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EXPONENTIALLY STABLE INTEGRAL MANIFOLDS, AVERAGING PRINCIPLE AND CONTINUOUS DEPENDENCE ON A PARAMETER

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The main tools that are used in this paper to the study of integral manifolds are some stability considerations and continuous depedence on a parameter so that differential equations in Banach spaces and functional differential equations are covered. The results on differential equations in Banach spaces are applied to special boundary value problems.

In abstract form the existence of exponentially stable integral manifolds is established for flows (section 2). By a flow a system of functions is meant that fulfils certain conditions, which are always fulfilled by the set of solutions of a differential equation. It is proved that there exists a unique (exponentially stable) integral manifold of every flow that is sufficiently near to a given flow, which fulfils some special conditions (Theorem 2,2). By means of some theorems on the continuous dependence on a parameter and a Stability Lemma this result is specialized to differential equations in Banach spaces. There exists a unique (exponentially stable) integral manifold of every differential equation the right hand side of which is sufficiently near (in a certain sense) to the right hand side of a given differential equation, which fulfils some special conditions (Theorem 1,2). The concept of being near is chosen in such a way as to cover the averaging principle.

In section 3 Theorem 2,2 and the Stability Lemma are applied to functional differential equations. As the Stability Lemma is proved for differential equations with no time lag, the existence of an exponentially stable integral manifold is established for a functional differential equation, the right hand side of which is sufficiently near to the right hand side of a differential equation (without time lag), which fulfils some special conditions.

Section 1 contains theorems on the continuous dependence on a parameter for differential equations in Banach spaces (Theorems 1,1–9,1). Theorems 1,1 and 3,1 only are needed in the specialization of Theorem 2,2 to Theorem 1,2. Two features are to be emphasized on these Theorems:

i) the usual condition that the difference of the right hand sides of the differential equations is small is weakened to the condition that the difference of the right hand sides integrated with respect to time is small,

ii) let x, y denote solutions of a differential equation depending on a parameter with the initial conditions $x(t_0) = \tilde{x}$, $y(t_0) = \tilde{y}$. An estimate is found for the change of $x(t) - y(t)$ with respect to the parameter. This change is found to be small compared to $\|\tilde{x} - \tilde{y}\|$.

In sections 4, 5 and 6 the results of sections 1 and 2 are applied to a boundary value problem for a weakly nonlinear wave equation (with one space variable). Section 4 contains preliminaries, in section 5 several examples are examined and in section 6 it is proved that a special problem of the above type has a smooth solution, which remains bounded for $t \to \infty$ and tends (uniformly) to a periodic function, which has discontinuous derivatives of the first order.

1. CONTINUOUS DEPENDENCE ON A PARAMETER

Let X be a Banach space, E_1 the real line, G a subset of X (G need not be open), K_1 a positive constant. Let $f_{(1)} = f_{(1)}(G, K_1)$ denote the class of functions f from $G \times E_1$ to X, which are continuous and fulfil the conditions

$$\|f(x, \tau)\| \leqq K_1 \tag{1,1}$$

$$\|f(x_1, \tau) - f(x_2, \tau)\| \leqq K_1\|x_1 - x_2\|\,, \quad x, x_1, x_2, \in G,\ \tau \in E_1\,. \tag{2,1}$$

The solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, \tau) \tag{3,1}$$

will be denoted by x, y, their values by $x(\tau), y(\tau)$; the initial conditions will be

$$x(\tilde{\tau}) = \tilde{x}\,, \quad y(\tilde{\tau}) = \tilde{y}\,. \tag{4,1}$$

Similarly the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = f_0(x, \tau) \tag{5,1}$$

will be denoted by x_0, y_0, their values by $x_0(\tau), y_0(\tau)$, with the initial conditions

$$x_0(\tilde{\tau}) = \tilde{x}, \quad y_0(\tilde{\tau}) = \tilde{y}\,. \tag{6,1}$$

The solutions $x, x_0, x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$ are unique, if $f, f_0 \in f_{(1)}$ and if in addition G is open, an existence theorem may be stated.

Theorem 1,1. *Let $K_1 > 0$ be given. There exists a function $\chi_1(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ such that*

$$\lim_{\zeta \to 0+} \chi_1(\zeta, T) = 0 , \tag{7,1}$$

and the following assertion takes place:

Let $f, f_0 \in f_{(1)}$, let x, x_0 be solutions of (3,1), (5,1), on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$ fulfilling (4,1), (6,1). Suppose that

$$\left\| \int_{\tau_1}^{\tau_2} [f(z, \tau) - f_0(z, \tau)] \, d\tau \right\| \leqq \zeta \quad \text{for} \quad z \in G, \ \tau_1 \leqq \tau_2 \leqq \tau_1 + 1 . \tag{8,1}$$

Then

$$\|x(\tau) - x_0(\tau)\| \leqq \chi_1(\zeta, T), \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T. \tag{9,1}$$

Note 1,1. The proof of Theorem 1,1 goes back to I. I. Gichman, [2], cf. also [1], Theorem 1,1.

Proof: It follows from (1,1) and (2,1) that $\|x(\tau) - x(\tau_1)\| \leqq K_1 |\tau - \tau_1|$ and that

$$x(\tau) = x(\tau_1) + \int_{\tau_1}^{\tau} f(x(\tau_1), \sigma) \, d\sigma + Z , \tag{10,1}$$

$$\|Z\| \leqq \tfrac{1}{2} K_1^2 |\tau - \tau_1|^2 \quad \text{for} \quad \tilde{\tau} \leqq \tau_1 \leqq \tau \leqq \tilde{\tau} + T.$$

Similar relations hold for x_0 also. Put $\tilde{\tau} = 0$ and let r, s denote integers, $0 < r \leqq s$, $T \leqq s$; let $0 < \tau \leqq T$. Then

$$x(r\tau/s) - x(0) = \sum_{p=0}^{r-1} \int_{p\tau/s}^{(p+1)\tau/s} f(x(p\tau/s), \sigma) \, d\sigma + Z_{r,s}$$

where $\|Z_{r,s}\| \leqq \frac{1}{2} K_1^2 r \tau^2 / s^2$ and a similar relation holds for x_0 also. Therefore

$$x(r\tau/s) - x_0(r\tau/s) = \sum_{p=0}^{r-1} \int_{p\tau/s}^{(p+1)\tau/s} [f(x(p\tau/s), \sigma) - f(x_0(p\tau/s), \sigma)] \, d\sigma +$$

$$+ \sum_{p=0}^{r-1} \int_{p\tau/s}^{(p+1)\tau/s} [f(x_0(p\tau/s), \sigma) - f_0(x_0(p\tau/s), \sigma)] \, d\sigma + Z'_{r,s} ,$$

where $\|Z'_{r,s}\| \leqq K_1^2 r \tau^2 / s^2$. According to (8,1) the norm of the second sum does not exceed $s\zeta$, so that

$$\|x(r\tau/s) - x_0(r\tau/s)\| \leqq (K_1 \tau / s) \sum_{p=0}^{r-1} \|x(p\tau/s) - x_0(p\tau/s)\| + s\zeta + K_1^2 T^2 / s .$$

Hence

$$\|x(r\tau/s) - x_0(r\tau/s)\| \leqq (s\zeta + K_1^2 T^2/s) (1 + K_1 \tau / s)^{r-1} ,$$

$$\|x(\tau) - x_0(\tau)\| \leqq (s\zeta + K_1^2 T^2/s) \, e^{K_1 \tau},$$

and Theorem 1,1 holds for

$$\chi_1(\zeta, T) = \min_{s \geqq T} \left(s\zeta + K_1^2 T^2/s\right) e^{K_1 T} \tag{11,1}$$

(s being an integer).

Note 2,1 The assumption in Theorem 1,1 that the solution x of (3,1) exists on $\langle\tilde{\tau}, \tilde{\tau} + T\rangle$ may be modified as follows: the solution x_0 of (5,1) is defined on $\langle\tilde{\tau}, \tilde{\tau} + T\rangle$ and $\tau \in \langle\tilde{\tau}, \tilde{\tau} + T\rangle$, $z \in X$, $\|z - x_0(\tau)\| \leqq \chi_1(\zeta, T) + \eta$ imply that $z \in G$, χ_1 being defined by (13,1), $\eta > 0$ being an arbitrary constant. If all other assumptions of Theorem 1,1 are fulfilled, then there exists the unique solution x of (3,1) fulfilling (4,1); this solution exists on $\langle\tilde{\tau}, \tilde{\tau} + T\rangle$ and fulfils (9,1).

Let ω be a nondecreasing function on $\langle 0, \infty)$, $\lim_{\xi \to 0+} \omega(\xi) = 0$. Denote by $f_{(2)} = f_{(2)}(G, K_1, \omega)$ the subset of those $f \in f_{(1)}$ that

$$\begin{gathered} \|f(x_2 + y, \tau) - f(x_1 + y, \tau) - f(x_2, \tau) + f(x_1, \tau)\| \leqq \\ \leqq \|x_2 - x_1\| \, \omega(\|y\|) \,, \quad x_1, x_2, x_1 + y, x_2 + y + G, \tau \in E_1 \,. \end{gathered} \tag{12,1}$$

If G is convex, if f is differentiable with respect to x and if

$$\left\|\frac{\partial f}{\partial x}(x_2, \tau) - \frac{\partial f}{\partial x}(x_1, \tau)\right\| \leqq \omega\left(\|x_2 - x_1\|\right), \quad x_2, x_1 \in G, \tau \in E_1 \,, \tag{13,1}$$

then (12,1) is fulfilled as $f(x_2 + y, \tau) - f(x_1 + y, \tau) - f(x_2, \tau) + f(x_1, \tau) = \int_0^1 [\partial f/\partial x(\sigma(x_2 - x_1) + x_1 + y, \tau) - \partial f/\partial x(\sigma(x_2 - x_1) + x_1, \tau)] \, d\sigma(x_2 - x_1)$.

Theorem 2,1. *Let the number $K_1 \geqq 0$, and the function ω be given. There exists a function $\chi_2(\zeta, T)$ defined for $\zeta > 0$ $T > 0$ nondecreasing in ζ such that*

$$\lim_{\zeta \to 0+} \chi_2(\zeta, T) = 0 \tag{14,1}$$

and the following assertion takes place:

Let $f, f_0 \in f_{(2)}$, let x, $y(x_0, y_0)$ be solutions of (3,1) ((5,1)) on $\langle\tilde{\tau}, \tilde{\tau} + T\rangle$, $T > 0$ fulfilling (4,1) ((6,1)). Suppose that (8,1) takes place and that

$$\left\|\int_{\tau_1}^{\tau_2} [f(z_2, \tau) - f(z_1, \tau) - f_0(z_2, \tau) + f_0(z_1, \tau)]\right\| d\tau \leqq \zeta \|z_2 - z_1\| \tag{15,1}$$

for $z_1, z_2 \in G$, $\tau_1 \leqq \tau_2 \leqq \tau_1 + 1$. Then

$$\begin{gathered} \|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\| \cdot \chi_2(\zeta, T) \,, \\ \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T \,. \end{gathered} \tag{16,1}$$

Note 3,1 (15,1) is fulfilled, if G is convex, if $\partial f/\partial x$, $\partial f_0/\partial x$ are continuous and if

$$(17,1) \qquad \left\| \int_{\tau_1}^{\tau_2} \left[\frac{\partial f}{\partial x}(z, \tau) - \frac{\partial f_0}{\partial x}(z, \tau) \right] d\tau \right\| \leqq \zeta\,, \quad z \in G,\ \tau_1 \leqq \tau_2 \leqq \tau_1 + 1\,.$$

Proof. Put $\tilde{\tau} = 0$.

$$(18,1) \qquad x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau) =$$

$$= \int_0^\tau [f(x(\sigma), \sigma) - f(x_0(\sigma) - y_0(\sigma) + y(\sigma), \sigma)]\, d\sigma +$$

$$+ \int_0^\tau [f(x_0(\sigma) - y_0(\sigma) + y(\sigma), \sigma) - f(y(\sigma), \sigma) - f(x_0(\sigma), \sigma) + f(y_0(\sigma)\, \sigma)]\, d\sigma +$$

$$+ \int_0^\tau [f(x_0(\sigma), \sigma) - f(y_0(\sigma), \sigma) - f_0(x_0(\sigma), \sigma) + f_0(y_0(\sigma), \sigma)]\, d\sigma = I_1 + I_2 + I_3\,.$$

It follows from (2,1) that

$$(19,1) \qquad \|I_1\| \leqq K_1 \int_0^\tau \|x(\sigma) - y(\sigma) - x_0(\sigma) + y_0(\sigma)\|\, d\sigma\,.$$

Taking (12,1) and (9,1) into account we obtain that

$$\|I_2\| \leqq \int_0^\tau \|x_0(\sigma) - y_0(\sigma)\|\, \omega(\|y(\sigma) - y_0(\sigma)\|)\, d\sigma \leqq$$

$$\leqq \omega(\chi_1(\zeta, T)) \int_0^\tau \|x_0(\sigma) - y_0(\sigma)\|\, d\sigma\,.$$

As $\|x_0(\sigma) - y_0(\sigma)\| \leqq \|\tilde{x} - \tilde{y}\|\, e^{K_1\sigma}$, it follows that

$$(20,1) \qquad \|I_2\| \leqq \|\tilde{x} - \tilde{y}\| \,.\, K_1^{-1} e^{K_1 T} \omega(\chi_1(\zeta, T))\,.$$

Choose an integer $s \geqq T$ and put $\sigma_j = j\tau/s$, $j = 0, 1, \ldots, s$. We have

$$(21,1)\, I_3 = \sum_{j=0}^{s-1} \int_{\sigma_j}^{\sigma_{j+1}} [f(x_0(\sigma_j), \sigma) - f(y_0(\sigma_j), \sigma) - f_0(x_0(\sigma_j), \sigma) + f_0(y_0(\sigma_j), \sigma)]\, d\sigma +$$

$$+ \sum_{j=0}^{s-1} \int_{\sigma_j}^{\sigma_{j+1}} [f(x_0(\sigma), \sigma) - f(y_0(\sigma), \sigma) - f(x_0(\sigma_j), \sigma) + f(y_0(\sigma_j), \sigma)]\, d\sigma +$$

$$+ \sum_{j=0}^{s-1} \int_{\sigma_j}^{\sigma_{j+1}} [-f_0(x_0(\sigma), \sigma) + f_0(y_0(\sigma), \sigma) + f_0(x_0(\sigma_j), \sigma) - f_0(y_0(\sigma_j), \sigma)]\, d\sigma\,.$$

According to (15,1) the norm of the first sum does not exceed

$$\zeta \sum_{j=0}^{s-1} \|x_0(\sigma_j) - y_0(\sigma_j)\| \leqq s\zeta e^{K_1 T} \|\tilde{x} - \tilde{y}\|\,.$$

The second sum and the third sum in (21,1) may be estimated as follows

$$\left\| \int_{\sigma_j}^{\sigma_{j+1}} [f(x_0, (\sigma), \sigma) - f(y_0(\sigma), \sigma) - f(x_0(\sigma_j), \sigma) + f(y_0(\sigma_j), \sigma)] \, d\sigma \right\| \leqq$$

$$\leqq \left\| \int_{\sigma_j}^{\sigma_{j+1}} [f(x_0(\sigma), \sigma) - f(x_0(\sigma_j) - y_0(\sigma_j) + y_0(\sigma), \sigma)] \, d\sigma \right\| +$$

$$+ \left\| \int_{\sigma_j}^{\sigma_{j+1}} [f(x_0(\sigma_j) - y_0(\sigma_j) + y_0(\sigma), \sigma) - f(y_0(\sigma), \sigma) -\right.$$

$$\left. - f(x_0(\sigma_j), \sigma) + f(y_0(\sigma_j), \sigma)] \, d\sigma \right\| \leqq$$

(cf. (2,1) and (12,1))

$$\leqq K_1 \int_{\sigma_j}^{\sigma_{j+1}} \|x_0(\sigma) - y_0(\sigma) - x_0(\sigma_j) + y_0(\sigma_j)\| \, d\sigma +$$

$$+ \|x_0(\sigma_j) - y_0(\sigma_j)\| \int_{\sigma_j}^{\sigma_{j+1}} \omega(\|y_0(\sigma) - y_0(\sigma_j)\|) \, d\sigma \leqq$$

$$\leqq K_1 \int_{\sigma_j}^{\sigma_{j+1}} \left\| \int_{\sigma_j}^{\sigma} [f_0(x_0(\lambda), \lambda) - f_0(y_0(\lambda), \lambda)] \, d\lambda \right\| d\sigma +$$

$$+ \|x_0(\sigma_j) - y_0(\sigma_j)\| \, . \, \omega(K_1 T/s) \, . \, T/s \leqq$$

$$\leqq (K_1 T/s) \, K_1 \int_{\sigma_j}^{\sigma_{j+1}} \|x_0(\lambda) - y_0(\lambda)\| \, d\lambda + \|x_0(\sigma_j) - y_0(\sigma_j)\| \, . \, \omega(K_1 T/s) \, . \, T/s \leqq$$

$$\leqq \|\tilde{x} - \tilde{y}\| \, e^{K_1 T} [K_1^2 T^2/s + T\omega(K_1 T/s)] \, . \, \frac{1}{s}$$

(as $\|x_0(\lambda) - y_0(\lambda)\| \leqq \|\tilde{x} - \tilde{y}\| \, e^{K_1 T}$). Therefore the norm of the second sum in (21,1) does not exceed $\|\tilde{x} - \tilde{y}\| \, e^{K_1 T}[K_1^2 T^2/s + T\omega(K_1 T/s)]$ and the same estimate holds for the third sum in (21,1) also. Therefore

$$\|I_3\| \leqq \|\tilde{x} - \tilde{y}\| \, e^{K_1 T}[s\zeta + 2K_1^2 T^2/s + 2T\omega(K_1 T/s)] \, .$$

Put $\chi_3(\zeta, T) = \min_{s \geqq T} e^{K_1 T}[s\zeta + 2K_1^2 T^2/s + 2T\omega(K_1 T/s)]$ (s being an integer). Obviously $\lim_{\zeta \to 0+} \chi_3(\zeta, T) = 0$ and

$$\|I_3\| \leqq \|\tilde{x} - \tilde{y}\| \, \chi_3(\zeta, T) \, . \tag{22,1}$$

From (18,1), (19,1), (20,1) and (22,1) it follows that

$$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq K_1 \int_0^\tau \|x(\sigma) - y(\sigma) - x_0(\sigma) + y_0(\sigma)\| \, d\sigma + \\ + \|\tilde{x} - \tilde{y}\| \cdot [K_1^{-1} e^{K_1 T} \omega(\chi_1(\zeta, T)) + \chi_3(\zeta, T)] \, .$$

Hence (16,1) holds, if

$$\chi_2(\zeta, T) = e^{K_1 T}[K_1^{-1} e^{K_1 T} \omega(\chi_1(\zeta, T)) + \chi_3(\zeta, T)]$$

and the proof of Theorem 2,1 is complete.

