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A NEW APPROACH TO SOME PROBLEMS IN THE THEORY OF NON-NEGATIVE MATRICES

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In the paper [11] I developed a semigroup treatment of some theorems concerning non-negative matrices. The substance of this method is the following.

Denote \( N = \{1, 2, \ldots, n\} \) and consider the set of all \( n \times n \) matrix units, i.e. the set of symbols \( \{e_{ij} \mid i \in N, j \in N\} \) together with a zero \( 0 \) adjoined. Define in \( S = \{0\} \cup \{e_{ij} \mid i \in N, j \in N\} \) a multiplication by

\[
e_{ij}e_{ml} = \begin{cases} e_{il} & \text{for } j = m, \\ 0 & \text{for } j \neq m,
\end{cases}
\]

the zero element having the usual properties of a multiplicative zero. The set \( S \) with this multiplication is a 0-simple semigroup containing \( n \) non-zero idempotents \( e_{11}, e_{22}, \ldots, e_{nn} \).

Let \( A = (a_{ij}) \) be a non-negative \( n \times n \) matrix. By the support \( C_A \) of \( A \) we shall mean the subset of \( S \) containing 0 and all \( e_{ij} \) for which \( a_{ij} > 0 \).

For any two non-negative \( n \times n \) matrices \( A, B \) we have \( C_{AB} = C_A C_B \), where the multiplication of subsets of \( S \) has the usual meaning used in the theory of semigroups.

Consider the sequence

\[ A, A^2, A^3, \ldots \]

The sequence of the corresponding supports

\[ C_A, C_A^2, C_A^3, \ldots \]

has clearly only a finite number of different members.

Let \( k = k(A) \) be the least positive integer such that \( C_A^k = C_A^{l_1} \) for some \( l_1 > k \).

Let further \( l = k + d \) \( [d = d(A) \geq 1] \) be the least positive integer for which \( C_A^k = C_A^{k+d} \) holds. Then the sequence (1) is of the form

\[ C_A, C_A^2, \ldots, C_A^{k-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid C_A^k, \ldots, C_A^{k+d-1} \mid \ldots \]

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The system of sets \( \{C^k, C^{k+1}, \ldots, C^{k+d-1}\} \) with respect to the multiplication of subsets of \( S \) forms a finite semigroup of order \( k + d - 1 \). It is well known from the elements of the theory of finite semigroups that \( \Theta_A = \{C^k, C^{k+1}, \ldots, C^{k+d-1}\} \) (with respect to the same multiplication) is a cyclic group of order \( d \). We mention by the way (though it will not be used in this paper) that the unit element of the group \( \Theta_A \) is the set \( C^d_s \), where \( q \) is the uniquely defined multiple \( q \) satisfying \( k \leq \tau d = q \leq k + d - 1 \).

In this manner we have associated to any non-negative matrix \( A \) three positive integers \( k = k(A), d = d(A), q = q(A) \).

A non-negative \( n \times n \) matrix \( A \) is called reducible if \( N \) can be decomposed in two non-empty disjoint subsets \( N = I \cup J, I \cap J = \emptyset \) such that \( a_{ij} = 0 \) for \( i \in I \) and \( j \in J \). Otherwise it is called irreducible.

In [11] we have shown: For an irreducible matrix \( A \) the number \( d = d(A) \) is simply the index of imprimitivity of \( A \) and we always have \( d \leq n \). [For a characterization of \( d(A) \) in the general case see [12].]

A matrix \( A \) is irreducible if and only if

\[ C_A \cup C_A^2 \cup \ldots \cup C_A^n = S. \]

It turns out that this is the case if and only if

\[ C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S. \] (2)

Note also that an irreducible matrix is primitive if and only if \( d(A) = 1 \).

In this paper we shall use a refinement of the argument used in [11] in order to find estimations for the number \( k = k(A) \) for any irreducible matrix.

For a primitive matrix it is well known that \( k(A) \leq (n - 1)^2 + 1 \) and that this result is sharp. (See [1]–[4], [6], [7], [8], [10], [11], [15].)

