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Czechoslovak Mathematical Journal, Vol. 16 (1966), No. 1, 72–75

Persistent URL: `http://dml.cz/dmlcz/100712`

ON HOMOMORPHISMS OF COMMUTATIVE INVERSE SEMIGROUPS[1])

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(Received January 19, 1965)

If S and T are semigroups, Hom (S, T) denotes the semigroup of all homomorphisms from S into T with respect to pointwise multiplication. The product of α and β in Hom (S, T) will always be denoted by $\alpha \,.\, \beta$, and function composition will be denoted by juxtaposition. A semigroup S is said to be an inverse semigroup if for each $x \in S$ there is a unique $x^{-1} \in S$ such that $xx^{-1}x = x^{-1}xx^{-1} = x$. For each inverse semigroup S, E_S denotes the maximal idempotent subsemigroup of S. If $e \in E_S$, then S_e denotes the maximal subgroup of S containing e.

The main result of this paper is the determination of Hom (S, T) in terms of the groups Hom (S_e, T_f) ($e \in E_S$ and $f \in E_T$) for commutative inverse semigroups S and T. In particular, we determine the character semigroup of a commutative inverse semigroup S in terms of the character groups of the groups S_e ($e \in E_S$). The latter result was obtained for finite S by SCHWARZ [2] and by WARNE and WILLIAMS [3] for inverse S whose idempotents satisfy the minimal condition.

Henceforth, S and T denote commutative inverse semigroups. Let α denote the homomorphism from S onto E_S defined by $x \to x^{-1}x$. Similarly, define β from T onto E_T. For each λ in Hom (E_S, E_T), let G_λ denote the set of all φ in Hom (S, T) such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & T \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ E_S & \xrightarrow{\lambda} & E_T \end{array}$$

is commutative.

[1]) The author is indebted to Professor PAUL D. HILL for his direction and advice in the preparation of this paper.

Lemma 1. *For each λ in* Hom (E_S, E_T), G_λ *is a subgroup of* Hom (S, T). *Furthermore,* Hom (S, T) *is the union of the collection of groups G_λ over the semilattice* Hom (E_S, E_T).

Proof. Suppose $\lambda \in$ Hom (E_S, E_T). If φ_1 and φ_2 are in G_λ, then $\beta(\varphi_1 \, . \, \varphi_2) = (\beta\varphi_1) \, . \, (\beta\varphi_2) = (\lambda\alpha) \, . \, (\lambda\alpha) = \lambda\alpha$ and $\varphi_1 \, . \, \varphi_2 \in G_\lambda$. The homomorphism $\lambda\alpha$ is in G_λ and is an identity for G_λ. If $\varphi \in G_\lambda$, then the group inverse of φ is the homomorphism defined by $x \to \varphi(x)^{-1}$. Thus G_λ is a group for each λ in Hom (E_S, E_T). Since the collection of groups $\{G_\lambda\}$ partitions Hom (S, T) regularly, the lemma follows.

If $e \in E_S$ and $f \in E_T$, let τ_e and τ_f be the translations of S and T defined by $x \to xe$ and $c \to xf$, respectively. For each λ in Hom (E_S, E_T), define H_λ to be the subgroup of $\prod_{e \in Es}$ Hom $(S_e, T_{\lambda(e)})$ consisting of those members $\varrho = \{\varrho_e\}$ of $\prod_{e \in Es}$ Hom $(S_e, T_{\lambda(e)})$ such that the diagram

$$\begin{array}{ccc} S_e & \xrightarrow{\varrho_e} & T_{\lambda(e)} \\ \downarrow{\scriptstyle \tau_e} & & \downarrow{\scriptstyle \tau_{\lambda(f)}} \\ S_f & \xrightarrow{\varrho_f} & T_{\lambda(f)} \end{array}$$

is commutative for all $e, f \in E_S$ such that $f \leqq e$ ($f \leqq e$ if and only if $ef = f$).

Lemma 2. *For each λ in* Hom (E_S, E_T), H_λ *is isomorphic to* G_λ.