The following Theorem 3,1 is the main tool in section 2; in contradistinction to Theorem 2,1 it is not necessary to assume (15,1). Let by G_d, $d > 0$ be denoted the set of such $x \in G$ that $y \in X$, $\|y - x\| \leqq d$ implies that $y \in G$.

Theorem 3,1. *Let the numbers $K_1 > 0$, $d > 0$ and the function ω be given. There exists such a function $\chi_4(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ that*

(23,1) $$\lim_{\zeta \to 0+} \chi_4(\zeta, T) = 0$$

and the following assertion takes place:

Let $f, f_0 \in f_{(2)}$, let x, $y(x_0, y_0)$ be solutions of (3,1) ((5,1)) on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$ fulfilling (4,1) ((6,1)), $x(\tau)$, $y(\tau)$, $x_0(\tau)$, $y_0(\tau) \in G_d$ for $\tau \in \langle \tilde{\tau}, \tilde{\tau} + T \rangle$. Suppose that (8,1) takes place. Then

(24,1) $$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\| \, \chi_4(\zeta, T) \, , \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T \, .$$

Lemma 1,1. *Let the numbers $K_1 > 0$, $d > 0$ and the function ω be given. Then there exists a function $\chi_5(\zeta)$ defined for $\zeta > 0$, nondecreasing in ζ that*

(25,1) $$\lim_{\zeta \to 0+} \chi_5(\zeta) = 0$$

and that the following assertion takes place:

Let $f, f_0 \in f_{(2)}$ and let (8,1) hold. Then

(26,1) $$\left\| \int_{\tau_1}^{\tau_2} [f(z_2, \tau) - f(z_1, \tau) - f_0(z_2, \tau) + f_0(z_1, \tau)] \, d\tau \right\| \leqq \chi_5(\zeta) \, \|z_2 - z_1\|$$

for $z_1, z_2 \in G_d$, $\tau_1 \leqq \tau_2 \leqq \tau_1 + 1$.

Proof of Lemma 1,1. Put $\zeta_1 = d^2/4$ and define $\chi_5(\zeta) = 2\omega(2\zeta^{\frac{1}{2}}) + 3\zeta^{\frac{1}{2}}$ for $0 < \zeta \leqq \zeta_1$, $\chi_5(\zeta) = \max(2K_1, \chi_5(\zeta_1))$ for $\zeta > \zeta_1$. It is obvious that (25,1) holds and that (26,1) holds, if $\zeta > \zeta_1$ or if $\chi_5(\zeta) \|z_2 - z_1\| \geqq 2\zeta$ (cf. (8,1)). Let there exist ζ, z_1, z_2,

τ_1, τ_2 in such a way that $0 < \zeta \leqq \zeta_1$, $0 < \|z_2 - z_1\| \leqq 2\zeta(\chi_5(\zeta))^{-1}$, $z_1, z_2 \in G_d$, $\tau_1 < \tau_2 \leqq \tau_1 + 1$ and

$$\left\|\int_{\tau_1}^{\tau_2}[f(z_2, \tau) - f(z_1, \tau) - f_0(z_2, \tau) + f_0(z_1, \tau)]\, d\tau\right\| \geqq \chi_5(\zeta)\,\|z_2 - z_1\|\,.$$

It follows from (12,1) that

$$\begin{aligned}(27,1)\qquad &\|f(z_1 + (k+1)(z_2 - z_1), \tau) - f(z_1 + k(z_2 - z_1), \tau -\\ &- f(z_2, \tau) + f(z_1, \tau)\| \leqq \|z_2 - z_1\|\,\omega(k\|z_2 - z_1\|)\,,\\ &k = 1, 2, \ldots,\quad k\|z_2 - z_1\| \leqq d\,.\end{aligned}$$

Hence

$$\begin{aligned}(28,1)\qquad &\|f(z_1 + k(z_2 - z_1), \tau) - f(z_1, \tau) - k(f(z_2, \tau) - f(z_1, \tau))\| \leqq\\ &\leqq k\|z_2 - z_1\|\,\omega(k\|z_2 - z_1\|)\,,\quad k = 1, 2, \ldots,\quad k\|z_2 - z_1\| \leqq d\,.\end{aligned}$$

As (27,1) and (28,1) is fulfilled by f_0, it follows that

$$\begin{aligned}&\left\|\int_{\tau_1}^{\tau_2}[f(z_1 + k(z_2 - z_1), \tau) - f(z_1, \tau) - f_0(z_1 + k(z_2 - z_1), \tau) + f_0(z_1, \tau)]\, d\tau -\right.\\ &\left.- k\int_{\tau_1}^{\tau_2}[f(z_2, \tau) - f(z_1, \tau) - f_0(z_2, \tau) + f_0(z_1, \tau)]\, d\tau\right\| \leqq\\ &\leqq 2k\|z_2 - z_1\|\,\omega(k\|z_2 - z_1\|)\,,\quad k = 1, 2, \ldots,\quad k\|z_2 - z_1\| \leqq d\,.\end{aligned}$$

Therefore

$$\begin{aligned}(29,1)\qquad &\left\|\int_{\tau_1}^{\tau_2}[f(z_1 + k(z_2 - z_1), \tau) - f_0(z_1 + k(z_2 - z_1), \tau)]\, d\tau\right\| \geqq\\ &\geqq k\left\|\int_{\tau_1}^{\tau_2}[f(z_2, \tau) - f(z_1, \tau) - f_0(z_2, \tau) + f_0(z_1, \tau)]\, d\tau\right\| -\\ &- \left\|\int_{\tau_1}^{\tau_2}[f(z_1, \tau) - f_0(z_1, \tau)]\, d\tau\right\| - 2k\|z_2 - z_1\|\,\omega(k\|z_2 - z_1\|) \geqq\\ &\geqq k\|z_2 - z_1\|\,[2\omega(2\zeta^{\frac{1}{2}}) + 3\zeta^{\frac{1}{2}}] - \zeta - 2k\|z_2 - z_1\|\,\omega(k\|z_2 - z_1\|)\,,\\ &k = 1, 2, \ldots,\quad k\|z_2 - z_1\| \leqq d\,.\end{aligned}$$

As $0 < \zeta \leqq \zeta_1 = d^2/4$, $\|z_2 - z_1\| \leqq 2\zeta(\chi_5(\zeta))^{-1} < \zeta^{\frac{1}{2}}$, there exists an integer k' $d \geqq 2\zeta^{\frac{1}{2}} \geqq k'\|z_2 - z_1\| \geqq \zeta^{\frac{1}{2}}$ and (29,1) implies that

$$\begin{aligned}&\left\|\int_{\tau_1}^{\tau_2}[f(z_1 + k'(z_2 - z_1), \tau) - f_0(z_1 + k'(z_2 - z_1), \tau)]\, d\tau\right\| \geqq\\ &\geqq 3k'\|z_2 - z_1\|\,\zeta^{\frac{1}{2}} - \zeta \geqq 2\zeta\,.\end{aligned}$$

This contradiction proves Lemma 1,1.

Theorem 3,1 is a consequence of Lemma 1,1 and Theorem 2,1 applied on G_d, if we put $\chi_4(\zeta, T) = \chi_2(\chi_5(\zeta) + \zeta, T)$.

Theorems 1,1, 2,1 and 3,1 may be modified as follows. Let $f_{(3)}$ denote the class of functions f from $G \times E_1 \times E_1$ to X, which are continuous and fulfil the conditions

$$(30,1) \qquad \|f(x, \tau, \sigma)\| \leqq K_1 ,$$

$$(31,1) \qquad \|f(x_2, \sigma, \sigma) - f(x_1, \sigma, \sigma)\| \leqq K_1\|x_2 - x_1\| ,$$

$$(32,1) \qquad \|f(x, \tau, \sigma) - f(x, \sigma, \sigma)\| \leqq K_1|\tau - \sigma| ,$$

$$x, x_1, x_2 \in G , \quad \tau, \sigma \in E_1 , \quad \tau \leqq \sigma \leqq \tau + 1 .$$

Let $f_{(4)}$ be the subset of those $f \in f_{(3)}$ that

$$(33,1) \qquad \|f(x_2 + y, \sigma, \sigma) - f(x_1 + y, \sigma, \sigma) - f(x_2, \sigma, \sigma) + f(x_1, \sigma, \sigma)\| \leqq \leqq \|x_2 - x_1\| \, \omega(\|y\|) ,$$

$$(34,1) \qquad \|f(x_2, \sigma, \sigma) - f(x_1, \sigma, \sigma) - f(x_2, \tau, \sigma) + f(x_1, \tau, \sigma)\| \leqq \leqq \|x_2 - x_1\| \, . \, \omega(|\tau - \sigma|) ,$$

$$x_2, x_1, x_2 + y, x_1 + y \in G , \quad \tau \leqq \sigma \leqq \tau + 1 .$$

Theorem 4,1. *Let K_1, be given. There exists a function $\chi_6(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ such that*

$$(35,1) \qquad \lim_{\zeta \to 0+} \chi_6(\zeta, T) = 0$$

and the following assertion holds:

Let $f, f_0 \in f_{(3)}$, let x, x_0 be solutions of

$$(36,1) \qquad \frac{dx}{d\tau} = f(x, \tau, \tau) ,$$

$$(37,1) \qquad \frac{dx}{d\tau} = f_0(x, \tau, \tau)$$

on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$ and let

$$(38,1) \qquad \left\| \int_{\sigma_2}^{\sigma_1} [f(z, \sigma_1, \sigma) - f_0(z, \sigma_1, \sigma)] \, d\sigma \right\| \leqq \zeta \quad \textit{for} \quad z \in G, \ \sigma_1 \leqq \sigma_2 \leqq \sigma_1 + 1 .$$

Then

$$(39,1) \qquad \|x(\tau) - x_0(\tau)\| \leqq \chi_6(\zeta, T) , \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T .$$

Theorem 5,1. *Let the number* K_1, *and the function* ω *be given. There exists a function* $\chi_7(\zeta, T)$ *defined for* $\zeta > 0$, $T > 0$ *nondecreasing in* ζ *such that*

$$\lim_{\zeta \to 0+} \chi_7(\zeta, T) = 0 \tag{40,1}$$

and the following assertion takes place:

Let $f, f_0 \in f_{(4)}$, *let* $x, y(x_0, y_0)$ *be solutions of* (36,1) ((37,1)) *on* $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$, $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$, *let* (38,1) *be fulfilled and let*

$$\left\| \int_{\sigma_1}^{\sigma_2} [f(x_2, \sigma_1, \sigma) - f(x_1, \sigma_1, \sigma) - f_0(x_2(x_2, \sigma_1, \sigma) + f_0(x_1, \sigma_1, \sigma) \, d\sigma \right\| \leqq \tag{41,1}$$
$$\leqq \zeta \|x_2 - x_1\| , \quad x_1, x_2 \in G, \ \sigma_1 \leqq \sigma_2 \leqq \sigma_1 + 1 .$$

Then

$$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\| \cdot \chi_7(\zeta, T) , \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T . \tag{42,1}$$

Theorem 6,1. *Let the numbers* $K_1 > 0$, $d > 0$ *and the function* ω *be given. There exists a function* $\chi_8(\zeta, T)$ *defined for* $\zeta > 0$, $T > 0$ *nondecreasing in* ζ *such that*

$$\lim_{\zeta \to 0+} \chi_8(\zeta, T) = 0 \tag{43,1}$$

and the following assertion takes place:

Let $f, f_0 \in f_{(4)}$, *let* $x, y(x_0, y_0)$ *be solutions of* (36,1) ((37,1)) *on* $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$, $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$, $x(\tau), y(\tau), x_0(\tau), y_0(\tau) \in G_d$ *for* $\tau \in \langle \tilde{\tau}, \tilde{\tau} + T \rangle$ *and let* (38,1) *be fulfilled. Then*

$$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\| \, \chi_8(\zeta, T) . \tag{44,1}$$

As the proofs of Theorems 4,1, 5,1 and 6,1 are simple modifications of the ones of Theorems 1,1, 2,1 and 3,1, they are omitted.

Note 3,1. Let

$$\frac{dr}{dt} = \varepsilon v(r, \psi, t) , \quad \frac{d\psi}{dt} = \omega + \varepsilon w(r, \psi, t) \tag{45,1}$$

be given. Suppose that v (w) is a continuous map from $G_1 \times E_m \times E_1$ to E_n (to E_m), G_1 being an open subset of E_n, $v(r, \psi + e_i, t) = v(r, \psi, t) = v(r, \psi, t + 1)$, $w(r, \psi + e_i, t) = w(r, \psi, t) = w(r, \psi, t + 1)$, $e_i = (\delta_{i1}, \ldots, \delta_{im})$, $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$, $i, j = 1, 2, \ldots, m$, the derivatives $\partial v/\partial r$, $\partial v/\partial \psi$, $\partial w/\partial r$, $\partial w/\partial \psi$ are continuous, (45,1) is transformed by $\psi = \varphi + \omega t$, $\varepsilon t = \tau$ to

$$\frac{dr}{d\tau} = v\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}\right), \quad \frac{d\varphi}{d\tau} = w\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}\right). \tag{46,1}$$

Put

$$v_0(r, \varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T v(r, \varphi + \omega t, t)\, dt\,, \quad w_0(r, \varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T w(r, \varphi + \omega t, t)\, dt\,.$$

Then to every compact subset $Q \subset G_1$ and $\zeta > 0$ there exists an $\varepsilon_0 > 0$ that

$$\left\| \int_{\tau_1}^{\tau_2} \left[v\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}\right) - v_0(r, \varphi) \right] d\tau \right\| \leqq \zeta\,,$$

$$\left\| \int_{\tau_1}^{\tau_2} \left[w\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}\right) - w_0(r, \varphi) \right] d\tau \right\| \leqq \zeta$$

for $r \in Q$, $\varphi \in E_m$, $0 \leqq \tau_2 - \tau_1 \leqq 1$, $0 < \varepsilon \leqq \varepsilon_0$ and Theorems 1,1 and 3,1 may be applied to (46,1) and

$$(47,1) \qquad \frac{dr}{d\tau} = v_0(r, \varphi)\,, \quad \frac{d\varphi}{d\tau} = w_0(r, \varphi)\,.$$

