrm{T} \theta, \quad (\theta, \sigma) \in \langle -\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda \rangle \times Q,$$

is $1 - 1$. Obviously h is continuous, and maps its compact domain onto F. Thus h is a homeomorphism, in fact an extension of q. The set F is a neighbourhood of S_0, and hence neither end-point of S_0 can be an end-point of S – this is quite obvious in the image set under h^{-1}.

Finally, if S is a closed curve, then omission of a suitable open subarc of $S - S_0$ results in a simple arc section as required. This completes the proof.

An interesting detail may be noticed in proposition 6 – that, under certain conditions, one obtains a locally connected S even though local connectedness was not assumed of S_0. We shall now exploit this to eliminate the local connectivity assumption of [1, theorem 1]:

Theorem 8. *Given, a local dynamical system on a 2-manifold P. Then every continuum section is locally connected and thus is a simple arc or a simple closed curve.*

Proof. Let S_0 be a continuum and a section. Apply proposition 6, obtaining a locally connected continuum section $S \supset S_0$. From [1, theorem 1], S is a simple arc or simple closed curve; in either case, S is hereditarily locally connected, so that $S_0 \subset S$ is locally connected.

Our method of proof of this latter result was rather roundabout, using theorem 1 of [1] (and hence dendrite theory) as an intermediate step. A more direct proof would be most satisfactory.

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Резюме

ПРОДОЛЖЕНИЕ СЕЧЕНИЙ В ЛОКАЛЬНЫХ ДИНАМИЧЕСКИХ СИСТЕМАХ

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Главные результаты: Пусть S_0 – компактное сечение лок. дин. системы в тихоновском пространстве P; тогда существует сечение $S \supset S_0$, которое порождает окрестность сечения S_0. (Классическая теорема Витней-Бебутова соответствует случаю, когда S_0 – единственная некритическая точка.) Если, далее, P лок. компактное и лок. связное, и S_0 связное, то существует континуум S. Если P метризуемо и обладает свойством $\mathscr{S}$, то существует лок. связное S (теоремы 5 и 6).

Другие результаты относятся к случаю, кода P – многообразие размерности 2. Всякое сечение – континуум является простой дугой или простой замкнутой кривой (обобщение теоремы 1 из [1]). Пусть S_0 – простая дуга и сечение; тогда существует $S \supset S_0$, являющееся простой дугой и сечением таким, что концевые точки S_0 не являются концевыми для S.