

Miloš Dostál

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ON FOURIER IMAGE OF THE SINGULAR SUPPORT
OF A DISTRIBUTION

MILOŠ DOSTÁL, Praha

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0° Let $\Phi \in \mathcal{D}'(R^n)$ be a distribution with compact support in R^n . If $p \in R^n$ we say that $\Phi \in C^\infty(p)$ if there exists a neighbourhood Ω of p and $f(x) \in C^\infty(R^n)$ such that for every $\varphi \in \mathcal{D}(\Omega)$

$$\Phi(\varphi) = \int_{R^n} f(x) \varphi(x) dx.$$

Then we put

$$\text{sing supp } \Phi = \text{supp } \Phi \setminus \{p : \Phi \in C^\infty(p)\},$$

where $\text{supp } \Phi$ means the support of Φ . In the present article we will give a precise description of the convex hull of $\text{sing supp } \Phi$ by means of the Fourier transform $\hat{\Phi}$ of Φ .

First of all we remember an analogous situation with the set $\text{supp } \Phi$. The well known Paley-Wiener theorem in Schwartz's modification says — roughly spoken — that the radius of the least sphere with center at 0 containing $\text{supp } \Phi$ is equal to the type of the entire function $\hat{\Phi}$. A more precise theorem has been proved by PLANCHEREL and PÓLYA ([1]). They introduced a notion of P -indicator of an entire function and proved that for $\Phi \in L^2(R^n)$ the corresponding P -indicator of $\hat{\Phi}$ is equal to the supporting function of the set $H(\text{supp } \Phi)$, where H means the convex hull. On the other side a theorem of Paley-Wiener's type is valid for the singular supports ([2]). We will find a corresponding variant of the Plancherel-Pólya theorem for the case of singular support i.e. we shall define a kind of indicator of $\hat{\Phi}$ and using it we shall completely describe the set $H(\text{sing supp } \Phi)$. After some notations we begin with a generalization of Paley-Wiener theorem in the Plancherel-Pólya direction (theorem 1) which will be necessary later (comp. [3], p. 130, Remarque 4°).

1° Let R^n ev. C^n be the real ev. complex n -dimensional space. We write the elements of C^n in the form $\zeta = \xi + i\eta = (\xi_1 + i\eta_1, \dots, \xi_n + i\eta_n)$, $\xi \in R^n$, $\eta \in R^n$. Let C_α denotes an arbitrary direction in R^n , i.e. $C_\alpha = (\cos \alpha_1, \dots, \cos \alpha_n)$ and $\sum \cos^2 \alpha_i = 1$. If

$H(M)$ is a closed convex hull of some $M \subset R^n$, denote by $\mathcal{K}_M(\alpha)$ the supporting function of the set $H(M)$ so that

$$\mathcal{K}_M(\alpha) = \sup_{x \in M} \langle x, C_\alpha \rangle$$

where $\langle x, C_\alpha \rangle = \sum x_k \cos \alpha_k$. For $\Phi \in \mathcal{E}'(R^n)$ we write $\mathcal{K}_\Phi(\alpha)$ instead of $\mathcal{K}_{\text{supp}\Phi}(\alpha)$. Conversely if $k(\alpha)$ is a function defined on 1-sphere such that for some $M \subset R^n$ we have $k(\alpha) \equiv \mathcal{K}_M(\alpha)$, we call $k(\alpha)$ to be a t.c. function (trigonometrically convex). If $\Phi \in \mathcal{E}'(R^n)$ the Fourier transform $\hat{\Phi}$ can be defined by $\hat{\Phi}(\zeta) = \Phi_x(e^{-i\langle x, \zeta \rangle})$.