An analogous question for irreducible (but not necessarily primitive) matrices has been recently treated in [5] and in some special cases in [10].

The refinement of our argument consists in the fact that instead of studying the global behaviour of the sequence (1) we shall first study the behaviour of a fixed “row” in the sequence (1).

To this end we introduce the following notations: We denote \( \{e_{i1}, e_{i2}, \ldots, e_{in}\} \cup \cup \{0\} = S_i \), so that \( S_1 \cup S_2 \cup \ldots \cup S_n = S \). If \( A \) is a given \( n \times n \) matrix, we further denote \( F_i = F_i(A) = S_i \cap C_A \). Hence \( F_i = F_i(A) \) is the “support of the \( i \)-th row of \( A \).” For further purposes note that \( F_i = e_iC_A \).

For brevity we shall occasionally say that \( F_i \) is “the \( i \)-th row of \( C_A \)” by considering hereby in a natural manner the set \( C_A \) (subset of \( S \)) written in the form of a square block with the non-zero entries \( e_{ij} \) on appropriate places. For instance for the matrix

\[ A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 4 & 3 \end{pmatrix} \]
we shall write \( C_A = \{0, e_1, e_2, e_3, e_4, e_5, e_6\} \) in the form

\[
C_A = \begin{pmatrix}
e_{11}, 0, & e_{13} \\
0, & e_{22}, 0 \\
e_{33}, & e_{32}, e_{33}
\end{pmatrix} \cup \{0\}.
\]

Here

\[
F_1 = \{0, e_{11}, e_{13}\}, \quad F_2 = \{0, e_{22}\}, \quad F_3 = \{0, e_{31}, e_{32}, e_{33}\}.
\]

Consider now the sequence

\[
F_i, F_iC_A, F_iC_A^2, \ldots
\]

and define \( F_iC_A^0 = F_i \). The members of this sequence are clearly the supports of the \( i \)-th rows in the sequence (1).

Again (3) contains only a finite numbers of different sets. Denote by \( k_i = k_i(A) \) the least integer such that \( F_iC_A^{k_i-1} \) occurs in (3) more than once. Let further \( d_i = d_i(A) \) be the least integer \( \geq 1 \) such that \( F_iC_A^{k_i-1} = F_iC_A^{k_i-1+d_i} \). Then the sequence (3) is of the form

\[
F_i, F_iC_A, \ldots, F_iC_A^{k_i-2} \big| F_iC_A^{k_i-1}, \ldots, F_iC_A^{k_i-1+d_i-1} \big| F_iC_A^{k_i-1}, \ldots
\]

Clearly \( k_i \leq k, d_i \leq d \) (for \( i = 1, 2, \ldots, n \)) so that, in particular, \( \max k_i \leq k \).

Conversely, if \( k* = \max k_i \), then the term \( F_iC_A^{k_i-1} \) (for any \( i \)) occurs in the sequence (3) more than once, hence \( F_iC_A^{k_i-1} = F_iC_A^{k_i-1+d_i} \) (for any \( i \)). This implies that for any integer \( \lambda_i \geq 1 \) we have \( F_iC_A^{k_i-1} = F_iC_A^{k_i-1+\lambda_id_i} \). Let \( d* \) be the least common multiple of the numbers \( d_1, d_2, \ldots, d_n \) and put \( \lambda_i = d*/d_i \). We then have \( F_iC_A^{k_i-1} = F_iC_A^{k_i-1+d*} \) and \( \bigcup_{i=1}^n F_iC_A^{k_i-1} = \bigcup_{i=1}^n F_iC_A^{k_i-1+d*} \), i.e. \( C_A^{k*} = C_A^{k*+d*} \). This shows that \( C_A^{k*} \) occurs in (1) more than once, so that \( k \leq k* \). Hence \( k = k* = \max k_i \).

**Remark 1.** By the way: \( C_A^{k*} = C_A^{k*+d*} \) immediately implies that \( d \leq d* \) and \( d \mid d* \). Since it is easy to see that \( d \mid d* \), we also have \( d \mid d* \), so that \( d = d* \). We shall not need this fact in the present paper.