Proof. Define a function F from G_λ into H_λ by $F(\varphi) = \{\varrho_e\}$ where $\varrho_e = \varphi \mid S_e$. It is easy to verify that F is an isomorphism into H_λ. We show that it is onto. Suppose that $\{\varrho_e\}$ is in H_λ. Let φ denote the function from S into T defined by $\varphi(x) = \varrho_e(x)$ if $x \in S_e$. Now if $x \in S_e$ and $y \in S_f$ for $e, f \in E_S$, then

$$\varphi(xy) = \varrho_{ef}(xy) = \varrho_{ef}(xef)\,\varrho_{ef}(yef) =$$
$$= \lambda(ef)\,\varrho_e(x)\,\lambda(ef)\,\varrho_f(y) = \varrho_e(x)\,\varrho_f(y) = \varphi(x)\,\varphi(y)\,.$$

Thus φ is in Hom (S, T). Moreover, $(\lambda\alpha)(x) = \lambda(e) = (\beta\varrho_e)(x) = (\beta\varphi)(x)$ if $x \in S_e$, so $\varphi \in G_\lambda$. Hence $F(\varphi) = \{\varphi \mid S_e\} = \{\varrho_e\}$ and F is onto.

Let $\mathcal{O}$ denote the set of all ordered pairs (e, f) of E_S such that $f \leqq e$. Define a relation $\precsim$ on $\mathcal{O}$ by $(e, f) \precsim (e', f')$ if and only if $e' \leqq e$ and $f \leqq f'$. The relation $\precsim$ is a partial order on $\mathcal{O}$ (but, in general, is not a direction). For λ in Hom (E_S, E_T) and $\alpha \precsim \beta$ in $\mathcal{O}$, define a function $\varphi_\alpha^\beta(\lambda)$ in the following way. If $\alpha = (e, f)$, $\beta = (e', f')$, and ψ is in Hom $(S_{e'}, T_{\lambda(f')})$, then $\varphi_\alpha^\beta(\lambda)(\psi) = (\tau_{\lambda(f)}\,\psi\tau_{e'}) \mid S_e$. The function $\varphi_\alpha^\beta(\lambda)$ is a homomorphism from Hom $(S_{e'}, T_{\lambda(f')})$ into Hom $(S_e, T_{\lambda(f)})$. We abbreviate $\varphi_\alpha^\beta(\lambda)$ to φ_α^β since it is always clear from the context which λ is associated with a given φ_α^β. Note that if $\alpha \precsim \beta \precsim \gamma$ in $\mathcal{O}$ then $\varphi_\alpha^\beta \varphi_\beta^\gamma = \varphi_\beta^\gamma$ and φ_α^α is the identity on its domain.

Theorem 1. *If* λ *is in* $\mathrm{Hom}(E_S, E_T)$, *then*

$$G_\lambda \cong \mathrm{invlim}\,[\{\mathrm{Hom}(S_e, T_{\lambda(f)})\}\,(e,f) \in \mathcal{O};\ \{\varphi_\alpha^\beta\}] \tag{1}$$

and

$$\mathrm{Hom}(S, T) \cong \bigcup_{\lambda \in \mathrm{Hom}(E_S, E_T)} \mathrm{invlim}\,[\{\mathrm{Hom}(S_e, T_{\lambda(f)})\}\,(e,f) \in \mathcal{O};\ \{\varphi_\alpha^\beta\}] \tag{2}$$

Proof. Let the group on the right hand side of 1) be denoted by L_λ. By Lemma 1 and Lemma 2, it suffices to show that $H_\lambda \cong L_\lambda$ for each λ in $\mathrm{Hom}(E_S, E_T)$. Let F denote the function from H_λ into $\prod_{(e,f)\in\mathcal{O}} \mathrm{Hom}(S_e, T_{\lambda(f)})$ defined by the condition: if $\varrho = \{\varrho_e\} \in H_\lambda$, then $F(\varrho) = \theta = \{\theta_\alpha\}_{\alpha\in\mathcal{O}}$ where $\theta_\alpha = \tau_{\lambda(f)}\varrho_e$ for each $\alpha = (e,f)$ in $\mathcal{O}$. Note that $\theta_{(e,e)} = \varrho_e$ if $e \in E_S$, and, therefore, F is one-one. It is immediate that F is a homomorphism. Now we show that F maps into L_λ. Suppose $\varrho \in H_\lambda$, $\theta = F(\varrho)$, and $\alpha \lesssim \beta$ where $\alpha = (e,f) \in \mathcal{O}$ and $\beta = (e', f') \in \mathcal{O}$. Then

$$\varphi_\alpha^\beta(\theta_\beta) = (\tau_{\lambda(f)}\tau_{\lambda(f')}\varrho_{e'}\tau_{e'}) \mid S_e =$$
$$= (\tau_{\lambda(f)}\varrho_{f'}\tau_{f'}\tau_{e'}) \mid S_e = \tau_{\lambda(f)}\tau_{\lambda(f')}\varrho_e = \theta_\alpha\,,$$