Theorem 1. a) Let $\Phi \in \mathcal{E}'(R^n)$ and $k(\alpha)$ be a t.c. function for which $k(\alpha) > \mathcal{K}_\Phi(\alpha)$ for every α . Then the following assertion holds:

(\mathcal{PW}) There exist constants $C > 0$, N (N integer depending only on Φ is the order of Φ and C depends on Φ and k) so that for every $\alpha, r > 0$ and $\xi \in R^n$

$$(1) \quad |\hat{\Phi}(\xi + iC_\alpha r)| \leq C(1 + |\xi|)^N e^{rk(\alpha)}$$

Conversely: Let F be an entire function and $k(\alpha)$ a bounded t. c. function such that for suitable C and N the condition (\mathcal{PW}) is satisfied, then $F = \hat{\Phi}$ for some $\Phi \in \mathcal{E}'(R^n)$ and $\mathcal{K}_\Phi(\alpha) \leq k(\alpha)$.

b) Let $\Phi \in \mathcal{D}(R^n)$ and let $k(\alpha)$ be a t.c. function for which $k(\alpha) \geq \mathcal{K}_\Phi(\alpha)$ then

(\mathcal{PW}^∞) For every integer N there exists $C_N > 0$ depending only on N and Φ such that for every $\alpha, r > 0$, $\xi \in R^n$

$$(2) \quad |\hat{\Phi}(\xi + iC_\alpha r)| \leq C_N(1 + |\xi|)^{-N} e^{rk(\alpha)}$$

and conversely as above.

The proof is a slight modification of Hörmander's proof of Paley-Wiener theorem ([2]) but we shall reproduce it only with the aim for comparing his main idea – translation of the integration domain from R_ξ^n to C^n – with a similar one in the case of singular supports, where the integration domain R_ξ^n has to be deformed in a more complicated way.

First we shall prove the necessity of (\mathcal{PW}^∞). If $\Phi \in \mathcal{E}'(R^n)$ then there exist constants C_1, N such that for every $\varphi \in \mathcal{D}(R^n)$

$$(3) \quad |\Phi(\varphi)| \leq C_1 \sum_{|\kappa| \leq N} \sup_x |D^\kappa \varphi(x)|$$

Furthermore for every $\chi \in \mathcal{E}'(R^n)$ such that $\chi \equiv 1$ in some neighbourhood of $\text{supp } \Phi$ we have

$$(4) \quad \hat{\Phi}(\zeta) = \Phi(e^{-i\langle x, \zeta \rangle} \chi(x))$$

Suppose C_α to be fixed. Put $\chi(x) = \psi(|\zeta| (\langle x, C_\alpha \rangle - \mathcal{K}_\Phi(\alpha)))$, where $\psi \in C^\infty(\mathbb{R}^n)$, $\psi(t) = 1$ for $t \leq 2^{-1}$ and $\psi(t) = 0$ for $t \geq 1$. Therefore we have

$$(5) \quad |\hat{\Phi}(\zeta)| \leq C_1 \sum_{|x| \leq N} \sup_x |D_x^k(e^{-i\langle x, \zeta \rangle} \chi(x))|$$

It is easy to obtain from (5) the estimate

$$(6) \quad |\hat{\Phi}(\zeta)| \leq C_2(1 + |\zeta|)^N e^{r\mathcal{K}_\Phi(\alpha)}$$

where C_2 is independent on α , and combining (6) with the obvious inequality $(1 + |\zeta|)^N \leq C_3(\varepsilon)(1 + |\xi|)^N e^{r\varepsilon}$, where $0 < \varepsilon < \inf(k(\alpha) - \mathcal{K}_\Phi(\alpha))$ we obtain (1).

Taking $\Phi \in \mathcal{D}(\mathbb{R}^n)$ and applying (1) with $N = 0$ on $(1 + \Delta)^M \Phi$ we obtain (\mathcal{PW}^∞) . Conversely if for some bounded t.c. function $k(\alpha)$ and an entire function $F \in (\mathcal{PW}^\infty)$ is valid, then $\Phi(x) = (2\pi)^{-n} \int F(\xi) e^{i\langle x, \xi \rangle} d\xi$ lies in \mathcal{S} (using the usual Paley-Wiener theorem and the boundedness of k , we see that $\Phi \in \mathcal{D}(\mathbb{R}^n)$). Take an arbitrary C_α and $r > 0$. Then for $\eta = C_\alpha r$ we have by the Cauchy-Poincaré formula

$$(7) \quad \Phi(x) = (2\pi)^{-n} \int F(\xi + i\eta) e^{i\langle x, \xi + i\eta \rangle} d\xi$$

so that for a great N we obtain from (2) and (7)

$$(8) \quad |\Phi(x)| \leq (2\pi)^{-n} C_N e^{r(k(\alpha) - \langle x, C_\alpha \rangle)} \int_{\mathbb{R}^n} (1 + |\xi|)^{-N} d\xi$$

and letting $r \rightarrow \infty$ we see that in the case of $x \in \text{supp } \Phi$ necessarily $k(\alpha) \geq \langle x, C_\alpha \rangle$ i.e. $\mathcal{K}_\Phi(\alpha) \leq k(\alpha)$.