**Remark 2.** If \( A \) is irreducible, then (2) implies that

\[
F_iC_A^{k_i-1} \cup F_iC_A^{k_i} \cup \ldots \cup F_iC_A^{k_i+d_i-2} = S_i
\]

for \( i = 1, 2, \ldots, n \). In particular, if \( A \) is primitive, then \( F_iC_A^{k_i-1} = S_i \).

**Remark 3.** It is easy to introduce in the sequence (3) a multiplication \( \odot \) so that (3) becomes a cyclic semigroup. To this end it is sufficient to define \( F_iC_A^* \odot F_iC_A^* = \)

\(^{1)}\) The set \( \{0\} \) can be omitted if \( A \) contains a zero entry.
Then the set \( \{ F_i C_A^{-1}, \ldots, F_i C_A^{-d_i-2} \} \) (with the same multiplication) is a cyclic group of order \( d_i \).

1. THE GENERAL CASE

The goal of this section is to prove some theorems, which hold for any non-negative irreducible matrix. Some of the lemmas are of independent interest.

All matrices considered below are \( n \times n \) matrices, \( n > 1 \).

We begin with the decisive

**Lemma 1.** Suppose that \( A \) is irreducible and \( M \) any proper subset of \( S_i \) containing 0 and at least one non-zero element. Then \( MC_A \) contains at least one non-zero element \( e \in S_i \), which is not contained in \( M \).

**Proof.** Let \( M = \{ 0, e_{ia}, e_{ib}, \ldots, e_{iv} \} \), \( \{ \alpha, \beta, \ldots, v \} \subseteq N \). Suppose for an indirect proof that for all elements \( e_{ia}, e_{ib}, \ldots, e_{iv} \) we have

\[
\{ e_{ia}, e_{ib}, \ldots, e_{iv} \} \subseteq \{ e_{ia}, e_{ib}, \ldots, e_{iv} \} \cup \{ 0 \}.
\]

If \( q \in \{ \alpha, \beta, \ldots, v \} \), we necessarily have \( \sigma \in \{ \alpha, \beta, \ldots, v \} \). In other words: If \( q \in \{ \alpha, \beta, \ldots, v \} \) and \( \sigma \in N - \{ \alpha, \beta, \ldots, v \} \), we have \( a_{q\sigma} = 0 \). This says that \( A \) is reducible, contrary to the assumption.

**Lemma 2.** Suppose that \( A \) is irreducible.

a) If \( F_i \) contains \( g \geq 1 \) non-zero elements \( e \in S_i \), we have

\[
F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-g} = S_i.
\]

b) In particular we always have

\[
F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-1} = S_i.
\]

c) If \( i \neq j \) we always have

\[
e_{ij} \in F_i \cup F_j C_A \cup \ldots \cup F_j C_A^{n-2}.
\]

**Proof.** a) By Lemma 1 \( F_i \cup F_i C_A \) contains at least \( g + 1 \) non-zero elements. Again by Lemma 1

\[
(F_i \cup F_i C_A) \cup (F_i \cup F_i C_A) C_A = F_i \cup F_i C_A \cup F_i C_A^2
\]

contains at least \( g + 2 \) non-zero elements. Repeating this argument we find that \( F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-g} \) contains at least \( n \) non-zero elements \( e \in S_i \), i.e. the whole set \( S_i \).
b) Follows from the fact that an irreducible matrix has in each row at least one element different from zero.

c) Since \(e_{ii} C_A\) contains at least one non-zero element \(\neq e_{ii}\), the set \(e_{ii} \cup e_{ii} C_A\) contains at least two non-zero elements in the set \(S_i\). Analogously \((e_{ii} \cup e_{ii} C_A) \cup (e_{ii} \cup e_{ii} C_A) C_A = e_{ii} \cup e_{ii} C_A \cup e_{ii} C_A^2\) contains at least 3 non-zero elements, and so on. We finally have

\[
e_{ii} \cup e_{ii} C_A \cup e_{ii} C_A^2 \cup \ldots \cup e_{ii} C_A^{n-1} = S_i.
\]

Since \(e_{ii} C_A = F_i\), the last equality can be written in the form

\[
e_{ii} \cup F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{n-2} = S_i,
\]

from which our assertion immediately follows.

