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THE SMALLEST GRAPH WHOSE GROUP IS CYCLIC[1])

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Frucht was the first to solve the problem posed by KÖNIG [3]: which abstract finite groups F have the property that there exists a graph G such that the automorphism group of G is isomorphic with F? The answer may be said to be affirmative since FRUCHT [1] showed (by constructive methods) that every group F has this property. Frucht later showed [2] that given F, there is a cubic graph G (in which every point has degree 3) whose group $\Gamma(G) \cong F$ (is isomorphic with F). In particular

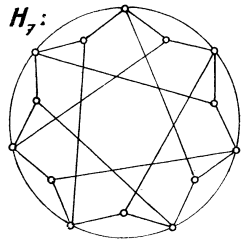

Fig. 1. Sabidussi's graph for C_7.

he proved that for the cyclic group C_n of order $n > 3$ there is a graph G_n with $p = 3n$ points and $q = (n^2 + 7n)/2$ lines such that $\Gamma(G_n) \cong C_n$. This suggests the problem of determining the smallest graph (with fewest points or lines) with given automorphism group, and as a special case, whose group is cyclic of order $n \geqq 3$.

For $n = 3$, Frucht [2] constructed a graph of order 10 whose group is isomorphic with C_3. For n a prime power $\geqq 7$, SABIDUSSI [4] showed that the minimum number of points in any graph with group C_n is $2n$, and he constructed graphs H_n for C_n with $2n$ points. In the three smallest cases, $n = 3, 4$, and 5, Sabidussi exhibited graphs H_n with group C_n such that each H_n has $3n$ points and $6n$ lines, a larger number of lines than in the graphs constructed by Frucht.

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The following three graphs M_n, $n = 3, 4, 5$ have the minimum number of lines as well as the other properties (group C_n and minimum order $3n$). It can be verified by exhaustion that there are exactly two other graphs for C_3 with $p = 9$ and $q = 15$. For $n > 5$, n a prime power, the corresponding graphs M_n have more points and lines than Sabidussi's graphs H_n. The determination of all connected graphs with group C_n, $n \geqq 4$, having minimum number of points or lines remains an unsolved problem.

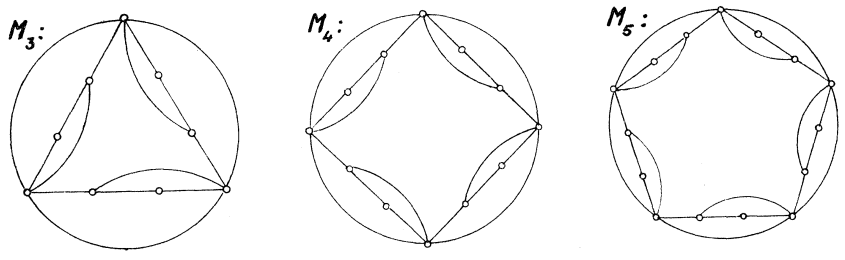

Fig. 2.

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Резюме

НАИМЕНЬШИЙ ГРАФ С ЦИКЛИЧЕСКОЙ ГРУППОЙ

Ф. ХАРАРЫ (F. Harary), Э. ПАЛМЕР (E. Palmer), Мичиган

Авторы предлагают примеры трех графов, для которых группы автоморфизмов циклические третьего, четвертого соотв. пятого порядка и числа ребер минимальны.