By means of a regularization it is easy to obtain the sufficiency of (\mathcal{PW}) .

Now we recall briefly the notion of P -indicator:

Definition. Let F be an entire function. Then for any direction C^α in \mathbb{R}_η^n we put

$$(9) \quad \mathcal{H}_F(\alpha) = \sup_{\xi \in \mathbb{R}^n} \overline{\lim}_{r \rightarrow \infty} r^{-1} \ln |F(\xi + iC_\alpha r)|$$

and call this function P -indicator of F .

Theorem 2. For $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ holds: $\mathcal{K}_\Phi(\alpha) \equiv \mathcal{H}_{\hat{\Phi}}(\alpha)$.

This is an easy consequence of the classical Plancherel-Pólya theorem and Theorem 1. Indeed, Theorem 1. gives $\mathcal{H}_{\hat{\Phi}}(\alpha) \leq \mathcal{K}_\Phi(\alpha)$ and for a regularization function $\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp } \varphi_\varepsilon = \{x : |x| \leq \varepsilon\}$ we have $\mathcal{K}_\Phi + \varepsilon = \mathcal{K}_{\Phi_\varepsilon} = \mathcal{H}_{\hat{\Phi}_\varepsilon} \leq \mathcal{H}_{\hat{\Phi}} + \mathcal{H}_{\hat{\Phi}_\varepsilon} = \mathcal{H}_{\hat{\Phi}} + \varepsilon$ where $\hat{\Phi}_\varepsilon = \hat{\Phi} * \varphi_\varepsilon$ and Plancherel-Pólya theorem, the theorem on supports and subadditivity of P -indicator were used.

Corollary. For every $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ there exists an integer N (= order of Φ) so that for every $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that for all $\alpha, r > 0, \xi \in \mathbb{R}^n$ we have

$$(10) \quad |\widehat{\Phi}(\xi + iC_\alpha r)| \leq C_\varepsilon (1 + |\xi|)^N e^{r(\mathcal{H}_\Phi(\alpha) + \varepsilon)}$$

Let us remark that this corollary cannot be obtained only from Theorem 1 and that on the other side it says more than Theorem 2.

2° Now take a function $\mathcal{H}_F(\alpha)$ in the case $n = 1$. Then there are obviously only two directions $C_\alpha = \pm 1$ in \mathbb{R}_η^1 corresponding to $\alpha = 0, \pi$ respectively. It is not hard to show ([1]) that then for every $\xi \in \mathbb{R}_\xi^1$ we have $\mathcal{H}_F(0) = \lim_{r \rightarrow \infty} r^{-1} \ln |F(\xi + ir)|$ and so $\mathcal{H}_F(0) = h_F(\frac{1}{2}\pi)$ where $h_F(\gamma)$ denotes the well known Phragmén-Lindelöf indicator of F (similar in the case $C_\alpha = -1$) Using the continuity of $h_F(\gamma)$ we obtain

$$(11) \quad \mathcal{H}_F(0) = \overline{\lim}_{t \rightarrow \infty} \overline{\lim}_{\xi \rightarrow \infty} \frac{\ln |F(\xi + i\xi t)|}{\xi t}$$

It is a special case of the following general situation. Take a fixed positive function $\eta(\xi, t)$ defined on $\mathbb{R}_\xi^n \times (0, +\infty)$ which tends to infinity in every variable separately. For every $t > 0$ and α the function $\eta(\xi, t)$ determines an n -dimensional complex manifold $\Gamma_{t,\alpha}$ in $C^n = \mathbb{R}^{2n} : \Gamma_{t,\alpha} = \{\zeta \in C^n : \zeta = \xi + iC_\alpha \eta(\xi, t)\}$. Now let α be fixed. The limit $\lim_{\zeta \in \Gamma_{t,\alpha}, |\xi| \rightarrow \infty} (\ln |F(\zeta)| / (|\eta(\xi, t)|))$ describes the growth of F on $\Gamma_{t,\alpha}$ at infinity and so the double limit $\overline{\lim}_{t \rightarrow \infty} \overline{\lim}_{\xi \rightarrow \infty} (\ln |F(\xi + iC_\alpha \eta(\xi, t))| / (|\eta(\xi, t)|)) = \mathfrak{R}_F(\alpha)$ gives us a simultaneous estimate of behaviour of F at infinity on the one parametrical system Γ_t in the direction C_α . It is to be expected that for some suitable choice of function $\eta(\xi, t)$ determining the system Γ_t we obtain an indicator \mathfrak{R}_F which will indicate a certain property of F .