**Lemma 3.** If \(A\) is irreducible, then there is an integer \(h = h(i)\) such that \(1 \leq h \leq n\) and \(F_i \in F_i C_A^h\). Here:

a) If \(e_{ii} \in F_i\), we may choose \(h = 1\).

b) If \(F_i\) contains \(g\) non-zero elements in the set \(S_i\), we may choose \(h \leq n - g + 1\).

**Proof.** a) If \(e_{ii} \in F_i\), then \(F_i = e_{ii} C_A \subseteq F_i C_A^h\), and our statement is true with \(h = 1\).

b) By Lemma 2b there is an integer \(u, 1 \leq u \leq n - g\) such that \(e_{ii} \in F_i C_A^u\). Multiplying by \(C_A\) we get \(F_i = e_{ii} C_A \subseteq F_i C_A^{u+1}\). Since \(u + 1 \leq n - g + 1\), our statement holds.

**Remark.** The example of the irreducible permutation matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & & & & \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}
\]

shows that \(F_i \subseteq F_i C_A^1\), but \(F_i \neq F_i C_A^h\) for \(h = 1, 2, \ldots, n - 1\). Hence the estimation \(h \leq n\) in Lemma 3 is — in general — the best possible.

**Theorem 1.** If \(A\) is irreducible, \(F_i\) contains \(g\) non-zero elements and \(F_i \in F_i C_A^h\), \(h \geq 1\), then \(k_i \leq (n - g) h + 1\).

**Proof.** The supposition implies

\[
F_i \subseteq F_i C_A^h \subseteq F_i C_A^{2h} \subseteq \ldots \subseteq F_i C_A^{(n-g)h} \subseteq F_i C_A^{(n-g+1)h} \subseteq \ldots
\]

Since \(F_i\) contains \(g\) non-zero elements in \(S_i\), the set \(F_i C_A^h\) is either equal to \(F_i\) or contains at least \(g + 1\) non-zero elements in \(S_i\). Further \(F_i C_A^{2h}\) is again either equal to \(F_i C_A^h\) or
contains at least \( g + 2 \) non-zero elements \( e_i \); and so on. The chain (4) cannot have more than \( n - g + 1 \) different members. There exists therefore a \( \tau, 0 \leq \tau \leq n - g \), such that \( F_i C_A^\tau = F_i C_A^{(\tau+1)\hat{h}} \). Hence \( k_i - 1 \leq \tau h \leq (n - g) h \). This proves our Theorem.

**Theorem 2.** If \( A \) is irreducible and \( F_i \) contains \( g \) non-zero elements \( e_i \), we have \( k_i \leq (n - g)^2 + (n - g) + 1 \).

**Proof.** By Lemma 3b we have \( h \leq n - g + 1 \), hence

\[
k_i \leq (n - g)(n - g + 1) + 1 = (n - g)^2 + (n - g) + 1 .
\]

**Remark.** The results of Theorem 1 and Theorem 2 cannot be — in general — sharpened. To show this consider the matrix \( A \) with

\[
C_A = \begin{pmatrix} 0 & e_{12} & 0 \\ 0 & 0 & e_{23} \\ e_{31} & e_{32} & 0 \end{pmatrix}
\]

and the third row \( F_3 = \{0, e_{31}, e_{32}\} \). Here \( n = 3, g = 2 \). We have \( F_3 C_A = \{0, e_{32}, e_{33}\}, F_3 C_A^2 = \{0, e_{31}, e_{32}, e_{33}\} \) so that \( k_3 = 3 \). On the other hand \( (n - g)^2 + (n - g) + 1 = 3 \).

With respect to the relation \( k(A) = \max_i k_i \) we immediately get:

**Corollary 1.** For any irreducible non-negative \( n \times n \) matrix \( A \) we always have \( k(A) \leq n^2 - n + 1 \).