In a similar way

$$(48,1) \qquad \frac{dr}{dt} = \varepsilon v^*(r, \psi, t, \varepsilon t)\,, \quad \frac{d\psi}{dt} = \omega + \varepsilon w^*(r, \psi, t, \varepsilon t)$$

may be treated provided that v^* (w^*) is a bounded continuous map from $G_1 \times E_m \times \times E_1 \times E_1$ to E_n (to E_m) the derivatives $\partial v^*/\partial r$, $\partial v^*/\partial \psi$, $\partial w^*/\partial r$, $\partial w^*/\partial \psi$ are bounded and uniformly continuous and $v^*(., ., ., \sigma)$ and $w^*(., ., ., \sigma)$ fulfil the same periodicity conditions as v and w, σ being fixed. (48,1) is transformed to

$$(49,1) \quad \frac{dr}{d\tau} = v^*\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}, \tau\right), \quad \frac{d\varphi}{d\tau} = w^*\left(r, \varphi + \omega \frac{\tau}{\varepsilon}, \frac{\tau}{\varepsilon}, \tau\right)$$

and Theorems 4,1 and 6,1 may be applied to (49,1) and

$$(50,1) \qquad \frac{dr}{d\tau} = v_0^*(r, \varphi, \tau)\,, \quad \frac{d\varphi}{d\tau} = w_0^*(r, \varphi, \tau)\,,$$

$$v_0^*(r, \varphi, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T v^*(r, \varphi + \omega t, t, \tau)\, dt\,,$$

$$w_0^*(r, \varphi, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T w^*(r, \varphi + \omega t, t, \tau)\, dt\,.$$

Let $f_j \in f_{(2)}$, $j = 1, 2, 3, \ldots$, $\zeta_j \to 0$ with $j \to \infty$ and let

$$\left\| \int_{\tau_1}^{\tau_2} [f_j(z, \tau) - f_k(z, \tau)]\, d\tau \right\| \leqq \zeta_j\,,$$

$$\left\| \int_{\tau_1}^{\tau_2} [f_j(z_2, \tau) - f_j(z_1, \tau) - f_k(z_2, \tau) + f_k(z_1, \tau)]\, d\tau \right\| \leqq \zeta_j \|z_2 - z_1\|$$

for $z, z_1, z_2 \in G$, $\tau_1 \leqq \tau_2 \leqq \tau_1 + 1$, $k \geqq j$. Let the solutions $x_j, y_j, x_j(\tilde{\tau}) = \tilde{x}$ $y_j(\tilde{\tau}) = \tilde{y}, j = 1, 2, \dots$ of

$$\frac{dx}{d\tau} = f_j(x, \tau)$$

be defined on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$. According to Theorem 1,1 $x_j(\tau)$ and $y_j(\tau)$ $\tau \in \langle \tilde{\tau}, \tilde{\tau} + T \rangle$ are Cauchy-sequences. Put $x^*(\tau) = \lim_{j\to\infty} x_j(\tau)$, $y^*(\tau) = \lim_{j\to\infty} y_j(\tau)$. According to Theorem 2,1

$$\|x^*(\tau) - y^*(\tau) - x_j(\tau) + y_j(\tau)\| \leqq \chi_2(\zeta_j, T) \,.\, \|\tilde{x} - \tilde{y}\| \,.$$

Nevertheless there need not exist a $f^* \in f_{(2)}$ that x^*, y^* are solutions of

$$\frac{dx}{d\tau} = f^*(x, \tau) \,.$$

Theorems 1,1, 2,1 and 3,1 may be extended in such a way that they are closed to the above limiting process. Put

$$\Delta_\tau^\sigma F(x, \tau) = F(x, \tau + \sigma) - F(x, \tau) \,, \quad \Delta_x^z F(x, \tau) = F(x + z, \tau) - F(x, \tau) \,.$$

Let $F_{(1)}$ be the set of functions F from $G \times E_1$ to X which fulfil the conditions

(51,1) $$\|\Delta_\tau^\sigma F(x, \tau)\| \leqq K_1 \sigma \,, \quad \|\Delta_\tau^\sigma \Delta_x^z F(x, \tau)\| \leqq K_1 \|z\| \,.\, \sigma$$

for $x, x + z \in G$, $\tau \in E_1$, $\sigma \geqq 0$.

Let $F \in F_{(1)}$. If v is a continuous function from $\langle \alpha, \beta \rangle$ to G, $\alpha, \beta \in E_1$, define $\int_{\alpha_1}^{\beta_1} D_\sigma F(v(\tau), \sigma)$ as the limit of the partial sums $\sum_{i=1}^{j} [F(v(\tau_i), \sigma_i) - F(v(\tau_i), \sigma_{i-1})]$, $\alpha \leqq \alpha_1 = \sigma_0 \leqq \tau_1 \leqq \sigma_1 \leqq \tau_2 \leqq \dots \leqq \tau_j \leqq \sigma_j = \beta_1 \leqq \beta$. This limit exists in the same sense as in the theory of the Riemann integral. The function (from $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$ to G) is said to be a solution of

(52,1) $$\frac{dx}{d\tau} = D\, F(x, \tau),$$

if it is continuous and if $x(\tau_2) - x(\tau_1) = \int_{\tau_1}^{\tau_2} D_\sigma F(x(\tau), \sigma)$, $\tau_1, \tau_2 \in \langle \tilde{\tau}, \tilde{\tau} + T \rangle$. It is verified easily that

$$\left\| \int_{\tau_1}^{\tau_2} D_\sigma F(u(\tau), \sigma) - \int_{\tau_1}^{\tau_2} D_\sigma F(v(\tau), \sigma) \right\| \leqq K_1 \int_{\tau_1}^{\tau_2} \|u(\sigma) - v(\sigma)\| \, d\sigma \,,$$

if u, v are continuous functions from $\langle \tau_1, \tau_2 \rangle$ to G. Therefore existence and uniqueness of solutions of (52,1) is established by means of successive approximations similarly as for equation (3,1). If u has the same meaning as above, $f \in f_{(1)}$, $F(x, \tau) =$

$= \int_0^\tau f(x, \sigma)\, d\sigma$ then $F \in F_{(1)}$ and $\int_{\tau_1}^{\tau_2} D_\sigma F(u(\tau), \sigma) = \int_{\tau_1}^{\tau_2} f(u(\tau), \tau)\, d\tau$; therefore every solution of (3,1) is simultaneously a solution of (52,1) and the converse also holds, as the solutions of (52,1) are unique. Theorem 1,1 may be given in the following form:

Theorem 7,1. *Let K_1 be given. There exists a function $\chi_1(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ such that $\lim_{\zeta \to 0+} \chi_1(\zeta, T) = 0$ and the following assertion takes place:*

Let $F, F_0 \in F_{(1)}$, let x be a solution of (52,1) let x_0 be a solution of

$$\frac{dx}{d\tau} = D F_0(x, \tau) \tag{53,1}$$

on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $T > 0$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$ and let

$$\|\Delta_\tau^\sigma [F(z, \tau) - F_0(z, \tau)]\| \leqq \zeta \quad \text{for} \quad z \in G,\ \tau \in E_1,\ 0 \leqq \sigma \leqq 1\,. \tag{54,1}$$

Then

$$\|x(\tau) - x_0(\tau)\| \leqq \chi_1(\zeta, T)\,, \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T\,.$$

Let $F_{(2)}$ be the set of those $F \in F_{(1)}$ that

$$\|\Delta_\tau^\sigma \Delta_x^u \Delta_x^v\, F(x, \tau)\| \leqq \omega(\|u\|) \,.\, \|v\| \,.\, \sigma \tag{55,1}$$

for $x, x + v, x + u, x + u + v \in G$, $\tau \in E_1$, $0 \leqq \sigma \leqq 1$, ω having the same meaning as in Theorem 2,1. Theorem 2,1 may be extended as follows:

Theorem 8,1. *Let the number K_1, and the function ω be given. There exists a function $\chi_2(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ such that $\lim_{\zeta \to 0+} \chi_2(\zeta, T) = 0$ and the following assertion holds:*

Let $F, F_0 \in F_{(2)}$, let x, y (x_0, y_0) be solutions of (52,1) ((53,1)) on $\langle \tilde{\tau}, \tilde{\tau} + T \rangle$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$, $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$, let (54,1) hold and let

$$\|\Delta_\tau^\sigma \Delta_x^z [F(x, \tau) - F_0(x, \tau)]\| \leqq \zeta \|z\| \quad \text{for} \quad x, x + z \in G\,, \quad \tau \in E_1\,, \quad 0 \leqq \sigma \leqq 1\,. \tag{56,1}$$

Then

$$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\|\, \chi_2(\zeta, T),\ \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T\,. \tag{57,1}$$

Theorem 9,1. *Let the numbers $K_1 > 0$, $d > 0$ and the function ω be given. There exists a function $\chi_4(\zeta, T)$ defined for $\zeta > 0$, $T > 0$ nondecreasing in ζ such that $\lim_{\zeta \to 0+} \chi_4(\zeta, T) = 0$ and the following assertion takes place:*

Let $F, F_0 \in F_{(2)}$ *let* $x, y(x_0, y_0)$ *be solutions of* (52,1) ((53,1)) *on* $\langle\tilde{\tau}, \tilde{\tau} + T\rangle$, $x(\tilde{\tau}) = \tilde{x} = x_0(\tilde{\tau})$, $y(\tilde{\tau}) = \tilde{y} = y_0(\tilde{\tau})$, $x(\tau), y(\tau), x_0(\tau), y_0(\tau) \in G_a$ *for* $\tau \in \langle\tilde{\tau}, \tilde{\tau} + T\rangle$ *and let* (54,1) *hold. Then*

$$\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq \|\tilde{x} - \tilde{y}\|\, \chi_4(\zeta, T) \quad \text{for} \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T\,.$$

The proofs of Theorems 7,1, 8,1 and 9,1 are omitted as they are quite analogous to those of Theorems 1,1,2,1 and 3,1. Theorems 6 and 7 announced in [3] follow from Theorems 7,1 and 8,1. If the functions $F \in F_{(1)}$ are normed by the condition $F(x, 0) = 0$, then from $\lim_{i\to\infty, j\to\infty} \sup_{0 \leqq \sigma \leqq 1} \|\Delta_\tau^\sigma[F_i(x, \tau) - F_j(x, \tau)]\| = 0$ follows that $\lim_{i\to\infty, j\to\infty} \|F_i(x, \tau) - F_j(x, \tau)\| = 0$. Put $F^*(x, \tau) = \lim_{i\to\infty} F_i(x, \tau)$. If $F_i \in F_{(1)}$, $i = 1, 2, \ldots$ then $F^* \in F_{(1)}$, if $F_i \in F_{(2)}$, $i = 1, 2, \ldots$, then $F^* \in F_{(2)}$ and Theorems 7,1, 8,1 and 9,1 are closed with respect to the above limiting process.

2. INTEGRAL MANIFOLDS

The purpose of this section is to establish existence theorems for integral manifolds. First a theorem (on the existence of integral manifolds) is formulated for differential equations in Banach spaces (Theorem 1,2). Then the concept of the differential equation is replaced by a more general concept of a flow (a flow is a set of functions, which fulfil some conditions, which are always fulfilled by the set of solutions of a differential equation). The existence of integral manifolds is proved in the case of flows (Theorem 2,2) and the properties of integral manifolds are studied (Theorems 3,2–5,2, Note 6,2). Theorem 1,2 is deduced from Theorem 2,2 by means of a Stability Lemma (Lemma 11,1) and a specialization to a finitedimensional system is made (Note 8,2). In a similar manner the existence of integral manifolds is established for generalized differential equations (Theorem 6,2).

Let $X = C \times \mathscr{C}$, $C, \mathscr{C}$ being Banach spaces. For $x \in X$, $c \in C$, $\gamma \in \mathscr{C}$ let $\|x\|$, $\|c\|$, $\|\gamma\|$ denote the norm of x, c, γ in the respective space. Without the loss on generality we may assume that

$$\|x\| = \|c\| + \|\gamma\|\,, \quad x = (c, \gamma) \in X\,. \tag{1,2}$$

Let $G = \mathscr{E}[(c, \gamma), c \in C, \|c\| < \varkappa, \gamma \in \mathscr{C}]$, $\varkappa > 0$ being fixed. Let equations (3,1), (5,1) be given in the form

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = a(c, \gamma, \tau)\,, \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \alpha(c, \gamma, \tau)\,, \tag{2,2}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = a_0(c, \gamma, \tau)\,, \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \alpha_0(c, \gamma, \tau)\,. \tag{3,2}$$

Solutions of (2,2) ((3,2)) will be denoted by (c, γ), $((c_0, \gamma_0))$, their values by $c(\tau)$,

$\gamma(\tau)\,(c_0(\tau), \gamma_0(\tau))$; the initial conditions will be usually written in the form $c(\tilde{\tau}) = \tilde{c} = c_0(\tilde{\tau}), \gamma(\tilde{\tau}) = \tilde{\gamma} = \gamma_0(\tilde{\tau})$.

Theorem 1,2. *Let the functions $f = (a, \alpha)$, $f_0 = (a_0, \alpha_0)$ fulfil the following conditions*:

$$(4,2) \qquad \|f(x, \tau)\| \leqq K_1\,, \quad \|f_0(x, \tau)\| \leqq K_1\,, \quad x = (c, \gamma) \in G\,, \quad \tau \in E_1\,,$$

(5,2) *f, f_0 are continuous in (x, τ), differentiable with respect to x and*

$$(6,2) \qquad \left\|\frac{\partial f}{\partial x}(x, \tau)\right\| \leqq K_1\,, \quad \left\|\frac{\partial f_0}{\partial x}(x, \tau)\right\| \leqq K_1\,, \quad x \in G\,, \quad \tau \in E_1\,,$$

$$\left\|\frac{\partial f}{\partial x}(x_2, \tau) - \frac{\partial f}{\partial x}(x_1, \tau)\right\| \leqq \omega(\|x_2 - x_1\|)\,,$$

$$\left\|\frac{\partial f_0}{\partial x}(x_1, \tau) - \frac{\partial f_0}{\partial x}(x_2, \tau)\right\| \leqq K_1\|x_2 - x_1\|^\mu\,, \quad x_2, x_1 \in G\,, \quad \tau \in E_1$$

$\mu > 0$ being fixed, ω being a nondecreasing function, $\omega(\xi) \geqq K_1\xi^\mu$ for $\xi \geqq 0$, $\lim_{\xi \to 0} \omega(\xi) = 0$.

Let (8,1) be fulfilled. Suppose in addition that

$$(7,2) \qquad a_0(0, \gamma, \tau) = 0\,, \quad \alpha_0(0, \gamma, \tau) = \alpha^*(\tau) \quad \text{is independent of } \gamma\,.$$

Put $A(\gamma, \tau) = \partial a_0/\partial c\,(0, \gamma, \tau)$ and suppose that the solutions of the linear equation

$$(8,2) \qquad \frac{\mathrm{d}c}{\mathrm{d}\tau} = A\left(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,\mathrm{d}\sigma + \delta, \tau\right) c\,, \quad \tilde{\tau} \in E_1\,, \quad \delta \in \mathscr{C}$$

may be estimated by

$$(9,2) \qquad \|c(\tau)\| \leqq K_1 e^{-\nu(\tau - \tilde{\tau})}\,.\,\|\tilde{c}\|\,, \quad \tau \geqq \tilde{\tau}\,, \quad c(\tilde{\tau}) = \tilde{c}\,, \quad \nu > 0\,,$$

K_1 and ν being independent of $\tilde{c}$, $\tilde{\tau}$ and δ.

Then there exist positive constants $\zeta_1, \varkappa_2, L, K', \nu', K''$ depending on $K_1, \varkappa, \mu, \nu$ only, $\varkappa_2 < \varkappa$, $L \leqq (6K_1)^{-1}$ in such a way that $0 < \zeta \leqq \zeta_1$ implies that there exists a map p from $\mathscr{C} \times E_1$ to C and the following assertions hold:

(i) $\|p(\gamma, \tau)\| \leqq \varkappa_2$, $\gamma \in \mathscr{C}$, $\tau \in E_1$,

(ii) $\|p(\gamma, \tau) - p(\beta, \tau)\| \leqq L\|\gamma - \beta\|$, $\gamma, \beta \in \mathscr{C}$, $\tau \in E_1$,

(iii) *If $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, $\tilde{c} = p(\tilde{\gamma}, \tilde{\tau})$, then the solution (c, γ) of (2,2), $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ exists for $\tau \in E_1$ and $c(\tau) = p(\gamma(\tau), \tau)$ for $\tau \in E_1$.*

(iv) *If* $\tilde{c} \in C$, $\|\tilde{c}\| \leqq \varkappa_2$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, *then the solution* (c, γ) *of* (2,2), $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ *exists on* $\langle\tilde{\tau}, \infty)$ *and* $\|c(\tau)\| \leqq 3\varkappa/4$, $\|c(\tau) - p(\gamma(\tau), \tau)\| \leqq K' e^{-\nu'(\tau - \tilde{\tau})}$. $\cdot \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|$ *for* $\tau \geqq \tilde{\tau}$.

(v) *If* p' *is a map from* $\mathscr{C} \times E_1$ *to* C, *which fulfils* (i) *and* (iii), *then* $p' = p$.

(vi) *If* (c, γ) *has the same meaning as in* (iv), *then there exists a solution* (b, β) *of* (2,2) *on* E_1, $b(\tau) = p(\beta(\tau), \tau)$ *for* $\tau \in E_1$ *and* $\|c(\tau) - b(\tau)\| + \|\gamma(\tau) - \beta(\tau)\| \leqq$ $\leqq K'' e^{-\nu'(\tau - \tilde{\tau})} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|$ *for* $\tau \geqq \tilde{\tau}$.