We put now $\eta(\xi, t) = t \log(1 + |\xi|)$ and the corresponding function $\mathfrak{R}_F(\alpha)$ denote $\mathcal{H}_F^s(\alpha)$ and call it the singular indicator of F . We shall consider \mathcal{H}_F^s only for $F = \widehat{\Phi}, \Phi \in \mathcal{E}'(\mathbb{R}^n)$. Further put $\mathcal{H}_\Phi^s(\alpha) = \mathcal{H}_{\text{singsupp}\Phi}^s(\alpha)$

Theorem 3. Let $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ and $k(\alpha)$ be a t.c. function so that $k(\alpha) > \mathcal{H}_\Phi^s(\alpha)$. Then

($\mathcal{P}\mathcal{P}$) There exist an integer N and a positive function $C(t)$ such that for every $\alpha, t > 0, \xi \in \mathbb{R}^n$ we have

$$(12) \quad |\widehat{\Phi}(\xi + iC_\alpha \eta(\xi, t))| \leq C(t) (1 + |\xi|)^N e^{k(\alpha)\eta(\xi, t)}$$

Conversely if for a $\widehat{\Phi} \in \widehat{\mathcal{E}}'(\mathbb{R}^n)$ the condition ($\mathcal{P}\mathcal{P}$) holds with some t.c. function k , then $\mathcal{H}_\Phi^s(\alpha) \leq k(\alpha)$ for all α .

We shall prove this theorem together with the following.

Theorem 4. For every $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ we have $\mathcal{H}_{\Phi}^s(\alpha) \equiv \mathcal{H}_{\Phi}^s \Phi(\alpha)$.

Proof: The main idea is due to L. Ehrenpreis (see [4]). First of all we prove the necessity of $(\mathcal{P}\mathcal{P})$. For $0 < \varepsilon < \inf_{\alpha} (k(\alpha) - \mathcal{H}_{\Phi}^s(\alpha))$ we can find by means of a suitable partition of unity the distributions $\Phi_i \in \mathcal{E}'(\mathbb{R}^n)$ ($i = 1, 2$) such that

$$(13) \quad \Phi = \Phi_1 + \Phi_2, \quad \Phi_2 \in \mathcal{D}(\mathbb{R}^n), \quad \mathcal{H}_{\Phi_1}(\alpha) = \mathcal{H}_{\Phi}^s(\alpha) + \varepsilon/2$$

From Theorem 1 we obtain for all $\alpha, r > 0, \xi \in \mathbb{R}^n$

$$(14) \quad |\hat{\Phi}_1(\xi + iC_x r)| \leq C(1 + |\xi|)^N e^{rk(\alpha)}$$

and further we obtain that there exists a constant $R > 0$ such that for every integer M we have for some C_M

$$(15) \quad |\hat{\Phi}_2(\xi)| \leq C_M(1 + |\xi|)^{-M} e^{Rr} \quad (r = |\operatorname{Im} \xi|)$$

for all $\xi \in \mathbb{C}^n$. Take a fixed t and α and chose $M \geq (R - R_0)t - N$, where $R_0 = \inf_{\alpha} k(\alpha)$ (evidently $R_0 > -\infty$). Then for $\eta = \eta(\xi, t)$ we have

$$(16) \quad |\hat{\Phi}_2(\xi)| \leq \tilde{C}_t(1 + |\xi|)^N e^{k(\alpha)\eta(\xi, t)}$$