**Proof.** Since \( g \geq 1 \), we have \( k(A) \leq (n - 1)^2 + (n - 1) + 1 = n^2 - n + 1 \).

**Corollary 2.** If \( A \) is irreducible and each row contains at least two non-zero elements, we have \( k(A) \leq n^2 - 3n + 3 \).

**Proof.** Follows from \( k(A) = \max_i k_i \leq (n - 2)^2 + (n - 2) + 1 = n^2 - 3n + 3 \).

The result of Corollary 1 is not the best possible. It is intuitively clear that a possible sharpening of this estimation depends on the possibility to sharpen Theorem 1 for the rows containing a unique non-zero element.

Note first: If \( A \) is irreducible and \( F_i \) contains a unique non-zero element \( e_i \) there cannot hold \( F_i = \{0, e_i\} \) since such a matrix is reducible. Therefore in the following Theorem 3 we may suppose \( F_i = \{0, e_i\} \) with \( i \neq j \).

**Theorem 3.** Suppose that \( A \) is irreducible and \( F_i \) contains exactly one non-zero element \( e_i \). Let \( h_i \) be the least integer \( \geq 1 \) such that \( F_i \subset F_i C_A^h_i \).

A) If \( h_i \leq n - 1 \), we have \( k_i \leq (n - 1) h_i + 1 \leq (n - 1)^2 + 1 \).

B) If \( h_i = n \), we have \( k_i \leq n^2 - 3n + 4 \).
Proof. A) This follows from Theorem 1 by putting \( g = 1 \) and \( h = n - 1 \).

B) We first show that in this case \( e_{ii} \in F_i C_A^n - 1 \) and \( e_{ii} \notin F_i C_A^h \) with \( h \leq n - 2 \).

By Lemma 2b we have \( e_{ii} \in F_i C_A^h \) with \( 1 \leq h \leq n - 1 \). If there were \( h \leq n - 2 \), we would have \( e_{ii} C_A \subseteq F_i C_A^{h + 1} \), i.e. \( F_i \subseteq F_i C_A^{h + 1} \) with \( h + 1 \leq n - 1 \), contrary to the assumption.

Next we show that for \( t = 1, 2, \ldots, n \) the set \( F_i C_A^t \) contains exactly one element \( e S_i \) which is not contained in the union \( F_i \cup F_i C_A \cup \ldots \cup F_i C_A^{t - 1} \). (Hereby \( F_i C_A^0 = F_i \).

By the same argument as in the proof of Lemma 2a it follows that \( F_i \cup \ldots \cup F_i C_A^{t - 1} \) contains at least \( t \) different non-zero elements \( e S_i \). Suppose for an indirect proof that \( F_i C_A^t \) has at least two non-zero elements not contained in \( F_i \cup \ldots \cup F_i C_A^{t - 1} \). Then \( F_i \cup \ldots \cup F_i C_A^t \) contains at least \( t + 2 \) non-zero elements \( e S_i \). By Lemma 1 \( (F_i \cup \ldots \cup F_i C_A^t) \cup (F_i \cup \ldots \cup F_i C_A^t) C_A = F_i \cup \ldots \cup F_i C_A^{t + 1} \) contains at least \( t + 3 \) non-zero elements, and repeating this process we obtain that \( F_i \cup \ldots \cup F_i C_A^{t - 2} = S_i \).

Hence \( e_{ii} \in F_i C_A^h \) with \( h \leq n - 2 \), which has been shown impossible.

In particular: \( F_i C_A^t \) contains exactly one element not contained in \( F_i \). But since \( F_i \notin F_i C_A \), we conclude that \( F_i C_A^t \) contains exactly one non-zero element \( e S_i \).

Consider now the finite sequence \( F_i, F_i C_A, \ldots, F_i C_A^{t - 1}, F_i C_A^t \), and let \( l_0 \) be the least integer such that \( F_i C_A^{l_0} \) contains more than one non-zero element \( e S_i \). We have just seen that \( l_0 > 1 \).