(vii) p *is uniformly continuous in* γ, τ.

Note 1,2. According to (iii) the set of such $(c, \gamma, \tau) \in X \times E_1$ that $c = p(\gamma, \tau)$ may be interpreted as an integral manifold of (2,2). Solutions of (2,2) lying on this manifold fulfil obviously

$$\frac{d\gamma}{d\tau} = \alpha(p(\gamma, \tau), \gamma, \tau) .$$

Note 2,2. (8,2) is equivalent to

$$\frac{dc}{d\tau} = A(\gamma, \tau)\, c\,, \quad \frac{d\gamma}{d\tau} = \alpha^*(\tau) .$$

Note 3,2. If $f_0 = (a_0, \alpha_0)$ does not depend on τ, then the notations are simplified: $A(\gamma, \tau) = A(\gamma) = \partial o_0/\partial c\,(0, \gamma)$, α^* is a constant and (8,2) is replaced by $dc/d\tau = A(\alpha^*(\tau - \tilde{\tau}) + \delta)\, c$, $\delta \in \mathscr{C}$, $\tilde{\tau} \in E_1$.

Note 4,2. Let $K_1, \varkappa, \mu, \nu$ be given. There exist positive functions $\bar{L}(\zeta)$, $\bar{\varkappa}_1(\zeta)$ on $(0, \zeta_1\rangle$ such that $\lim_{\zeta \to 0+} \bar{L}(\zeta) = 0$, $\lim_{\zeta \to 0+} \bar{\varkappa}_2(\zeta) = 0$ and the following assertion holds: if $0 < \zeta \leqq \zeta_1$ and if all assumptions of Theorem 1,2 are fulfilled, then the map p fulfils (i) and (ii) with $\varkappa_2$ and L replaced by $\bar{\varkappa}_2(\zeta)$ and $\bar{L}(\zeta)$. This situation is a consequence of Theorem 1, 2, as to every set of positive numbers $K_1^*, \varkappa^*, \mu, \nu$ there exist positive numbers $\zeta_1^*, \varkappa_2^*, L^*, K'^*, \gamma^{*\prime}, K''^*, \varkappa_2^* < \varkappa^*, L^* \leqq (6K_1^*)^{-1}$ that Theorem 1,2 holds, if $K_1, \varkappa, \zeta_1, \varkappa_2, L, K', \gamma', K''$ are replaced by $K_1^*, \varkappa^*, \zeta_1^*, \varkappa_2^*, L^*, K'^*, \gamma'^*, K''^*$.

Let X, C, $\mathscr{C}$ have the same meaning as above. Let $\hat{G}$ be a subset of X. Let $\mathscr{X}$ be a set of functions x from intervals $J(x)$ in E_1 to $\hat{G}$ (each $x \in \mathscr{X}$ is defined on an interval $J(x)$ (closed, open or halfclosed, bounded or unbounded), which may vary with x). $\mathscr{X}$ will be called a flow (in $\hat{G}$), if the following conditions are fulfilled:

(I) If $x \in \mathscr{X}$, $J \subset J(x)$, J being an interval, and if y is defined by $y(\tau) = x(\tau)$ for $\tau \in J$, then $y \in \mathscr{X}$ $(J(y) = J)$.

(II) To every $\tilde{x} \in \hat{G}$ and $\tilde{\tau} \in E_1$ there exists an $x \in \mathscr{X}$, $J(x) = \langle\tilde{\tau}, \tilde{\tau}_1\rangle$, $\tilde{\tau} < \tilde{\tau}_1$, $x(\tilde{\tau}) = \tilde{x}$.

(III) Let $x_i \in \mathscr{X}$, $i = 1, 2$; if there is $\tau_1 \in J(x_1) \cap J(x_2)$ that $x_1(\tau_1) = x_2(\tau_1)$ then $x_1(\tau) = x_2(\tau)$ for $\tau \in J(x_1) \cap J(x_2) \cap \langle \tau_1, \infty)$.

(IV) Let $x_i \in \mathscr{X}$, $\tilde{\tau} \in J(x_i)$, $x_i(\tau) = x_j(\tau)$ for $\tau \in J(x_i) \cap J(x_j)$, $i, j = 1, 2, 3, \ldots$ Then there exists an $x \in \mathscr{X}$, $J(x) = \bigcup_i J(x_i)$, $x(\tau) = x_i(\tau)$ for $\tau \in J(x_i)$, $i = 1, 2, \ldots$

(V) Let $x \in \mathscr{X}$, $J(x) = \langle \tau_1, \tau_2)$, $\tau_2 > \tau_1$, $\lambda > 0$ and let $z \in X$, $\tau \in \langle \tau_1, \tau_2)$, $\|z - x(\tau)\| < \lambda$ imply that $z \in \hat{G}$. Then there exists a $y \in X$, $J(y) = \langle \tau_1, \tau_2 \rangle$, $y(\tau) = x(\tau)$ for $\tau \in J(x)$.

The elements of a flow $\mathscr{X}$ will be called solutions, as they have many properties in common with the solutions of differential equations. A solution $x \in \mathscr{X}$ will be said to exist on $\langle \tau_1, \tau_2 \rangle$, if $J(x) = \langle \tau_1, \tau_2 \rangle$. One of these properties is stated explicitely in the following Lemma.

Lemma 1,2. *Let $\mathscr{X}$ be a flow, let x_0 be a map from $\langle \tau_1, \tau_2 \rangle$ to $\hat{G}$, $\tau_1 < \tau_2$, $\xi_2 > \xi_1 > 0$ and let the following assumptions be fulfilled:*

(10,2) $$z \in X,\ \tau \in \langle \tau_1, \tau_2 \rangle,\ \|z - x_0(\tau)\| \leqq \xi_2 \quad \textit{imply that} \quad z \in \hat{G}\,,$$

(11,2) $$x \in X,\ \tau_1 \in J(x),\ x(\tau_1) = x_0(\tau_1) \quad \textit{imply that} \quad \|x(\tau) - x_0(\tau)\| \leqq \xi_1$$
$$\textit{for} \quad \tau \in J(x) \cap \langle \tau_1, \tau_2 \rangle\,.$$

Then there exists a $y \in \mathscr{X}$ such that $J(y) = \langle \tau_1, \tau_2 \rangle$, $y(\tau_1) = x_0(\tau_1)$ (and $\|y(\tau) - x_0(\tau)\| \leqq \xi_1$ for $\tau \in \langle \tau_1, \tau_2 \rangle$).

The proof is omitted, as it is quite analogous to the one in case of differential equations. In the proof conditions (I)–(V) are needed.

Let $\mathscr{X}$, $\mathscr{X}_0$ be flows, $T > 0$. Let T-distance $d_T(\mathscr{X}, \mathscr{X}_0)$ of $\mathscr{X}$, $\mathscr{X}_0$ be defined by

(12,2) $$d_T(\mathscr{X}, \mathscr{X}_0) = \sup_{x, x_0, \tau} \|x(\tau) - x_0(\tau)\| +$$
$$+ \sup_{y, z, y_0, z_0} \|\tilde{y} - \tilde{z}\|^{-1} \|y(\tau) - z(\tau) - y_0(\tau) + z_0(\tau)\|\,,$$
$$x, y, z \in \mathscr{X}\,,\ x_0, y_0, z_0 \in \mathscr{X}_0\,,$$
$$J(x) = J(y) = J(z) = J(x_0) = J(y_0) = J(z_0) = \langle \tau_1, \tau_2 \rangle\,,$$
$$\tau_1 < \tau_2 \leqq \tau_1 + T\,, \quad \tau \in \langle \tau_1, \tau_2 \rangle\,,$$
$$x(\tau_1) = x_0(\tau_1)\,, \quad y(\tau_1) = y_0(\tau_1) = \tilde{y}\,, \quad z(\tau_1) = z_0(\tau_1) = \tilde{z}\,.$$

Theorem 2,2. *Let the flow $\mathscr{X}_0$ (in $\hat{G}$) fulfil the following conditions:*

(Ω_1) *To every $\tilde{y} \in \mathscr{C}$, $\tilde{\tau} \in E_1$ there exists a solution $(c_0, y_0) \in \mathscr{X}_0$, $J(c_0, y_0) = \langle \tilde{\tau}, \infty)$, $c_0(\tau) = 0$ for $\tau \geqq \tilde{\tau}$, $y_0(\tilde{\tau}) = \tilde{y}$.*

(Ω_2) *There exist* $\hat{K}_2 > 1$ *and* $\nu_1 > 0$ *that to every* $\tilde{c} \in C$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, $\|\tilde{c}\| < \hat{K}_2^{-1}\hat{\varkappa}$ *there exists a solution* $(c_0, \gamma_0) \in \mathscr{X}_0$, $J(c_0, \gamma_0) = \langle\tilde{\tau}, \infty)$, $c_0(\tilde{\tau}) = \tilde{c}$, $\gamma_0(\tilde{\tau}) = \tilde{\gamma}$ *and*

$$\|c_0(\tau)\| \leqq \hat{K}_2 e^{-\nu_1(\tau-\tilde{\tau})}\|\tilde{c}\|\,, \quad \tau \geqq \tilde{\tau}\,. \tag{13,2}$$

(Ω_3) *To every* ϱ, $0 < \varrho \leqq 1$ *there exists a* $\varkappa_1$, $0 < \varkappa_1 \leqq (2\hat{K}_2)^{-1}\,\hat{\varkappa}$ *that the estimates*

$$\|c_0(\tau) - b_0(\tau)\| \leqq \hat{K}_2 e^{-\nu_1(\tau-\tilde{\tau})}[\|\tilde{c} - \tilde{b}\| + \|\tilde{\gamma} - \tilde{\beta}\|]\,, \quad \tau \geqq \tilde{\tau}\,, \tag{14,2}$$

$$\|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| \leqq \hat{K}_2[\|\tilde{c} - \tilde{b}\| + \varrho\|\tilde{\gamma} - \tilde{\beta}\|]\,, \quad \tau \geqq \tilde{\tau} \tag{15,2}$$

hold for any couple of solutions $(c_0, \gamma_0), (b_0, \beta_0) \in \mathscr{X}_0$, $c_0(\tilde{\tau}) = \tilde{c}$, $\gamma_0(\tilde{\tau}) = \tilde{\gamma}$, $b_0(\tilde{\tau}) = \tilde{b}$, $\beta_0(\tilde{\tau}) = \tilde{\beta}$, $J(c_0, \gamma_0) = J(b_0, \beta_0) = \langle\tilde{\tau}, \infty)$, $\|\tilde{c}\|, \|\tilde{b}\| \leqq \varkappa_1$.

Then there exist positive constants $D = (72\hat{K}_2)^{-1}$, $L = (6\hat{K}_2)^{-1}$, T, $\varkappa_2$, K', ν', K'' *depending on* $\hat{\varkappa}$, $\hat{K}_2$, ν_1 *only*, $\varkappa_2 \leqq \frac{1}{2}\hat{\varkappa}$, *that to every flow* $\mathscr{X}$ *for which* $d_T(\mathscr{X}, \mathscr{X}_0) \leqq D$ *there exist a map* p *from* $\mathscr{C} \times E_1$ *to* C *and the following assertions hold*:

(i′) $\|p(\gamma, \tau)\| \leqq \varkappa_2$, $\gamma \in \mathscr{C}$, $\tau \in E_1$.

(ii′) $\|p(\gamma, \tau) - p(\beta, \tau)\| \leqq L\|\gamma - \beta\|$, $\gamma, \beta \in \mathscr{C}$, $\tau \in E_1$.

(iii′) *If* $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, $\tilde{c} = p(\tilde{\gamma}, \tilde{\tau})$, *then there exists a* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = E_1$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ *and* $c(\tau) = p(\gamma, \tau), \tau)$ *for* $\tau \in E_1$.

(iv′) *If* $\tilde{c} \in C$, $\|\tilde{c}\| \leqq \varkappa_2$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, *then there exists a* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle\tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ *and*
$\|c(\tau)\| \leqq \frac{3}{4}\hat{\varkappa}$, $\|c(\tau) - p(\gamma(\tau), \tau)\| \leqq K' e^{-\nu'(\tau-\tilde{\tau})}\|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|\,, \quad \tau \geqq \tilde{\tau}$

(v′) *If* p' *is a map from* $\mathscr{C} \times E_1$ *to* C, *which fulfils* (i′)) *and* (iii′), *then* $p' = p$.

(vi′) *If* (c, γ) *has the same meaning as in* (iv′), *then there exists a solution* $(b, \beta) \in \mathscr{X}$, $J(b, \beta) = E_1$, $b(\tau) = p(\beta(\tau), \tau)$ *for* $\tau \in E_1$ *and*
$\|c(\tau) - b(\tau)\| + \|\gamma(\tau) - \beta(\tau)\| \leqq K'' e^{-\nu'(\tau-\tilde{\tau})}\|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|\,, \quad \tau \geqq \tilde{\tau}\,.$

(vii′) *If solutions* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = E_1$, $c(\tau) = p(\gamma(\tau), \tau)$ *for* $\tau \in E_1$ *are equicontinuous, then* $p(\gamma, \tau) = p(\tau)(\gamma)$ *is uniformly continuous.*

In order to prove Theorem 2,2 several Lemmas will be needed.

Lemma 2,2. *Let* ϑ *be a map from* $\mathscr{C}$ *into* $\mathscr{C}$ *and let*

$$l_2^{-1}\|\gamma - \beta\| \leqq \|\vartheta(\gamma) - \vartheta(\beta)\| \leqq l_2\|\gamma - \beta\|\,, \quad \gamma, \beta \in \mathscr{C}\,, \tag{16,2}$$

$$\|\vartheta(\gamma) - \vartheta(\beta) - \gamma + \beta\| \leqq l_1\|\gamma - \beta\|\,, \quad \gamma, \beta \in \mathscr{C} \tag{17,2}$$

with $0 < l_1 < 1 < l_2$. *Then* ϑ *maps* $\mathscr{C}$ *onto itself.*

Proof. Choose $\delta \in \mathscr{C}$, put $\gamma_1 = \delta$ and efine

$$\gamma_{i+1} = \gamma_i + \delta - \vartheta(\gamma_i)\,, \quad i = 1, 2, 3, \ldots$$

As $\|\vartheta(\gamma_{i+1}) - \vartheta(\gamma_i) - \gamma_{i+1} + \gamma_i\| \leqq l_1\|\gamma_{i+1} - \gamma_i\|$,

$$\gamma_{i+1} - \gamma_i = \delta - \vartheta(\gamma_i), \quad \vartheta(\gamma_{i+1}) - \vartheta(\gamma_i) - \gamma_{i+1} + \gamma_i = \vartheta(\gamma_{i+1}) - \delta,$$

we obtain $\|\vartheta(\gamma_{i+1}) - \delta\| \leqq l_1\|\vartheta(\gamma_i) - \delta\|$ and $\vartheta(\gamma_i) \to \delta$ with $i \to \infty$. (16,2) implies that γ_i is a Cauchy sequence, hence $\gamma_i \to \gamma$ and $\vartheta(\gamma_i) \to \delta = \vartheta(\gamma)$.

Let us choose

$$(18,2) \qquad \varrho = (3\hat{K}_2)^{-1}, \quad L = (6\hat{K}_2)^{-1}, \quad D = (72\hat{K}_2)^{-1}$$

and let $\varkappa_2$ be the number $\varkappa_1$ that corresponds to $\varrho = (3\hat{K}_2)^{-1}$ according to condition (Ω_3) $(\varkappa_2 \leqq (2\hat{K}_2)^{-1}\,\hat{\varkappa} < \frac{1}{2}\hat{\varkappa})$.

Lemma 3,2. *Let* $\tilde{c} \in C$, $\|\tilde{c}\| \leqq \varkappa_2$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$. *If*

$$(19,2) \qquad \hat{K}_2 e^{-\nu_1 T} \leqq \tfrac{1}{3}, \quad d_{2T}(\mathscr{X}, \mathscr{X}_0) \leqq \tfrac{1}{2}\varkappa_2,$$

then there exists a solution $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle\tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ *and*

$$(20,2) \qquad \|c(\tau)\| \leqq \tfrac{3}{4}\hat{\varkappa}, \quad \tau \in \langle\tilde{\tau}, \tilde{\tau} + T\rangle,$$

$$(21,2) \qquad \|c(\tau)\| \leqq \varkappa_2, \quad \tau \in \langle\tilde{\tau} + T, \infty).$$

Proof. Let (c_0, γ_0) be defined by (Ω_2). Apply Lemma 1,2 with $x_0(\tau) = (c_0(\tau), \gamma_0(\tau))$ on $\langle\tilde{\tau}, \tilde{\tau} + 2T\rangle = \langle\tau_1, \tau_2\rangle$, $\xi_1 = \frac{1}{2}\hat{\varkappa}_2$, $\xi_2 = \frac{1}{2}\hat{\varkappa}$. As conditions (10,2) and (11,2) are fulfilled (cf. (12,2), (19,2) and $\varkappa_2 < \frac{1}{2}\hat{\varkappa}$), there exists a solution $(c_1, \gamma_1) \in \mathscr{X}$, $c_1(\tilde{\tau}) = \tilde{c}$, $\gamma_1(\tilde{\tau}) = \tilde{\gamma}$ and $\|c_1(\tau)\| \leqq \|c_0(\tau)\| + \|c_1(\tau) - c_0(\tau)\| \leqq \varkappa_2\hat{K}_2 e^{-\nu_1(\tau-\tilde{\tau})} + \frac{1}{2}\varkappa_2$, $\tau \in \langle\tilde{\tau}, \tilde{\tau} + 2T\rangle$.