and (16) together with (14) gives (12). From (12) we obtain immediately

$$(17) \quad \mathcal{H}_{\Phi}^s(\alpha) \leq \mathcal{H}_{\Phi}^s(\alpha)$$

Now let us prove the inverse inequality. It is sufficient to prove the following assertion: for every $\varepsilon > 0$ we have: if α is an arbitrary but fixed, then for every $x \in \mathbb{R}^n$ such that $\langle x, C_x \rangle > \mathcal{H}_{\Phi}^s(\alpha) + \varepsilon$ holds $\Phi \in C^\infty(x)$, that is $\mathcal{H}_{\Phi}^s(\alpha) \leq \mathcal{H}_{\Phi}^s(\alpha) + \varepsilon$ for every $\varepsilon > 0, \alpha$. Now take an arbitrary integer $j > 0$ and put $T_\delta = \{x : \langle x, C_x \rangle > \mathcal{H}_{\Phi}^s(\alpha) + \varepsilon + \delta\}$ for $\delta > 0$. It is sufficient to prove that $\Phi \in C^j(T_\delta)$ for every $\delta > 0$. Take an arbitrary $\delta > 0$. From the definition of \mathcal{H}_{Φ}^s follows that there exists $t_0 > 0$ so that for $t \geq t_0$

$$(18) \quad \overline{\lim}_{|\xi| \rightarrow \infty} \frac{\log |\hat{\Phi}(\xi)|}{\eta(\xi, t)} < \mathcal{H}_{\Phi}^s(\alpha) + \varepsilon$$

Put now

$$(19) \quad t_1 = \max(t_0, (n + j + 1)\delta^{-1}).$$

Further for some $\xi_0(t_1) > 0$ we have

$$(20) \quad |\hat{\Phi}(\xi + iC_x t_1 \log(1 + |\xi|))| < \exp\{\eta(\xi, t_1)(\mathcal{H}_{\Phi}^s(\alpha) + \varepsilon)\}$$

for all $|\xi| \geq \xi_0(t_1)$ so that for some constant $C_1(t_1)$ holds

$$(21) \quad |\hat{\Phi}(\xi, t_1)| \leq C_1(t_1)(1 + |\xi|)^{t_1(\mathcal{H}_{\Phi}^s(\alpha) + \varepsilon)}$$

for all $\xi \in R_\xi^n$. We put

$$(22) \quad \chi(x) = \int_{\Gamma_{t_1, x}} e^{i\langle x, \xi \rangle} \hat{\Phi}(\xi) d\xi_1 \wedge \dots \wedge d\xi_n = \int_{R_\xi^n} e^{i\langle x, \xi \rangle} \hat{\Phi}(\xi, t_1) \frac{\partial(\xi_1, \dots, \xi_n)}{\partial(\xi_1, \dots, \xi_n)} d\xi$$

and if we take ι to be an arbitrary multiindex with the length $|\iota| \leq j$ then estimating the ι -th derivative of the last integrand absolutely and uniformly in T_δ by a summable function we shall prove that $\chi(x) \in C^j(T_\delta)$ and so $\chi(x) \in C^j(T_0)$.

Evidently we have for some $C_i(t_1)$ ($i = 2, 3$)

$$(23) \quad |D_x^\iota(e^{i\langle x, \xi \rangle})| \leq C_2(t_1) (1 + |\xi|)^{j + t_1 \langle x, C_x \rangle}; \quad x \in T_\delta, \quad \xi \in R_\xi^n$$

$$(24) \quad \left| \frac{\partial(\xi_1, \dots, \xi_n)}{\partial(\xi_1, \dots, \xi_n)} \right| \leq C_3(t_1)$$

Combining (21), (23), (24) we see that with regard to (19)

$$(25) \quad \int_{R_\xi^n} \left| D_x^\iota(e^{i\langle x, \xi \rangle}) \hat{\Phi}(\xi, t_1) \frac{\partial(\xi_1, \dots, \xi_n)}{\partial(\xi_1, \dots, \xi_n)} \right| d\xi \leq C_4(t_1) \int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|)^{n+1}}$$

so $D_x^\iota(\chi(x)) = \int_{\Gamma} D_x^\iota(e^{i\langle x, \xi \rangle}) \hat{\Phi}(\xi) d\xi$ and $\chi(x) \in C^j(T)$. Especially from our conclusions follows for every $\varphi(x) \in \mathcal{D}(T)$:

$$(26) \quad \int_{R_{x^n}} \int_{R_\xi^n} \left| e^{i\langle x, \xi \rangle} \hat{\Phi}(\xi, t_1) \frac{\partial(\xi_1, \dots, \xi_n)}{\partial(\xi_1, \dots, \xi_n)} \varphi(x) \right| dx d\xi < +\infty$$