α) If \( l_0 = n \), then each of the sets \( F_i, \ldots, F_i C_A^{n - 1} \), contains a unique element and since \( e_{ii} \in F_i C_A^{n - 1} \), we have \( \{0, e_{ii}\} = F_i C_A^{n - 1} \). Therefore \( e_{ii} C_A = F_i C_A^n \), i.e. \( F_i = F_i C_A^n \), so that \( k_i = 1 \).

β) Suppose next \( l_0 \leq n - 1 \) and let \( F_i = \{0, e_{ia}\}, F_i C_A = \{0, e_{ib}\}, \ldots, F_i C_A^{l_0 - 1} = \{0, e_{ik}\} \). Since \( F_i C_A^{l_0} \) contains at least two non-zero elements \( e S_i \) and only one not contained in \( \{e_{ia}, e_{ib}, \ldots, e_{ik}\} \), there is necessarily an index \( \xi \in \{a, b, \ldots, \lambda\} \) such that \( e_{ii} \in F_i C_A^{l_0} \). Consequently: There is an integer \( \tau, 1 \leq \tau \leq l_0 \), such that

\[
\{0, e_{ii}\} = F_i C_A^{l_0 - \tau} \subseteq F_i C_A^{l_0}.
\]

Now \( \tau \) cannot be \( l_0 \) since \( F_i \subseteq F_i C_A^{l_0} \) with \( l_0 \leq n - 1 \) contradicts our assumption. Therefore we have \( 1 \leq \tau \leq l_0 - 1 \). The relation (5) implies

\[
F_i C_A^{l_0 - \tau} \subseteq F_i C_A^{l_0} \subseteq F_i C_A^{l_0 + \tau} \subseteq \ldots \subseteq F_i C_A^{l_0 + (n - 1) \tau}.
\]

This chain of \( n + 1 \) sets cannot have all members different one from the other. There is therefore an integer \( u, -1 \leq u \leq n - 2 \), such that

\[
F_i C_A^{l_0 + ut} = F_i C_A^{l_0 + (u + 1) \tau}.
\]

Hence

\[
k_i - 1 \leq l_0 + ut \leq l_0 + u(l_0 - 1) \leq n - 1 + (n - 2)(n - 2) = n^2 - 3n + 3.
\]

This proves Theorem 3.
Remark. The result \( k_i \leq n^2 - 3n + 4 \) cannot be — in general — sharpened. To show this consider the matrix \( A \) with

\[
C_A = \begin{cases}
0, & e_{12}, 0 \\
0, & 0, e_{23} \\
e_{31}, & 0, e_{32}
\end{cases}
\]

We have

\[
C_A^2 = \begin{cases}
0, & 0, e_{13} \\
e_{21}, & 0, e_{23} \\
e_{31}, & e_{32}, e_{33}
\end{cases},
\quad C_A^3 = \begin{cases}
e_{11}, & 0, e_{13} \\
e_{21}, & e_{22}, e_{23} \\
e_{31}, & e_{32}, e_{33}
\end{cases},
\quad C_A^4 = \begin{cases}
e_{11}, & e_{12}, e_{13} \\
e_{21}, & e_{22}, e_{23} \\
e_{31}, & e_{32}, e_{33}
\end{cases} \cup \{0\},
\]

so that \( A \) is primitive (hence irreducible). Now

\( F_1 = \{0, e_{12}\}, \quad F_1C_A = \{0, e_{13}\}, \quad F_1C_A^2 = \{0, e_{11}, e_{13}\}, \quad F_1C_A^3 = \{0, e_{11}, e_{12}, e_{13}\} \)

so that indeed \( F_1 \subset F_1C_A \) and \( k_1 = 4 \). On the other hand \( n^2 - 3n + 4 \) for \( n = 3 \) is equal to 4.