Therefore (cf. $\varkappa_2 < \frac{1}{2}\hat{\varkappa}$) c_1 fulfils (20,2) and $\|c_1(\tau)\| \leqq \varkappa_2$ for $\tau \in \langle\tilde{\tau} + T, \tilde{\tau} + 2T\rangle$. It may be proved by induction that there exists a solution $(c_i, \gamma_i) \in \mathscr{X}$, $J(c_i, \gamma_i) = \langle\tilde{\tau}, \tilde{\tau} + (i + 1)\,T\rangle$, $c_i(\tilde{\tau}) = \tilde{c}$, $\gamma_i(\tilde{\tau}) = \tilde{\gamma}$ that c_i fulfils (20,2) and $\|c_i(\tau)\| \leqq \varkappa_2$ for $\tau \in \langle\tilde{\tau} + T, \tilde{\tau} + (i + 1)\,T\rangle$, $i = 1, 2, 3, \ldots$ Lemma 3,2 holds (cf. (IV)).

Lemma 4,2. *Let* $\tilde{c}, \tilde{b} \in C$, $\|\tilde{c}\|, \|\tilde{b}\| \leqq \varkappa_2$, $\tilde{\gamma}, \tilde{\beta} \in \mathscr{C}$, $\tilde{\tau} \in E_1$, $T > 0$ *and let* (19,2) *hold. If*

$$(22,2) \qquad e^{\nu_1 T} \geqq 72\hat{K}_2^2, \quad \mathrm{d}_T(\mathscr{X}, \mathscr{X}_0) \leqq D,$$

$$(23,2) \qquad \|\tilde{c} - \tilde{b}\| \leqq L\|\tilde{\gamma} - \tilde{\beta}\|,$$

(D, L *being defined in* (18,2)), *then there exist solutions* (c, γ), $(b, \beta) \in \mathscr{X}$, $J(c, \gamma) = J(b, \beta) = \langle\tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$, $b(\tilde{\tau}) = \tilde{b}$, $\beta(\tilde{\tau}) = \tilde{\beta}$ *and*

$$(24,2) \qquad \|c(\tau) - b(\tau)\| \leqq L\|\gamma(\tau) - \beta(\tau)\|, \quad \tau \geqq \tilde{\tau} + T.$$

Proof. It follows from Lemma 3,2 that solutions $(c, \gamma), (b, \beta) \in \mathscr{X}$ exist on $\langle \tilde{\tau}, \infty)$. Therefore the following estimates hold on $\langle \tilde{\tau} + T, \tilde{\tau} + 2T \rangle$ (cf. (12,2), (14,2), (15,2), (18,2), (22,2)), (b_0, β_0) being the solution of $\mathscr{X}_0$ on $\langle \tilde{\tau}, \infty)$, $b_0(\tilde{\tau}) = \tilde{b}$, $\tilde{b}_0(\tilde{\tau}) = \tilde{\beta}$)

$$\|c(\tau) - b(\tau)\| \leqq \|c_0(\tau) - b_0(\tau)\| + \|c(\tau) - b(\tau) - c_0(\tau) + b_0(\tau)\| \leqq$$
$$\leqq \hat{K}_2 e^{-\nu_1 T}[\|\tilde{c} - \tilde{b}\| + \|\tilde{\gamma} - \tilde{\beta}\|] + D[\|\tilde{c} - \tilde{b}\| + \|\tilde{\gamma} - \tilde{\beta}\|] \leqq$$
$$\leqq [(72\hat{K}_2)^{-1} + (72\hat{K}_2)^{-1}](1 + L)\|\tilde{\gamma} - \tilde{\beta}\| \leqq (18\hat{K}_2)^{-1} \cdot \|\tilde{\gamma} - \tilde{\beta}\|,$$

$$\|\gamma(\tau) - \beta(\tau)\| \geqq \|\tilde{\gamma} - \tilde{\beta}\| - \|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| - \|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| \geqq$$
$$\geqq \|\tilde{\gamma} - \tilde{\beta}\| - \hat{K}_2(\|\tilde{c} - \tilde{b}\| + \varrho\|\tilde{\gamma} - \tilde{\beta}\|) - D[\|\tilde{c} - \tilde{b}\| + \|\tilde{\gamma} - \tilde{\beta}\|] \geqq$$
$$\geqq \|\tilde{\gamma} - \tilde{\beta}\| [1 - \hat{K}_2(L + \varrho) - D(L + 1)] \geqq \|\tilde{\gamma} - \tilde{\beta}\| [1 - \tfrac{1}{6} - \tfrac{1}{3} - \tfrac{2}{72}] \geqq$$
$$\geqq \tfrac{1}{3}\|\tilde{\gamma} - \tilde{\beta}\|.$$

Consequently $\|c(\tau) - b(\tau)\| \leqq L\|\gamma(\tau) - \beta(\tau)\|$ for $\tau \in \langle \tilde{\tau} + T, \tilde{\tau} + 2T \rangle$ and (24,2) holds by induction.

Denote by Q the set of maps q from $\mathscr{C}$ to C fulfilling the conditions

$$\|q(\gamma)\| \leqq \varkappa_2, \quad \|q(\gamma_2) - q(\gamma_1)\| \leqq L\|\gamma_2 - \gamma_1\| \quad \text{for} \quad \gamma, \gamma_1, \gamma_2 \in \mathscr{C} \tag{25,2}$$

Let $\|q_1\|$, $q_1 + q_2$, λq_1 be defined in the usual way, q_1, q_2 being bounded maps from $\mathscr{C}$ to C, $\lambda \in E_1$.

Let q be a map from $\mathscr{C}$ to C, $\tau_1 \geqq \tilde{\tau}$. Let there exist a $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \tau_1 \rangle$, $c(\tilde{\tau}) = q(\tilde{\gamma})$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ for every $\tilde{\gamma} \in \mathscr{C}$. Denote by $U_{\tau_1, \tilde{\tau}}\, q$ the set of all $(c(\tau_1), \gamma(\tau_1))$. If $U_{\tau_1, \tilde{\tau}} q$ is a graph of a map from $\mathscr{C}$ to C, then this map will be denoted by $P_{\tau_1, \tilde{\tau}} q$.

Lemma 5,2. *Let* (22,2) *be fulfilled,* $q \in Q$, $\tilde{\tau} \leqq \tau \leqq \tilde{\tau} + 2T$. *Let* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = q(\tilde{\gamma})$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$. *Put* $\vartheta(\tilde{\gamma}) = \gamma(\tau)$. *Then* ϑ *is a map of* $\mathscr{C}$ *onto* $\mathscr{C}$ *and* ϑ *fulfils* (16,2) *for* $l_2 = 3$.

Proof. Let $\tilde{\beta} \in \mathscr{C}$; then $\vartheta(\tilde{\beta}) = \beta(\tau)$, (b, β) being defined by the conditions $(b, \beta) \in \mathscr{X}$, $J(b, \beta) = \langle \tilde{\tau}, \infty)$, $b(\tilde{\tau}) = q(\tilde{\beta})$, $\beta(\tilde{\tau}) = \tilde{\beta}$. It follows from (12,2), (15,2), (18,2) and (22,2) that

$$\|\vartheta(\tilde{\gamma}) - \vartheta(\tilde{\beta})\| = \|\gamma(\tau) - \beta(\tau)\| \leqq$$
$$\leqq \|\tilde{\gamma} - \tilde{\beta}\| + \|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| + \|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| \leqq$$
$$\leqq \|\tilde{\gamma} - \tilde{\beta}\| + \hat{K}_2[\|q(\tilde{\gamma}) - q(\tilde{\beta})\| + \varrho\|\tilde{\gamma} - \tilde{\beta}\|] + D[\|q(\tilde{\gamma}) - q(\tilde{\beta})\| + \|\tilde{\gamma} - \tilde{\beta}\|] \leqq$$
$$\leqq \|\tilde{\gamma} - \tilde{\beta}\| [1 + \hat{K}_2(L + \varrho) + D(L + 1)] \leqq \|\tilde{\gamma} - \tilde{\beta}\| [1 + \tfrac{1}{2} + \tfrac{1}{36}]$$

Similarly

$$\|\vartheta(\tilde{\gamma}) - \vartheta(\tilde{\beta})\| = \|\gamma(\tau) - \beta(\tau)\| \geqq \tag{26,2}$$
$$\geqq \|\tilde{\gamma} - \tilde{\beta}\| - \|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| - \|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| \geqq$$
$$\geqq \|\tilde{\gamma} - \tilde{\beta}\| [1 - K_2(L + \varrho) - D(L + 1)] \geqq \|\tilde{\gamma} - \tilde{\beta}\| [1 - \tfrac{1}{2} - \tfrac{1}{36}].$$

Therefore (16,2) is fulfilled. As

$$\|\vartheta(\tilde{\gamma}) - \vartheta(\tilde{\beta}) - \tilde{\gamma} + \tilde{\beta}\| \leqq$$
$$\leqq \|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| + \|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| \leqq$$
$$\leqq D[\|q(\tilde{\gamma}) - q(\tilde{\beta})\| + \|\tilde{\gamma} - \tilde{\beta}\|] + \hat{K}_2[\|q(\tilde{\gamma}) - q(\tilde{\beta})\| + \varrho\|\tilde{\gamma} - \tilde{\beta}\|] \leqq$$
$$\leqq \|\tilde{\gamma} - \tilde{\beta}\| \, [2(72\hat{K}_2)^{-1} + \hat{K}_2[(6\hat{K}_2)^{-1} + (3\hat{K}_2)^{-1}] \leqq \tfrac{2}{3}\|\tilde{\gamma} - \tilde{\beta}\| \, ,$$

(17,2) is fulfilled and ϑ is a map of $\mathscr{C}$ onto $\mathscr{C}$ (cf: Lemma 2,2).

Corollary of Lemma 5,2. *Let* (22,2) *be fulfilled,* $q \in Q$, $\tilde{\tau} \leqq \tau \leqq \tilde{\tau} + 2T$. *Let* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = q(\tilde{\gamma})$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$. *Put* $\vartheta(\tilde{\gamma}) = \gamma(\tau)$. *Then* ϑ *is a one-to-one map of* $\mathscr{C}$ *onto itself. The map* ϑ *fulfils a Lipschitz condition.*

Lemma 6,2. *Let* (22,2) *be fulfilled,* $q \in Q$. *Then* $P_{\tau,\tilde{\tau}}q$ *is defined for* $\tau \geqq \tilde{\tau}$, $P_{\tau,\tilde{\tau}}q \in Q$ *for* $\tau \geqq \tilde{\tau} + T$ *and*

$$P_{\tau_2,\tau_1}(P_{\tau_2,\tilde{\tau}}q) = P_{\tau_2,\tilde{\tau}}q \, , \quad \tau_2 \geqq \tau_1 \geqq \tilde{\tau} \, . \tag{27,2}$$

Proof. Let $\tilde{\tau} \leqq \tau \leqq \tilde{\tau} + 2T$. $U_{\tau,\tilde{\tau}}q$ is the set of such $(c(\tau), \gamma(\tau))$ that $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$. It follows from Lemma 5,2 ((16,2) implies that ϑ is a one-to-one map) that $U_{\tau,\tilde{\tau}}q$ is a graph of a map; this map is denoted by $P_{\tau,\tilde{\tau}}q$. (24,2) implies that $P_{\tau,\tilde{\tau}}q \in Q$ if $\tilde{\tau} + T \leqq \tau$. (27,2) is a consequence of the definition of $P_{\tau,\tilde{\tau}}q$.

Lemma 7,2. *Let* (22,2) *be fulfilled,* $q_1, q_2 \in Q$, $\tilde{c} \in C$, $\|\tilde{c}\| \leqq \varkappa_2$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$. *Let* $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ *((c, γ) exists according to Lemma* 3,2). *Then*

$$\|c(\tau) - (P_{\tau,\tilde{\tau}}q_1)(\gamma(\tau))\| \leqq (\hat{K}_2 + 1)\| \tilde{c} - q_1(\tilde{\gamma}, \tilde{\tau})\| \quad \text{for} \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T, \tag{28,2}$$

$$\|c(\tau) - (P_{\tau,\tilde{\tau}}q_1)(\gamma(\tau))\| \leqq \tfrac{1}{3}\|\tilde{c} - q_1(\tilde{\gamma})\| \quad \text{for} \quad \tilde{\tau} + T \leqq \tau \leqq \tilde{\tau} + 2T, \tag{29,2}$$

$$\|P_{\tau,\tilde{\tau}}q_2 - P_{\tau,\tilde{\tau}}q_1\| \leqq \tfrac{1}{3}\|q_2 - q_1\| \quad \text{for} \quad \tilde{\tau} + T \leqq \tau \leqq \tilde{\tau} + 2T. \tag{30,2}$$

Proof. Put $\tilde{\beta} = \tilde{\gamma}$, $\tilde{b} = q_1(\tilde{\beta})$; let (b, β), (c_0, γ_0) and (b_0, β_0) have the usual meaning, $\tau \in \langle \tilde{\tau}, \tilde{\tau} + 2T \rangle$. As $(P_{\tau,\tilde{\tau}}q_1)(\beta(\tau)) = b(\tau)$, it follows from (12,2), (14,2), (15,2), (18,2) and (22,2) that

$$\|c(\tau) - (P_{\tau,\tilde{\tau}}q_1)(\gamma(\tau))\| \leqq \|c(\tau) - b(\tau)\| + \|(P_{\tau,\tilde{\tau}}q_1)(\beta(\tau)) - (P_{\tau,\tilde{\tau}}q_1)(\gamma(\tau))\| \leqq$$
$$\leqq \|(c(\tau) - b(\tau) - c_0(\tau) + b_0(\tau)\| + \|c_0(\tau) - b_0(\tau)\| +$$
$$+ L[\|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| + \|\gamma_0(\tau) - \beta_0(\tau)\|] \leqq$$
$$\leqq D\|\tilde{c} - \tilde{b}\| + \hat{K}_2 e^{-\nu_1(\tau - \tilde{\tau})}\|\tilde{c} - \tilde{b}\| + \hat{K}_2 L\|\tilde{c} - \tilde{b}\| \leqq$$
$$\leqq [\tfrac{1}{72} + \hat{K}_2 + \tfrac{1}{6}] \, \|\tilde{c} - q_1(\tilde{\gamma}, \tilde{\tau})\|$$

and (28,2) holds. If $\tilde{\tau} + T \leqq \tau \leqq \tilde{\tau} + 2T$, then

$$\|c(\tau) - (P_{,\tau\tilde{\tau},}q_1)(\gamma(\tau))\| \leqq [D + \hat{K}_2 e^{-\nu_1 T} + \hat{K}_2 L] \|\tilde{c} - \tilde{b}\| \leqq$$
$$\leqq [\tfrac{1}{72} + \tfrac{1}{72} + \tfrac{1}{6}] \|\tilde{c} - q_1(\tilde{\gamma}, \tilde{\tau})\| .$$

(29,2) holds. Substitute $\tilde{c} = q_2(\tilde{\gamma})$ into (29,2). As $c(\tau) = (P_{\tau,\tilde{\tau}}q_2)(\gamma(\tau))$, we obtain

$$\|(P_{\tau,\tilde{\tau}}q_2)(\gamma(\tau)) - (P_{\tau,\tilde{\tau}}q_1)(\gamma(\tau))\| \leqq \tfrac{1}{3}\|q_2(\tilde{\gamma}) - q_1(\tilde{\gamma})\| \leqq \tfrac{1}{3}\|q_2 - q_1\|$$

According to Lemma 5,2 $\gamma(\tau) = \vartheta(\tilde{\gamma})$ accepts all values from $\mathscr{C}$, if $\tilde{\gamma}$ runs through $\mathscr{C}$, $\tilde{\tau} + T \leqq \tau \leqq \tilde{\tau} + 2T$, τ being fixed. Consequently (30,2) holds too.