so that we can change the order of integrations and using the formulas of Plancherel and of Cauchy-Poincaré¹⁾ we obtain finally

$$(27) \quad \begin{aligned} \Phi(\varphi) (2\pi)^{-n} &= \int_{R_\xi^n} \hat{\Phi}(\xi) \hat{\varphi}(-\xi) d\xi = \int_{\Gamma} \hat{\Phi}(\xi) \hat{\varphi}(-\xi) d\xi = \\ &= \int_{R_{x^n}} \int_{R_\xi^n} \hat{\Phi}(\xi, t_1) e^{i\langle x, \xi \rangle} \varphi(x) \frac{\partial(\xi_1, \dots, \xi_n)}{\partial(\xi_1, \dots, \xi_n)} d\xi dx = \\ &= \int_{R_{x^n}} \left(\int_{\Gamma} \hat{\Phi}(\xi, t_1) e^{i\langle x, \xi \rangle} d\xi \right) \varphi dx = \int_{R_{x^n}} \chi(x) \varphi(x) dx \end{aligned}$$

which means that $\Phi = \chi$ on T and so $\Phi \in C^j(T)$ and Theorem 4 is therefore proved. If we suppose $(\mathcal{P}\mathcal{P})$ to be valid for some $k(\alpha)$, then we have obviously $\mathcal{H}_{\hat{\Phi}}^s(\alpha) \leq k(\alpha)$ which is by the Theorem 4 the same as $\mathcal{H}_{\Phi}^s(\alpha) \leq k(\alpha)$.

¹⁾ $\hat{\varphi}$ decreases very fast, see $(\mathcal{P}\mathcal{W}^\infty)$.

Added in the proofs: Theorem 2 is proved in [5].

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Author's address: Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).

Резюме

ОБ ОБРАZE ФУРЬЕ НОСИТЕЛЯ СИНГУЛЯРНОСТЕЙ ОБОБЩЕННОЙ ФУНКЦИИ

МИЛОШ ДОСТАЛ, (Miloš Dostál), Прага

Пусть $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ является обобщенной функцией с компактным носителем. Обозначим через $\text{sing supp } \Phi$ ее носитель сингулярностей. Для целой функции $\hat{\Phi}$ (преобразование Фурье Φ) мы положим

$$\mathcal{H}_{\Phi}^s(\alpha) = \overline{\lim}_{t \rightarrow \infty} \overline{\lim}_{|\xi| \rightarrow \infty} (\ln |\hat{\Phi}(\xi + iC_{\alpha}\eta(\xi, t))| \cdot |\eta(\xi, t)|^{-1}),$$

где $\eta(\xi, t) = t \log(1 + |\xi|)$ и $C_{\alpha} = (\cos \alpha_1, \dots, \cos \alpha_n)$ — единичный вектор в n -мерном вещественном пространстве \mathbb{R}^n . Функция $\mathcal{H}_{\Phi}^s(\alpha)$ описывает рост функции $\hat{\Phi}$ в направлении C_{α} на однопараметрической системе $(\Gamma_t)_{t>0}$ многообразий $\Gamma_t = \{\xi \in \mathbb{C}^n : \xi = \xi + iC_{\alpha}\eta(\xi, t)\}$ (\mathbb{C}^n n -мерное комплексное пространство). Если мы теперь положим

$$\mathcal{H}_{\Phi}^s(\alpha) = \sup_{x \in \text{sing supp } \Phi} \langle x, C_{\alpha} \rangle,$$

то имеет место следующая теорема:

Теорема 4. Для каждого $\Phi \in \mathcal{E}'(\mathbb{R}^n)$ имеем $\mathcal{H}_{\Phi}^s(\alpha) \equiv \mathcal{H}_{\Phi}^s(\alpha)$. Теорема 3 приводит другую формулировку этого утверждения при помощи неравенств. Для доказательств этих теорем нужна теорема Пэйли Винер-Шварца, сформулированная более точным образом. Это Теорема 1, которая аналогична теореме 3. Теорема 2, аналогом которой является теорема 4, представляет обобщение теоремы Планшерель-Пойа на случай обобщенных функций.