Theorems 2 and 3 allow the following conclusions. If \( n \geq 2 \), we have for the rows with at least two non-zero elements

\[
k_i \leq (n-g)^2 + (n-g) + 1 \leq (n-2)^2 + (n-2) + 1 = n^2 - 3n + 3.
\]

For the rows with a unique non-zero element we have (with \( h_i \) defined above)

\[
\text{either } k_i \leq n^2 - 3n + 4 \quad \text{if } h_i = n,
\]

\[
\text{or } k_i \leq (n-1)h_i+1 \leq (n-1)^2 + 1 \quad \text{if } h_i \leq n-1.
\]

Since (for \( n \geq 2 \)) we have

\[
(n-1)(n-2) + 1 = (n-2)^2 + (n-2) + 1 = n^2 - 3n + 3 < n^2 - 3n + 4 \leq (n-1)^2 + 1,
\]

we get with respect to \( k(A) = \max_i k_i \):

**Theorem 4.** For any non-negative irreducible matrix \( A \) we always have \( k(A) \leq (n-1)^2 + 1 \).

**Theorem 5.** Let \( A \) be irreducible. Denote \( h_i \) the least positive integer for which \( F_i \subset F_iC_A^{h_i} \). If for every row \( F_i \) containing a unique non-zero element we have \( h_i \leq n-1 \) (i.e. either \( h_i = n \) or \( h_i \leq n - 2 \)), then \( k(A) \leq n^2 - 3n + 4 \).

**Remark 1.** The result of Theorem 4 is the best possible for it is known that to every \( n \geq 2 \) there is a primitive matrix \( A \) with \( k(A) = (n-1)^2 + 1 \). This property has the "Wielandt matrix", which is a matrix with \( C_A = \{0, e_{12}, e_{23}, e_{34}, \ldots, e_{n-1,n}, e_{n1}, e_{n2}\} \).
Remark 2. Also the result of Theorem 5 cannot be — in general — sharpened. This shows the example in the Remark after Theorem 3. Here \( F_1 = \{0, e_{12}\} \) and \( h_1 = 3 \), \( F_2 = \{0, e_{23}\} \) and \( h_2 = 1 \) so that the suppositions of Theorem 5 are satisfied. On the other hand \( k(A) = 4 = n^2 - 3n + 4 \).

2. THE CASE OF A PRIMITIVE MATRIX

We shall now apply our results to the case of a primitive matrix. For a primitive matrix \( A \) the set \( F_iC_A^{k-1} \) is the whole set \( S_i \).

Theorem 6. If \( A \) is primitive, then \( k(A) \leq n - 1 + \min_i k_i \).

Proof. Let \( e_{ix} \) be any element \( e \in S_i \). Take \( j \neq i \) and write \( e_{ix} = e_{ij}e_{ja} \). By Lemma 2 \( e_{ij} \in F_iC_A^t \), where \( t = t(i, j) \) satisfies \( 0 \leq t \leq n - 2 \). By definition of the number \( k_j \) we have (for any \( \alpha \)) \( e_{ja} \in S_j = F_jC_A^{k_j-1} \). Hence

\[
S_i = \{0, e_{i1}, e_{i2}, \ldots, e_{in}\} \subseteq F_iC_A^tF_jC_A^{k_j-1} = F_iC_A^{t+k_j}.
\]

Therefore \( k_i - 1 \leq t + k_j \), i.e. \( k_i \leq t + 1 + k_j \). (This is, of course, trivially true also for \( i = j \).) Since \( j \) is arbitrary, we have \( k_i \leq (n - 2) + 1 + \min_j k_j = n - 1 + \min_j k_j \). Hence

\[
\max_{i,j} k_i = \max_{i,j} k_i - \min_{i,j} k_i \leq n - 1.
\]

By the way we have also proved\(^2\):

Theorem 7. For any primitive \( n \times n \) matrix \( A \) we always have

\[
\max_i k_i - \min_i k_i \leq n - 1.
\]

Remark. The result of Theorem 6 is sharp in the following sense. In any primitive matrix there is at least one row, say \( j \)-th row, containing at least \( q = 2 \) non-zero elements. By Theorem 2 \( k_j \leq n^2 - 3n + 3 \). Hence by Theorem 6 \( k(A) \leq (n - 1) + (n^2 - 3n + 3) = n^2 - 2n + 2 \) and the "Wielandt matrix" attains this upper bound.

Also simple examples show that the result of Theorem 7 is the best possible. The following result described in Theorem 8 is known. (See [1], [4], [11].)