Lemma 8,2. *Let (22,2) be fulfilled, $q_i \in Q$, $i = 1, 2, \ldots$ $q \in Q$, $\|q_i - q\| \to 0$ with $i \to \infty$, $\tau_1 \geqq \tilde{\tau}$, $P_{\tau_1,\tilde{\tau}}q \in Q$, $i = 1, 2, \ldots$ Then $\|P_{\tau_1,\tilde{\tau}}q_i - P_{\tau_1,\tilde{\tau}}q\| \to \infty$ with $i \to \infty$.*

Proof. Let $\tilde{\gamma} = \tilde{\beta} \in \mathscr{C}$, $\tilde{c} = q_i(\tilde{\gamma})$, $\tilde{b} = q(\tilde{\beta})$, $\tau \in E_1$, let (c, γ), (c_0, γ_0), (b, β), (b_0, β_0) have the usual meaning. Suppose in addition that $\tau_1 \leqq \tilde{\tau} + 2T$. According to (12,2), (14,2), (15,2), (18,2) and (23,2) we obtain

$$\|c(\tau_1) - b(\tau_1)\| + \|\gamma(\tau_1) - \beta(\tau_1)\| \leqq$$
$$\leqq \|c_0(\tau_1) - b_0(\tau_1)\| + \|c(\tau_1) - b(\tau_1) - c_0(\tau_1) + b_0(\tau_1)\| +$$
$$+ \|\gamma_0(\tau_1) - \beta_0(\tau_1)\| + \|\gamma(\tau_1) - \beta(\tau_1) - \gamma_0(\tau_1) + \beta_0(\tau_1)\| \leqq$$
$$\leqq 2\hat{K}_2\|\tilde{c} - \tilde{b}\| + D\|\tilde{c} - \tilde{b}\| \leqq (2\hat{K}_2 + 1)\|q_i - q\| .$$

As $c(\tau_1) = (P_{\tau_1,\tilde{\tau}}q_i)(\gamma(\tau_1))$, $b(\tau_1) = (P_{\tau_1,\tilde{\tau}}q)(\beta(\tau_1))$, $P_{\tau_1,\tilde{\tau}}q_i \in Q$, it follows that

$$\|(P_{\tau_1,\tilde{\tau}}q_i)(\beta(\tau_1)) - (P_{\tau_1,\tilde{\tau}}q)(\beta(\tau_1))\| \leqq (1 + L)(2\hat{K}_2 + 1)\|q_i - q\| .$$

According to Lemma 5,2 $\beta(\tau_1)$ accepts all values from $\mathscr{C}$, if $\tilde{\beta}$ runs through $\mathscr{C}$ and Lemma 8,2 holds (the additional restriction $\tau_1 \leqq \tilde{\tau} + 2T$ being removed by induction, cf. (27,2)).

Lemma 9,2. *Let (22,2) be fulfilled. Then the limit*

(31,2) $$\lim_{\tilde{\tau} \to -\infty} P_{\tau,\tilde{\tau}}q = p(\tau), \quad q \in Q, \ \tau \in E_1$$

exists and does not depend on q and

(32,2) $$p(\tau) \in Q, \qquad \tau \in E_1,$$

(33,2) $$P_{\tau_2,\tau_1}\, p(\tau_1) = p(\tau_2), \quad \tau_1 \leqq \tau_2 .$$

Proof. Let $q \in Q$, $\tilde{\tau}_1 = \tau$, $\tilde{\tau}_{\tau+1} \leqq \tilde{\tau}_i - T$, $i = 1, 2, \ldots$ Let $j > i > 1$. Then according to (26,2) it follows that

(34,2) $$P_{\tau,\tilde{\tau}_j} = P_{\tau,\tilde{\tau}_i}(P_{\tilde{\tau}_i,\tilde{\tau}_j}q)$$

and (30,2) together with (27,2) imply that

$$\|P_{\tau,\tilde{\tau}_i}q - P_{\tau,\tilde{\tau}_j}q\| = \|P_{\tilde{\tau}_1,\tilde{\tau}_2} \dots P_{\tilde{\tau}_{i-1},\tilde{\tau}_i}(P_{\tilde{\tau}_i,\tilde{\tau}_j}q) - P_{\tilde{\tau}_1,\tilde{\tau}_2} \dots P_{\tilde{\tau}_{i-1},\tilde{\tau}_i}q\| \leqq$$
$$\leqq (\tfrac{1}{3})^{i-1} \|P_{\tilde{\tau}_i,\tilde{\tau}_j}q - q\| \leqq (\tfrac{1}{3})^{i-1} 2\varkappa_2 .$$

Therefore $\lim_{i\to\infty} P_{\tau,\tilde{\tau}_i}q$ exists for every sequence $\tilde{\tau}_i$ and consequently the limit in (31,2) exists. In a similar way one may prove that this limit does not depend on q. (32,2) is fulfilled, as Q is closed. (33,2) is a consequence of Lemma 8,2 and (34,2).

Let us finish the proof of Theorem 2,2 with the exception of (vi′) and (vii′). Let (22,2) hold, let p be defined by Lemma 9,2 and put $p(\gamma, \tau) = p(\tau)(\gamma)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$. Remind that ϱ, L, D were defined in (18,2) and that $\varkappa_2$ was defined by condition (Ω_3).

Assertions (i′) and (ii′) of Theorem 2,2 are consequences of (32,2).

Let $\tilde{\tau} \in E_1$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{c} = p(\tilde{\gamma}, \tilde{\tau})$. As $\|\tilde{c}\| \leqq \varkappa_2$, it follows from Lemma 3,2 that there exists a solution $(c_1, \gamma_1) \in \mathscr{X}$, $J(c_1, \gamma_1) = \langle\tilde{\tau}, \infty)$, $c_1(\tilde{\tau}) = \tilde{c}$, $\gamma_1(\tilde{\tau}) = \tilde{\gamma}$ and the definition of $P_{\tau,\tilde{\tau}}q$ together with (33,2) and $p(\tilde{\tau})(\tilde{\gamma}) = \tilde{c}$ imply that $c_1(\tau) = p(\gamma_1(\tau), \tau)$ for $\tau \geqq \tilde{\tau}$. It is a consequence of the definition of $P_{\tau,\tilde{\tau}}q$ and of (33,2) that to every $i = 1, 2, \dots$ there exists a $(b_i, \beta_i) \in \mathscr{X}$, $J(b_i, \beta_i) = \langle\tilde{\tau} - i, \infty)$, $b_i(\tilde{\tau}) = \tilde{c}$, $\beta_i(\tilde{\tau}) = \tilde{\gamma}$ and that $b_i(\tau) = p(\beta_i(\tau), \tau)$ for $\tau \geqq \tilde{\tau}$. It follows from Corollary of Lemma 5,2 that $b_i(\tau) = b_j(\tau)$, $\beta_i(\tau) = \beta_j(\tau)$ for $\tau \geqq \tilde{\tau} - i$, $i \leqq j$. Hence assertion (iii′) holds (cf. (III), (IV) in the definition of a flow).

Let $\tilde{c} \in C$, $\|\tilde{c}\| \leqq \varkappa_2$, $\tilde{\gamma} \in \mathscr{C}$, $\tau \in E_1$. According to Lemma 3,2 there exists a solution $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle\tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$ and $\|c(\tau)\| \leqq \frac{3}{4}\varkappa$ for $\tau \geqq \tilde{\tau}$ $(\varkappa_2 \leqq \frac{1}{2}\varkappa)$. It follows by induction from Lemma 7,2 and from (33,2) that

$$\|c(\tau) - p(\gamma(\tau), \tau)\| \leqq (\tfrac{1}{3})^i \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| , \quad \tilde{\tau} + iT \leqq \tau \leqq \tilde{\tau} + (i + 1)\, T,$$
$$i = 1, 2, 3, \dots ,$$
$$\|c(\tau) - p(\gamma(\tau), \tau)\| \leqq (\hat{K}_2 + 1) \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| , \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T.$$

Hence (iv′) holds with $K' = 3(K_2 + 1)$, $\nu' = T^{-1} \log 3$.

Let p' be a map from $\mathscr{C} \times E_1$ to C, which fulfils (i′) and (iii′). Let $\tilde{\beta} \in \mathscr{C}$, $\tau_1 \in E_1$, $\tilde{b} = p'(\tilde{\beta}, \tau_1)$; then there exists a solution $(b, \beta) \in \mathscr{X}$, $J(b, \beta) = E_1$, $b(\tau_1) = \tilde{b}$, $\beta(\tau_1) = \tilde{\beta}$ and $b(\tau) = p'(\beta(\tau), \tau)$ for $\tau \in E_1$. Put $\tau = \tau_1$, $\tilde{\tau} < \tau_1$, $(c, \gamma) = (b, \beta)$ on $\langle\tilde{\tau}, \infty)$ in the second inequality in (iv′); it follows that

$$\text{(35,2)} \qquad \|p'(\tilde{\beta}, \tau_1) - p(\tilde{\beta}, \tau_1)\| \leqq \|b(\tau_1) - p(\beta(\tau_1), \tau_1)\| \leqq$$
$$\leqq K' e^{-\nu'(\tau_1 - \tilde{\tau})} . \|b(\tilde{\tau}) - p(\beta(\tilde{\tau}), \tilde{\tau})\| \leqq K' e^{-\nu'(\tau_1 - \tilde{\tau})} 2\varkappa_2 .$$

(v′) holds, as (35,2) is fulfilled for every $\tilde{\tau} \leqq \tau_1$.

Assertions (i′) – (v′) of Theorem 2,2 are proved. In order to prove (vi′) the following Lemma will be needed:

Lemma 10,2. *Let* $(c_i, \gamma_i) \in \mathscr{X}$, $J(c_i, \gamma_i) = \langle \tilde{\tau}, \infty)$, $c_i(\tilde{\tau}) = \tilde{c}_i$, $\gamma_i(\tilde{\tau}) = \tilde{\gamma}_i$, $\tilde{c}_i = p(\tilde{\gamma}_i, \tilde{\tau})$, $i = 1, 2$, $v_3 = (2T)^{-1} \log (\frac{36}{17})$. *Then*

$$\|\gamma_2(\tau) - \gamma_1(\tau)\| \geqq \tfrac{17}{36} e^{-v_3(\tau - \tilde{\tau})} \|\tilde{\gamma}_2 - \tilde{\gamma}_1\|$$

Proof. It follows from (26,2) that

$$\|\gamma_2(\tau) - \gamma_1(\tau)\| \geqq \tfrac{17}{36} \|\tilde{\gamma}_2 - \tilde{\gamma}\| \quad \text{for} \quad \tau \in \langle \tilde{\tau}, \tilde{\tau} + 2T \rangle .$$

As $c_i(\tau) = p(\gamma_i(\tau), \tau)$, $\tau \geqq \tilde{\tau}$, $i = 1, 2$, ti follows by induction that

$$\|\gamma_2(\tau) - \gamma_1(\tau)\| \geqq (\tfrac{17}{36})^i \|\tilde{\gamma}_2 - \tilde{\gamma}_1\| \quad \text{for} \quad \tau \in \langle \tilde{\tau} + 2(i-1) T, \tilde{\tau} + 2iT \rangle , \quad i = 1, 2, 3, \ldots$$

and Lemma 10,2 holds.

Let us prove (vi′). Let $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\|\tilde{c}\| \leqq \varkappa_2$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$. According to (iii′) there exist $(b_i, \beta_i) \in \mathscr{X}$,

$$J(b_i, \beta_i) = E_1, \ \beta_i(\tilde{\tau} + iT) = \gamma(\tilde{\tau} + iT), \ b_i(\tilde{\tau} + iT) = p(\gamma(\tilde{\tau} + iT), \tilde{\tau} + iT), \quad i = 0, 1, 2, \ldots$$

(iv′) implies that

$$\|c(\tilde{\tau} + iT) - b_i(\tilde{\tau} + iT)\| \leqq K' e^{-v' iT} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| . \tag{36,2}$$

Let $(c_{i0}, \gamma_{i0}), (b_{i0}, \beta_{i0}) \in \mathscr{X}_0$, $J(c_{i0}, \gamma_{i0}) = J(b_{i0}, \beta_{i0}) = \langle \tilde{\tau} + iT, \infty)$, $c_{i0}(\tilde{\tau} + iT) = c(\tilde{\tau} + iT)$, $\gamma_{i0}(\tilde{\tau} + iT) = \gamma(\tilde{\tau} + iT)$, $b_{i0}(\tilde{\tau} + iT) = b_i(\tilde{\tau} + iT)$, $\beta_{i0}(\tilde{\tau} + iT) = \beta_i(\tilde{\tau} + iT)$, $i = 0, 1, 2, \ldots$

($(c_{i0}, \gamma_{i0}), (b_{i0}, \beta_{i0})$ exist, cf. (Ω_2), (i′), (21,2) and the definition of $\varkappa_2$). Then (cf. (15,2) and (36,2))

$$\|\gamma(\tau) - \beta_{i-1}(\tau)\| \leqq \|\gamma_{i-1,0}(\tau) - \beta_{i-1,0}(\tau)\| + \|\gamma(\tau) - \beta_{i-1}(\tau) - \gamma_{i-1,0}(\tau) + \beta_{i-1,0}(\tau)\| \leqq (\hat{K}_2 + D) \|c(\tilde{\tau} + (i-1)T) - b_{i-1}(\tilde{\tau} + (i-1)T)\| \leqq (\hat{K}_2 + 1) K' e^{-v'(i-1)T} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|, \ \tilde{\tau} + (i-1)T \leqq \tau \leqq \tilde{\tau} + iT, \quad i = 1, 2, \ldots \tag{37,2}$$

$\beta_i(\tilde{\tau} + iT) = \gamma(\tilde{\tau} + iT)$; therefore

$$\|\beta_i(\tilde{\tau} + iT) - \beta_{i-1}(\tilde{\tau} + iT)\| \leqq (\hat{K}_2 + 1) K' e^{-v'(i-1)T} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|$$

and Lemma 10,2 implies that

$$\|\beta_i(\tau_1) - \beta_{i-1}(\tau_1)\| \leqq \tfrac{36}{17} e^{v_3(\tilde{\tau} + iT - \tau_1)} (\hat{K}_2 + 1) K' e^{-v'(i-1)T} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| \leqq K_3 e^{-(v' - v_3) iT - v_3(\tau_1 - \tilde{\tau})} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| \tag{38,2}$$

for $\tilde{\tau} \leqq \tau_1 \leqq \tilde{\tau} + iT, K_3 = \frac{36}{17}(\hat{K}_2 + 1) K' e^{\nu' T}$. As $\nu' > \nu_3$, the sequence β_i converges uniformly on every bounded interval $\langle \tilde{\tau}, \tau_2 \rangle$, $\tau_2 > \tilde{\tau}$ and b_i also converges uniformly on $\langle \tilde{\tau}, \tau_2 \rangle$ (cf. (ii'), $b_i(\tau) = p(\beta_i(\tau), \tau)$). Put $b(\tau) = \lim_{i \to \infty} b_i(\tau)$, $\beta(\tau) = \lim_{i \to \infty} \beta_i(\tau)$, $\tau \geqq \tilde{\tau}$. Obviously $b(\tau) = p(\beta(\tau), \tau)$, $\tau \geqq \tilde{\tau}$. The continuity of the map ϑ from Corollary 5,2 implies that $(b, \beta) \in \mathscr{X}$. It follows from (38,2) that

$$\|\beta(\tau_1) - \beta_{i-1}(\tau_1)\| \leqq K_3[1 - e^{-(\nu' - \nu_3)T}]^{-1} \,.\, e^{-(\nu' - \nu_3)iT - \nu_3(\tau_1 - \tilde{\tau})} \,.\, \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| \,,$$
$$\tilde{\tau} \leqq \tau_1 \leqq \tilde{\tau} + iT \,, \quad i = 1, 2, 3, \ldots$$

(37,2) implies that

$$\|\beta_{i-1}(\tau) - \gamma(\tau)\| \leqq K_3 e^{-\nu'(i-1)T} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\|, \; \tilde{\tau} + (i-1)\,T \leqq \tau \leqq \tilde{\tau} + iT \,,$$
$$i = 1, 2, 3, \ldots$$

Therefore

$$\|\beta(\tau) - \gamma(\tau)\| \leqq K_4 e^{-\nu'(\tau - \tilde{\tau})} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| \,, \quad \tau \geqq \tilde{\tau} \,,$$
$$K_4 = K_3 e^{\nu' T}(1 + [1 - e^{-(\nu' - \nu_3)T}]^{-1}) \,,$$
$$\|c(\tau) - b(\tau)\| \leqq \|c(\tau) - p(\gamma(\tau), \tau)\| + \|p(\gamma(\tau), \tau) - p(\beta(\tau), \tau)\| \leqq$$
$$\leqq (K' + LK_4)\, e^{-\nu'(\tau - \tilde{\tau})} \|\tilde{c} - p(\tilde{\gamma}, \tilde{\tau})\| \,, \quad \tau \geqq \tilde{\tau} \,.$$

(vi') holds, if $K'' = K' + (L + 1) K_4$.