and $\theta \in L_\lambda$. In order to show that F is onto L_λ, suppose that $\theta \in L_\lambda$. Define $\varrho = \{\varrho\}_{e\in E_S}$ where $\varrho_e = \theta_{(e,e)}$ for all $e \in E_S$. We show that $\varrho \in H_\lambda$ and that $F(\varrho) = \theta$. Suppose $f \leqq e$ in E_S, $\alpha = (e, e)$, $\beta = (f, f)$, and $\gamma = (e, f)$. Then $\gamma \lesssim \alpha$, $\gamma \lesssim \beta$, and

$$\tau_{\lambda(f)}\varrho_e = \tau_{\lambda(f)}\theta_\alpha = \varphi_\gamma^\alpha(\theta_\alpha) = \theta_\gamma = \varphi_\gamma^\beta(\theta_\beta) = \theta_\beta\tau_f \mid S_e = \varrho_f\tau_f \mid S_e\,.$$

From the above equations, one has that $\varrho \in \mathrm{H}_\lambda$. It follows from similar calculations that $F(\varrho) = \theta$, which completes the proof of the theorem.

A character of S is a homomorphism χ from S into the multiplicative semigroup C of complex numbers such that $\chi(1) \neq 0$ if 1 is an identity of S. We let S^* denote the semigroup of characters of S with respect to pointwise multiplication. If S is a group, then S^* is the usual character group of S.

If $\lambda \in E_S^*$, the set of $e \in E_S$ such that $\lambda(e) \neq 0$ forms a directed set with respect to the order $\geqq$ on E_S. For each pair (e,f) of elements from this directed set such that $f \leqq e$, define π_e^f to be the adjoint of the translation from S_e into S_f, that is, $\pi_e^f(\varphi) = (\varphi\tau_f) \mid S_e$ for each φ in S_f^*. It follows that $[\{S_e^*\}_{\lambda(e)\neq 0};\ \{\pi_e^f\}]$ is an inverse system of groups.

Theorem 2. $S^* \cong \bigcup_{\lambda\in E_S^*} \mathrm{invlim}\,[(\{S_e^*\}_{\lambda(e)\neq 0};\ \{\pi_e^f\}]$ *provided that the inverse limit of a void collection of groups is defined to be the zero group.*

Proof. If in Theorem 1 we let $T = C$, we have only to show that

$$L_\lambda = \mathrm{invlim}\,[\{\mathrm{Hom}(S_e, T_{\lambda(f)})\}_{(e,f)\in\mathcal{O}};\ \{\varphi_\alpha^\beta\}]$$

is isomorphic to invlim $[\{S_e^*\}_{\lambda(e)\neq 0}; \{\pi_e^f\}]$ for each λ in E_S^* such that λ is not identically zero. The function F from L_λ onto invlim $[\{S_e^*\}_{\lambda(e)\neq 0}; \{\pi_e^f\}]$ defined by $F(\{\theta_\alpha\}) = \{\varrho_e\}$ where $\varrho_e = \theta_{(e,e)}$ for each $e \in E_S$ with $\lambda(e) \neq 0$ is an isomorphism. The details of the proof that F is an isomorphism are similar to those in the proof of Theorem 1.

The following corollary of Theorem 2 is essentially Theorem 5.63 in [1].

Corollary. *If E_S satisfies the minimal condition, then $S^* \cong \bigcup_{\lambda \in E_S} S^*_{e(\lambda)}$ where $e(\lambda)$ is the minimal e such that $\lambda(e) \neq 0$.*

Proof. Since the set $\{e \in E_S \mid \lambda(e) \neq 0\}$ has a minimal element $e(\lambda)$,

$$\text{invlim}\,[\{S_e^*\}_{\lambda(e)\neq 0}; \{\pi_e^f\}] = S^*_{e(\lambda)}.$$

More precisely, the map $\{\varrho_e\} \to \varrho_{e(\lambda)}$ is an isomorphism from invlim $[\{S_e^*\}_{\lambda(e)\neq 0}; \{\pi_e^f\}]$ onto $S^*_{e(\lambda)}$.

References

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Резюме

ГОМОМОРФИЗМЫ КОММУТАТИВНЫХ ИНВЕРСНЫХ ПОЛУГРУПП

РОНАЛЬД О. ФУЛП Ronald O. Fulp), Атланта

Пусть S, T — коммутативные инверсные полугруппы, E_S, E_T — подполугруппы идемпотентов. Если $e \in E_S$, $f \in E_T$, то пусть S_e, S_f — максимальные группы, принадлежащие идемпотентам e, f.

Целью статьи является изучение строения $\mathrm{Hom}(S, T)$ с помощью групп $\mathrm{Hom}(S_e, S_f)$.

Как следствие получаются некоторые результаты, касающиеся характеров инверсных полугрупп.