Lemma 4. If \( A \) is irreducible and \( e_{jj} \in F_j \), then \( k_j \leq n - 1 \).

Remark. It is well known that in this case irreducibility implies primitivity.

\(^2\) (Added in proofs, May 1966.) In a forthcoming paper ([16]) we shall show that Theorem 7 holds for any non-negative irreducible matrix \( A \) and we use it to obtain estimates for \( k(A) \) in the case of imprimitive matrices.
Proof. By supposition \( e_{jj} \in F_j \), hence \( F_j = e_{jj}C_A \subseteq F_jC_A \). This implies \( F_j \subseteq F_jC_A \subseteq F_jC_A^2 \subseteq \cdots \subseteq F_jC_A^{n-2} \). By Lemma 2c we have for \( j \neq \alpha \)

\[
e_{ja} \in F_j \cup F_jC_A \cup \cdots \cup F_jC_A^{n-2} = F_jC_A^{n-2}, \quad \text{i.e.} \quad S_j = F_jC_A^{n-2}.
\]

Hence there is a \( \tau, 0 \leq \tau \leq n - 2 \), such that \( F_jC_A^\tau = F_jC_A^{\tau + 1} \). Therefore \( k_j - 1 \leq \tau \), i.e. \( k_j - 1 \leq \tau + 1 \leq (n - 2) + 1 = n - 1 \).

Remark. The result of Lemma 4 is sharp, since e.g. \( A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \) is primitive and direct computation shows that \( k_2 = k_3 = 2(n - 1) \).

Under the suppositions of Lemma 4 we have \( \min_i k_i \leq n - 1 \). This combined with Theorem 6 gives the following

Corollary. If \( A \) is irreducible and contains a non-zero element in the main diagonal, then \( k(A) \leq 2n - 2 \).

In the proof of the next Theorem 8 we shall again use the inequality \( k_i \leq t(i, j) + 1 + k_j \) (proved in the proof of Theorem 6).

Theorem 8. If \( A \) is primitive and contains \( r \geq 1 \) non-zero elements in the main diagonal, we have \( k(A) \leq 2n - r - 1 \).

Proof. Suppose that \( \{e_{j_1, i_1}, e_{j_2, i_2}, \ldots, e_{j_r, i_r}\} \subseteq C_A \). Then \( k_{j_i} \leq n - 1 \), \( i \leq n - 1 \).

If \( r = n \), then \( k(A) = \max_j k_j \leq n - 1 \), and our statement holds.

Suppose \( r < n \) and choose an index \( i \notin \{j_1, j_2, \ldots, j_r\} \). Since

\[
e_{ii} \cup e_{ii}C_A \cup \cdots \cup e_{ii}C_A^{n-r} = e_{ii} \cup F_i \cup F_iC_A \cup \cdots \cup F_iC_A^{n-r-1}
\]

contains at least \( n - r + 1 \) non-zero elements \( \in S_i \) and \( \{e_{ii}, e_{ij_1}, \ldots, e_{ij_r}\} \) contains exactly \( r \) elements, these sets intersect and there is a \( j \), say \( j_1 \), such that \( e_{ij_1} \in F_iC_A^t \) with \( 0 \leq t(i, j_1) \leq n - r - 1 \). Now \( k_i \leq t(i, j_1) + 1 + k_{j_1} \) implies \( k_i \leq (n - r - 1) + + 1 + (n - 1) = 2n - r - 1 \). Hence \( k(A) = \max_i k_i \leq 2n - r - 1 \), q.e.d.

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Резюме

НОВЫЙ МЕТОД РЕШЕНИЯ НЕКОТОРЫХ ВОПРОСОВ ТЕОРИИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

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Пусть $A$ — квадратная неотрицательная матрица. Распределение нулевых и ненулевых элементов в последовательности $A, A^2, A^3, \ldots$, начиная с некоторой степени $k(A)$, периодически повторяется. Цель статьи — получить оценки для числа $k(A)$ в случае неразложимых матриц. При этом используется новый метод, являющийся уточнением метода, использованного автором в работе [11].