Let solutions $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = E_1$, $c(\tau) = p(\gamma, \tau), \tau)$ for $\tau \in E_1$, be equicontinuous. Let $\beta', \tilde{\beta} \in \mathscr{C}$, $\tau', \tilde{\tau} \in E_1$ and let $(b, \beta) \in \mathscr{X}$, $J(b, \beta) = E_1$, $b(\tilde{\tau}) = p(\tilde{\beta}, \tilde{\tau})$, $\beta(\tilde{\tau}) = \tilde{\beta}$. Then

$$\|p(\beta', \tau') - p(\tilde{\beta}, \tilde{\tau})\| \leqq \|p(\beta', \tau') - p(\beta(\tau'), \tau')\| + \|b(\tau') - b(\tilde{\tau})\| \leqq$$
$$\leqq L[\|\beta' - \tilde{\beta}\| + \|\beta(\tau') - \beta(\tilde{\tau})\|] + \|b(\tau') - b(\tilde{\tau})\| \,.$$

Hence (vii') holds and Theorem 2,2 is proved completely.

Note 5,2. Theorem 2,2 may be strenghtened as follows: Let $\mathscr{X}_0$ fulfil the conditions of Theorem 2,2. Then there exist positive functions $\varkappa^*(\eta)$, $L^*(\eta)$ on $(0, D\rangle$ $\lim_{\eta \to 0+} \varkappa^*(\eta) = 0 = \lim_{\eta \to 0+} L^*(\eta)$ and the following assertion holds: if $\mathscr{X}$ is a flow and if $d_{\eta^{-1}T}(\mathscr{X}, \mathscr{X}_0) \leqq \eta$, then map p fulfils (i') and (ii') with $\varkappa_2$ and L replaced by $\varkappa^*(\eta)$ and $L^*(\eta)$. The proof is carried out, if Theorem 2,2 is applied $\hat{K}_2$ being replaced by $i\hat{K}_2$, $i = 1, 2, 3, \ldots$

Flow $\mathscr{X}$ is called periodic in γ with the period δ, $\delta \in \mathscr{C}$ if $(c, \gamma) \in \mathscr{X}$ implies that $(b, \beta) \in \mathscr{X}$, (b, β) being defined by $b(\tau) = c(\tau)$, $\beta(\tau) = \gamma(\tau) + \delta$, $\tau \in J(c, \gamma) = J(b, \beta)$. $\mathscr{X}$ is said to be periodic in τ with the period σ, if $x \in \mathscr{X}$, $J(x) = \langle \tau_1, \tau_2 \rangle$ implies that $y \in \mathscr{X}$, y being defined by $y(\tau) = x(\tau + \sigma)$ for $\tau \in \langle \tau_1 - \sigma, \tau_2 - \sigma \rangle = J(y)$. $\mathscr{X}$ will be called almost periodic, if there is a sequence of flows $\mathscr{X}_i$, $i = 1, 2, \ldots$ and $d_{T_1}(\mathscr{X}, \mathscr{X}_i) \to 0$ with $i \to \infty$ for every $T_1 > 0$, $\mathscr{X}_i$ being periodic in τ.

If $\mathscr{X}$, $\mathscr{Y}$ are flows in $\hat{G}$, $T_1 > 0$, put

$$d^*_{T_1}(\mathscr{X}, \mathscr{Y}) = \sup_{x,y,\alpha} \|x(\tau) - y(\tau)\| ,$$

$$x \in \mathscr{X},\ y \in \mathscr{Y},\ J(x) = J(y) = \langle \tau_1, \tau_2 \rangle,\ \tau_1 < \tau_2 \leqq \tau_1 + T_1,$$

$$\tau \in \langle \tau_1, \tau_2 \rangle,\ x(\tau_1) = y(\tau_1) .$$

Theorem 3,2. *Let the assumptions of Theorem 2,2 be fulfilled, let $d_T(\mathscr{X}, \mathscr{X}_0) \leqq D$ and let $\mathscr{X}$ be periodic in γ with the period δ. Then $p(\gamma - \delta, \tau) = p(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$.*

Proof. Define $p'(\gamma, \tau) = p(\gamma - \delta, \tau)$, $\gamma \in \mathscr{C}$, $\tau \in E_1$. It may be shown that p' fulfils assertions (i') and (iii') of Theorem 2,2; therefore $p' = p$ according to (v').

The following Theorem is proved in the same manner.

Theorem 4,2. *Let the assumptions of Theorem 2,2 be fulfilled, $d_T(\mathscr{X}, \mathscr{X}_0) \leqq D$, $\sigma > 0$ and let $\mathscr{X}$ be periodic in τ with the period σ. Then $p(\gamma, \tau + \sigma) = p(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$.*

Theorem 5,2. *Let $\mathscr{X}_i$, $i = 1, 2, \ldots$, $\mathscr{X}^*$ and $\mathscr{X}_0$ be flows in $\hat{G}$, let the assumptions of Theorem 2,2 be fulfilled, if the couple $\mathscr{X}, \mathscr{X}_0$ is replaced by $\mathscr{X}_i, \mathscr{X}_0$, $i = 1, 2, \ldots$ and by $\mathscr{X}^*, \mathscr{X}_0$ (K_2 and v_1 being fixed and $\varkappa_1$ in (Ω_3) being independent of i). Let $d_T(\mathscr{X}_i, \mathscr{X}_0) \leqq D$, $d_T(\mathscr{X}^*, \mathscr{X}_0) \leqq D$. Suppose in addition that $d^*_{T_1}(\mathscr{X}_i, \mathscr{X}^*) \to 0$ with $i \to \infty$ for every $T_1 > 0$. Let p_i, $i = 1, 2, \ldots$ and p^* correspond to $\mathscr{X}_i$, $i = 1, 2, \ldots$ and $\mathscr{X}^*$ in the same way as p corresponds to $\mathscr{X}$ in Theorem 2,2. Then*

$$\sup_{\gamma \in \mathscr{C}, \tau \in E_1} \|p_i(\gamma, \tau) - p^*(\gamma, \tau)\| \to 0 \quad \text{with} \quad i \to \infty .$$

Proof. Let the operations $\overset{(i)}{P}_{\tau,\tilde{\tau}}$, $i = 1, 2, \ldots$ $\overset{*}{P}_{\tau,\tilde{\tau}}$ be in the same relation to $\mathscr{X}_i$, $i = 1, 2, \ldots$, $\mathscr{X}^*$ as $P_{\tau,\tilde{\tau}}$ relates to $\mathscr{X}$ (cf. the definition of $P_{\tau,\tilde{\tau}}$ after Lemma 4,2). Let $q \in Q$, $\tau \in E_1$. Then $\overset{*}{P}_{\tilde{\tau}+T,\tilde{\tau}} q \in Q$, $i = 1, 2, \ldots$, $\overset{*}{P}_{\tilde{\tau}+T,\tilde{\tau}} q \in Q$ (cf. Lemma 6,2). Let $\tilde{\gamma} \in \mathscr{C}$, $\tilde{c} = q(\tilde{\gamma})$. According to Lemma 3,2 there exist $x_i = (c_i, \gamma_i) \in \mathscr{X}_i$, $x^* = (c^*, \gamma^*) \in \mathscr{X}^*$, $J(x_i) = J(x^*) = \langle \tilde{\tau}, \infty)$, $c_i(\tilde{\tau}) = \tilde{c} = c^*(\tilde{\tau})$, $\gamma_i(\tilde{\tau}) = \tilde{\gamma} = \gamma^*(\tilde{\tau})$, $i = 1, 2, 3, \ldots$ Therefore

$$\|(\overset{(i)}{P}_{\tilde{\tau}+T,\tilde{\tau}} q)(\gamma^*(\tilde{\tau} + T)) - (\overset{*}{P}_{\tilde{\tau}+T,\tilde{\tau}} q)(\gamma^*(\tilde{\tau} + T))\| \leqq$$

$$\leqq \|(\overset{(i)}{P}_{\tilde{\tau}+T,\tilde{\tau}} q)(\gamma^*(\tilde{\tau} + T)) - (\overset{(i)}{P}_{\tilde{\tau}+T,\tilde{\tau}})(\gamma_i(\tilde{\tau} + T))\| + \|c_i(\tilde{\tau} + T) - c^*(\tilde{\tau} + T)\| \leqq$$

$$\leqq (1 + L)\, d^*_T(\mathscr{X}_i, \mathscr{X}^*) .$$

According to Lemma 5,2 $\gamma^*(\tilde{\tau} + T)$ accepts all values from $\mathscr{C}$ if $\tilde{\gamma}$ runs through $\mathscr{C}$; hence

$$\|\overset{(i)}{P}_{\tilde{\tau}+T,\tilde{\tau}} q - \overset{*}{P}_{\tilde{\tau}+T,\tilde{\tau}} q\| \leqq (1 + L)\, d^*_T(\mathscr{X}_i, \mathscr{X}^*) . \tag{39,2}$$

Let $q_1, q_2 \in Q$. Lemma 7,2 and (39,2) imply that

$$\|P^{(i)}_{\tilde{\tau}+T,\tilde{\tau}} q_1 - P^{*}_{\tilde{\tau}+T,\tilde{\tau}} q_2\| \leqq \|P^{(i)}_{\tilde{\tau}+T,\tilde{\tau}} q_1 - P^{*}_{\tilde{\tau}+T,\tilde{\tau}} q_1\| +$$
$$+ \|P^{*}_{\tilde{\tau}+T,\tilde{\tau}} q_1 - P^{*}_{\tilde{\tau}+T,\tilde{\tau}} q_2\| \leqq (1 + L)\, d^*_T(\mathscr{X}_i, \mathscr{X}^*) + \tfrac{1}{3}\|q_1 - q_2\|$$

It follows by induction that

(40,2)
$$\|P^{(i)}_{\tau,\tau-jT} q_1 - P^{*}_{\tau,\tau-jT} q_2\| \leqq$$
$$\leqq 2d^*_T(\mathscr{X}_i, \mathscr{X}^*)\,[1 + \tfrac{1}{3} + \dots (\tfrac{1}{3})^{j-1}] + (\tfrac{1}{3})^j\, \|q_2 - q_1\|\,, \quad j = 1, 2, 3, \dots$$

As $p_i(\tau) = \lim_{\tilde{\tau}\to-\infty} P^{(i)}_{\tau,\tilde{\tau}} q$, $p^*(\tau) = \lim_{\tilde{\tau}\to-\infty} P^{*}_{\tau,\tilde{\tau}} q$ (cf. Lemma 9,2), it follows from (40,2) that $\|p_i(\tau) - p^*(\tau)\| \leqq 3d^*_T(\mathscr{X}_i, \mathscr{X}^*)$ and Theorem 5,2 is proved.

Note 6,2. If the assumptions of Theorem 2,2 are fulfilled, $d_T(\mathscr{X}, \mathscr{X}_0) < D$ and if $\mathscr{X}$ is almost periodic (in τ), then according to Theorems 4,2 and 5,2 there exist maps p_i from $\mathscr{C} \times E_1$ to C, which are periodic in τ and $p_i(\gamma, \tau) \to p(\gamma, \tau)$ uniformly. If each p_i is uniformly continuous (cf. (vii′)), then p is uniformly almost periodic.

The following Stability Lemma plays an important part in the proof of Theorem 1,2

Lemma 11,2. (Stability Lemma.) *Let $f_0 = (a_0, \alpha_0)$ fulfil the assumptions of Theorem 1,2 (i.e. (4,2), (5,2), (6,2), (7,2) and (9,2)). Let $0 < \nu_1 < \nu\mu$. Then there exists a $K_2 = K_2(K_1, \varkappa, \mu, \nu, \nu_1) \geqq \max(K_1, 1)$ and to every $\varrho > 0$ there exists a $\varkappa_5 = \varkappa_5(K_1, \varkappa, \mu, \nu, \nu_1, \varrho) > 0$, $K_2\varkappa_5 \leqq \frac{3}{4}\varkappa$ that the solutions (c_0, γ_0), (b_0, β_0) of (3,2), $c_0(\tilde{\tau}) = \tilde{c}$, $\gamma_0(\tilde{\tau}) = \tilde{\gamma}$, $b_0(\tilde{\tau}) = \tilde{b}$, $\beta_0(\tilde{\tau}) = \tilde{\beta}$, $\|\tilde{c}\|, \|\tilde{b}\| \leqq \varkappa_5$ exist on $\langle\tilde{\tau}, \infty)$ and the following estimates take place*

(41,2) $$\|c_0(\tau) - b_0(\tau)\| \leqq K_2 e^{-\nu_1(\tau-\tilde{\tau})}[\|\tilde{c} - \tilde{b}\| + \varrho\|\tilde{\gamma} - \tilde{\beta}\|]\,, \quad \tau \geqq \tilde{\tau}\,,$$

(42,2) $$\|\gamma_0(\tau) - \beta_0(\tau) - \tilde{\gamma} + \tilde{\beta}\| \leqq K_2[\|\tilde{c} - \tilde{b}\| + \|\tilde{\gamma} - \tilde{\beta}\|]\,, \quad \tau \geqq \tilde{\tau}\,.$$

Corollary 1,2. *Let $\tilde{b} = 0$, $\tilde{\gamma} = \tilde{\beta}$; then $b_0 = 0$ and (42,2) gives*

(43,2) $$\|c_0(\tau)\| \leqq K_2 e^{-\nu_1(\tau-\tilde{\tau})}\|\tilde{c}\|\,, \quad \tau \geqq \tilde{\tau}$$

for any solution (c_0, γ_0) of (3,2) $\|\tilde{c}\| \leqq \varkappa_5$, $\tilde{\gamma} \in \mathscr{C}$, $\varkappa_5(K_1, \varkappa, \mu, \nu, \nu_1, 1)$.

Proof of Lemma 11,2: As $f_0 = (a_0, \alpha_0)$ it follows from (1,2) that

(44,2) $$\left\|\frac{\partial \alpha_0}{\partial c}(c, \gamma, \tau)\right\| \leqq \left\|\frac{\partial f_0}{\partial c}(c, \gamma, \tau)\right\| \leqq \left\|\frac{\partial f_0}{\partial x}(c, \gamma, \tau)\right\| \leqq K_1,\ (c, \gamma) \in G,\ \tau \in E_1\,.$$

Similarly

(45,2) $$\left\|\frac{\partial a_0}{\partial c}(c_2, \gamma, \tau) - \frac{\partial a_0}{\partial c}(c_1, \gamma, \tau)\right\| \leqq K_1\, \|c_2 - c_1\|^\mu, (c_1, \gamma), (c_2, \gamma) \in G\,,\ \tau \in E_1\,.$$

Obviously

$$a_0(c, \gamma, \tau) = \int_0^1 \frac{\partial a_0}{\partial c}(\sigma c, \gamma, \tau)\, d\sigma c\,, \quad \alpha_0(c, \gamma, \tau) = \alpha^*(\tau) + \int_0^1 \frac{\partial \alpha_0}{\partial c}(\sigma c, \gamma, \tau)\, d\sigma c$$

and therefore (3,2) may be rewritten as

$$\frac{dc}{d\tau} = A(\gamma, \tau)\, c + Z_1\,, \quad \frac{d\gamma}{d\tau} = \alpha^*(\tau) + \mathscr{X}_1\,,$$

$Z_1, \mathscr{X}_1$ being estimated by

(46,2)
$$\|Z_1\| \leqq K_1 \|c\|^{1+\mu}\,, \quad \|\mathscr{X}_1\| \leqq K_1 \|c\|\,.$$

Let (c_0, γ_0) be a solution of (3,2) on $\langle \tilde{\tau}, \tilde{\tau} + T_1 \rangle$, $T_1 > 0$, $c_0(\tilde{\tau}) = \tilde{c}$, $\gamma_0(\tilde{\tau}) = \tilde{\gamma}$ and put $\psi(\tau) = \gamma_0(\tau) - \tilde{\gamma} - \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma$; (c_0, ψ) is a solution of

(47,2)
$$\frac{dc}{d\tau} = A\left(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma + \tilde{\gamma}, \tau\right) c + Z_2\,, \quad \frac{d\psi}{d\tau} = \mathscr{X}_1\,, \quad c_0(\tilde{\tau}) = \tilde{c}, \quad \psi(\tilde{\tau}) = 0\,,$$

Z_2 being defined by

(48,2)
$$Z_2 = \left[A\left(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma + \tilde{\gamma} + \psi, \tau\right) - A\left(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma + \tilde{\gamma}, \tau\right)\right] c + Z_1\,.$$

(46,2) and (47,2) imply that

(49,2)
$$\|\psi(\tau)\| \leqq K_1 \int_{\tilde{\tau}}^{\tau} \|c(\sigma)\|\, d\sigma\,.$$

Let $\Xi = \Xi(\tau, \sigma)$ be the operator solution of

$$\frac{d}{d\tau} \Xi = A\left(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma + \tilde{\gamma}, \tau\right) \Xi\,,$$

$\Xi(\sigma, \sigma) = I$ (I being the identity operator — i.e. the values of Ξ are bounded operators from C to C, $\Xi(\tau, \sigma)\, \tilde{c}$ is a solution of $dc/d\tau = A(\int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\, d\sigma + \tilde{\gamma}, \tau)\, c$, $\|\Xi(\tau, \sigma)\| = \sup_{\|c\| \leqq 1} \|\Xi(\tau, \sigma)\, c\|$. Ξ is continuous in (τ, σ) and according to (9,2)

(50,2)
$$\|\Xi(\tau, \sigma)\| \leqq K_1 e^{-\nu(\tau - \sigma)}\,, \quad \tau \geqq \sigma\,.$$

The first one of equations (47,2) together with the condition $c_0(\tilde{\tau}) = \tilde{c}$ is equivalent to

(51,2)
$$c_0(\tau) = \Xi(\tau, \tilde{\tau})\, \tilde{c} + \int_{\tilde{\tau}}^{\tau} \Xi(\tau, \sigma)\, Z_2\, d\sigma\,.$$

According to (51,2), (50,2), (48,2), (45,2), (49,2) and (46,2) we obtain the following estimate:

(52,2) $$\|c_0(\tau)\| \leqq K_1 e^{-\nu(\tau-\tilde{\tau})}\|\tilde{c}\| +$$

$$+ \int_{\tilde{\tau}}^{\tau} K_1 e^{-\nu(\tau-\sigma)} \left[K_1^{1+\mu} \left(\int_{\tilde{\tau}}^{\sigma} \|c_0(\sigma_1)\| \, d\sigma_1 \right)^{\mu} \|c_0(\sigma)\| + K_1 \|c_0(\sigma)\|^{1+\mu} \right] d\sigma \, .$$

Put $\nu_2 = \frac{1}{2}(\nu_1 \mu^{-1} + \nu)$, $\xi_1(\tau) = e^{\nu_2(\tau-\tilde{\tau})}\|c_0(\tau)\|$. Then

$$\xi_1(\tau) \leqq K_1\|\tilde{c}\| + K_1^2 \int_{\tilde{\tau}}^{\tau} e^{-\nu(\tau-\sigma)+\nu_2(\tau-\tilde{\tau})-\nu_2(\sigma-\tilde{\tau})} \, .$$

$$\cdot \left[\left(K_1 \int_{\tilde{\tau}}^{\sigma} e^{-\nu_2(\sigma_1-\tilde{\tau})} \xi_1(\sigma_1) \, d\sigma_1 \right)^{\mu} + (\xi_1(\sigma)^{\mu} \right] \xi_1(\sigma) \, d\sigma \, , \quad \tau \leqq \tau \leqq \tilde{\tau} + T \, .$$

Let $\|\tilde{c}\| > 0$ and let ξ_2 be the solution of

(53,2) $$\xi_2(\tau) = 2K_1\|\tilde{c}\| + K_1^2 \int_{\tilde{\tau}}^{\tau} e^{-(\nu-\nu_2)(\tau-\sigma)} \, .$$

$$\cdot \left[\left(K_1 \int_{\tilde{\tau}}^{\sigma} e^{-\nu_2(\sigma_1-\tilde{\tau})} \xi_2(\sigma_1) \, d\sigma_1 \right)^{\mu} + (\xi_2(\sigma))^{\mu} \right] \xi_2(\sigma) \, d\sigma$$

on $\langle \tilde{\tau}, \tilde{\tau} + T_1 \rangle$, $T_1 > 0$. Then ξ_2 is positive and nondecreasing on $\langle \tilde{\tau}, \tilde{\tau} + T_1 \rangle$ and as the right hand side of (53,2) is nondecreasing with respect to ξ_2, ξ_1 necessarily exists on $\langle \tilde{\tau}, \tilde{\tau} + T_1 \rangle$ and $\xi_2(\tau) \geqq \xi_1(\tau)$ on $\langle \tilde{\tau}, \tilde{\tau} + T_1 \rangle$. As ξ_2 is nondecreasing,

(54,2) $$\xi_2(\tau) \leqq 2K_1\|\tilde{c}\| + K_3 \int_{\tilde{\tau}}^{\tau} e^{-(\nu-\nu_2)(\tau-\sigma)} (\xi_2(\sigma))^{1+\mu} \, d\sigma \, , \quad \tilde{\tau} \leqq \tau \leqq \tilde{\tau} + T_1 \, ,$$

$$K_3 = K_1^2 [\nu_2^{-\mu} K_1^{\mu} + 1] \, .$$

Let $0 < \varkappa_3 < \varkappa$, $K_3 \varkappa_3 < \nu - \nu_2$, $\varkappa_3 < 1$ and let ξ_3 be the solution of

(55,2) $$\xi_3(\tau) = 3K_1\|\tilde{c}\| + K_3 \varkappa_3^{\mu} \int_{\tilde{\tau}}^{\tau} e^{-(\nu-\nu_2)(\tau-\sigma)} \xi_3(\sigma) \, d\sigma \, .$$

As ξ_3 is a solution of

$$\frac{d}{d\tau} \xi_3 = -(\nu - \nu_2 - K_3 \varkappa_3^{\mu}) \, \xi_3 + 3(\nu - \nu_2) \, K_1 \|\tilde{c}\| \, , \quad \xi_3(\tilde{\tau}) = 3K_1\|\tilde{c}\| \, ,$$

it is defined on $\langle \tilde{\tau}, \infty)$ and $\xi_3(\tau) \leqq K_4\|\tilde{c}\|$ on $\langle \tilde{\tau}, \infty)$, $K_4 = (\nu - \nu_2)(\nu - \nu_2 - K_3 \varkappa_3^{\mu})^{-1} \, 3K_1 + 3K_1$ Choose a $\varkappa_4$, $0 < \varkappa_4 < \varkappa$, $K_4 \varkappa_4 \leqq \varkappa_3$. Let $0 < \|\tilde{c}\| \leqq \varkappa_4$. Then $\xi_3(\tau) \leqq \varkappa_3$ on $\langle \tilde{\tau}, \infty)$, ξ_2 exists on $\langle \tilde{\tau}, \infty)$ and $\xi_2(\tau) \leqq \xi_3$ on $\langle \tilde{\tau}, \infty)$. Thus ξ_1 and c_0 together with γ_0 are defined on $\langle \tilde{\tau}, \infty)$ and

$$e^{-\nu_2(\tau-\tilde{\tau})}\|c_0(\tau)\| = \xi_1(\tau) \leqq K_4\|\tilde{c}\| \leqq \varkappa_3 < \varkappa \, , \quad \tau \geqq \tilde{\tau} \, .$$

Let us sum up: if $\|\tilde{c}\| \leqq \varkappa_4$, then the solution (c_0, γ_0) of (3,2) exists on $\langle\tilde{\tau}, \infty)$ and

(56,2) $$\|c_0(\tau)\| \leqq K_4 e^{-\nu_2(\tau-\tilde{\tau})}\|\tilde{c}\| \leqq \varkappa_3 < \varkappa\,, \quad \tau \geqq \tilde{\tau}\,,$$

($\tilde{\gamma}$ being arbitrary).

Let (c_0, γ_0), (b_0, β_0) be solutions of (3,2), $\|\tilde{c}\|$, $\|\tilde{b}\| \leqq \varkappa_4$. (c_0, γ_0) and (b_0, β_0) exist on $\langle\tilde{\tau}, \infty)$ and fulfil (56,2). Put $\psi(\tau) = \gamma_0(\tau) - \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma - \tilde{\gamma}$, $\varphi(\tau) = \beta_0(\tau) - \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma - \tilde{\beta}$. Then

(57,2)
$$\frac{d}{d\tau}(c_0 - b_0) = c_0\left(c_0, \psi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) - a_0\left(b_0, \varphi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\beta}, \tau\right),$$

(58,2)
$$\frac{d}{d\tau}(\psi - \varphi) = \alpha_0\left(c_0, \psi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) -$$
$$- \alpha_0\left(b_0, \psi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) + \alpha\left(b_0, \psi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) -$$
$$- \alpha_0\left(b_0, \varphi + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\beta}, \tau\right).$$

The following estimates are obtained in a similar way as (46,2) (using (1,2), (5,2), (6,2), (56,2) and $\partial\alpha_0/\partial\gamma\,(0, \gamma, \tau) = 0$):

$$\left\|\alpha_0\left(c_0(\tau), \psi(\tau) + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) - \alpha_0\left(b_0(\tau), \psi(\tau) + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right)\right\| \leqq$$
$$\leqq K_1\|c_0(\tau) - b_0(\tau)\|\,,$$

$$\left\|\alpha_0\left(b_0(\tau), \psi(\tau) + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\gamma}, \tau\right) - \alpha_0\left(b_0(\tau). \varphi(\tau) + \int_{\tilde{\tau}}^{\tau} \alpha^*(\sigma)\,d\sigma + \tilde{\beta}, \tau\right)\right\| \leqq$$
$$\leqq K_1(K_4 e^{-\nu_2(\tau-\tilde{\tau})}\|\tilde{b}\|)^{\mu}\,[\|\tilde{\gamma} - \tilde{\beta}\| + \|\psi(\tau) - \varphi(\tau)\|]\,.$$

Integrating (58,2) we get

$$\|\psi(\tau) - \varphi(\tau)\| \leqq K_1 \int_{\tilde{\tau}}^{\tau} \|c_0(\sigma) - b_0(\sigma)\|\,d\sigma + K_8\|\tilde{b}\|^{\mu}\,.\,\|\tilde{\gamma} - \tilde{\beta}\| +$$
$$+ K_5\|\tilde{b}\|^{\mu} \int_{\tilde{\tau}}^{\tau} e^{-\nu_2\mu(\sigma-\tilde{\tau})}\|\psi(\sigma) - \varphi(\sigma)\|\,d\sigma\,, \quad K_5 = K_1 K_4^{\mu}\,, \quad K_8 = K_5(\nu_2\mu)^{-1}\,.$$

Hence (cf. $\|\tilde{c}\| \leqq \varkappa_4 < 1$)

(59,2) $$\|\psi(\tau) - \varphi(\tau)\| \leqq \left[K_1 \int_{\tilde{\tau}}^{\tau} \|c_0(\sigma) - b_0(\sigma)\|\,d\sigma + K_8\,\|\tilde{b}\|^{\mu}\|\tilde{\gamma} - \tilde{\beta}\|\right] K_9\,,\ K_9 = e^{K_8}\,.$$

Let us write

$$(60,2)\qquad a_0\left(c_0(\tau), \psi(\tau) + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right) - a_0\left(b_0(\tau), \varphi(\tau) + \right.$$

$$\left. + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\beta}, \tau\right) = A\left(\left(\int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right)(c_0(\tau) - b_0(\tau)) + Z_3\,,\right.$$

$$Z_3 = a_0\left(b_0(\tau), \psi(\tau) + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right) - a_0\left(b_0(\tau), \varphi(\tau) + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\beta}, \tau\right) +$$

$$+ \int_0^1\left[\frac{\partial a_0}{\partial c}\left(b_0(\tau) + \sigma(c_0(\tau) - b_0(\tau)), \psi(\tau) + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right) - \frac{\partial a_0}{\partial c}(0, \psi(\tau) + \right.$$

$$\left. + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau)\right]\mathrm{d}\sigma(c_0(\tau) - b_0(\tau)) + \left[+ A\left(\psi(\tau) + \int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right) - \right.$$

$$\left. - A\left(\left(\int_{\tilde{\tau}}^{\tau}\alpha^*(\sigma)\,\mathrm{d}\sigma + \tilde{\gamma}, \tau\right)\right]\left(c_0(\tau) - b_0(\tau)\right).$$

It follows that (use (1,2), (6,2), (56,2), (49,2) and $\partial a_0/\partial\gamma\,(0, \gamma, \tau) = 0$ and suppose that $\|\tilde{b}\| \geqq \|\tilde{c}\|$)

$$(61,2)\qquad \|Z_3\| \leqq K_1(K_4 e^{-\nu_2(\tau-\tilde{\tau})}\|\tilde{b}\|)^{\mu}\,[\|\tilde{\gamma} - \tilde{\beta}\| + \|\psi(\tau) - \varphi(\tau)\|] +$$

$$+ K_1[(K_4 e^{-\nu_2(\tau-\tilde{\tau})}\|\tilde{b}\|)^{\mu} + \|\psi(\tau)\|^{\mu}]\,\|c_0(\tau) - b_0(\tau)\| \leqq$$

$$\leqq K_5 e^{-\nu_2\mu(\tau-\tilde{\tau})}\|\tilde{b}\|^{\mu}\,[\|\tilde{\gamma} - \tilde{\beta}\| + \|\psi(\tau) - \varphi(\tau)\|] +$$

$$+ [K_5 e^{-\nu_2(\tau-\tilde{\tau})}\|\tilde{b}\|^{\mu} + K_6\|\tilde{b}\|^{\mu}]\,\|c_0(\tau) - b_0(\tau)\|\,,$$

$$K_5 = K_1 K_4^{\mu} \quad K_6 = K_1(K_1 K_4 \nu_2^{-1})^{\mu}\,.$$

According to (60,2) equation (57,2) may be given the following form

$$c_0(\tau) - b_0(\tau) = \Xi(\tau - \tilde{\tau})\,(\tilde{c} - \tilde{b}) + \int_{\tilde{\tau}}^{\tau}\Xi(\tau, \sigma)\,Z_3\,\mathrm{d}\sigma$$

Taking (50,2) and (61,2) into account we obtain

$$(62,2)\qquad \|c_0(\tau) - b_0(\tau)\| \leqq K_1 e^{-\nu(\tau-\tilde{\tau})}\|\tilde{c} - \tilde{b}\| +$$

$$+ K_1 K_5\|\tilde{b}\|^{\mu}\int_{\tilde{\tau}}^{\tau} e^{-\nu(\tau-\sigma)-\nu_2\mu(\sigma-\tilde{\tau})}\mathrm{d}\sigma\|\tilde{\gamma} - \tilde{\beta}\| +$$

$$+ K_1 K_5\|\tilde{b}\|^{\mu}\int_{\tilde{\tau}}^{\tau} e^{-\nu(\tau-\sigma)-\nu_2\mu(\sigma-\tilde{\tau})}\|\psi(\sigma) - \varphi(\sigma)\|\,\mathrm{d}\sigma +$$

$$+ K_7\|\tilde{b}\|^{\mu}\int_{\tilde{\tau}}^{\tau} e^{-\nu(\tau-\sigma)}\|c_0(\sigma) - b_0(\sigma)\|\,\mathrm{d}\sigma\,,\quad K_7 = K_1(K_5 + K_6)$$

Substituting (59,2) into (62,2) we obtain after some arrangements

$$(63,2)\qquad \|c_0(\tau) - b_0(\tau)\| \leqq K_1 e^{-\nu(\tau-\tilde{\tau})}\|\tilde{c} - \tilde{b}\| + K_{10}\|\tilde{b}\|^{\mu} e^{-\nu_2\mu(\tau-\tilde{\tau})}\|\tilde{\gamma} - \tilde{\beta}\| + $$
$$+ K_{11}\|\tilde{b}\|^{\mu}\int_{\tilde{\tau}}^{\tau} e^{-\nu_2\mu(\tau-\sigma)}\|c_0(\sigma) - b_0(\sigma)\|\,\mathrm{d}\sigma\,,$$

$$K_{10} = K_1K_5(\nu - \nu_2\mu)^{-1}(1 + K_8K_9)\,,\quad K_{11} = K_1^2K_5K_9(\nu - \nu_2\mu)^{-1} + K_7\,.$$

Put $e^{\nu_1(\tau-\tilde{\tau})}\|c_0(\tau) - b_0(\tau)\| = \xi_4(\tau)$ and multiply (63,2) by $e^{\nu_1(\tau-\tilde{\tau})}$ (and use $\nu_2\mu - \nu_1 = \frac{1}{2}(\nu\mu - \nu_1) > 0$):

(64,2)

$$\xi_4(\tau) \leqq K_1\|\tilde{c} - \tilde{b}\| + K_{10}\|\tilde{b}\|^{\mu}\,\|\tilde{\gamma} - \tilde{\beta}\| + K_{11}\|\tilde{b}\|^{\mu}\int_{\tilde{\tau}}^{\tau} e^{-(\nu_2\mu-\nu_1)(\tau-\sigma)}\xi_4(\sigma)\,\mathrm{d}\sigma\,.$$

It follows from (64,2) that

$$\xi_4 \leqq [K_1\|\tilde{c} - \tilde{b}\| + K_{10}\|\tilde{b}\|^{\mu}\,\|\tilde{\gamma} - \tilde{\beta}\|]\,.\,K_{12}\,,\quad K_{12} = \exp\{K_{11}(\nu_2\mu - \nu_1)^{-1}\}\,.$$

Therefore

$$(65,2)\qquad \|c_0(\tau) - b_0(\tau)\| \leqq [K_1K_{12}\|\tilde{c} - \tilde{b}\| + K_{10}K_{12}\|\tilde{b}\|^{\mu}\,\|\tilde{\gamma} - \tilde{\beta}\|]\,e^{-\nu_1(\tau-\tilde{\tau})}$$

and (59,2) together with (65,2) imply that

$$(66,2)\qquad \|\psi(\tau) - \varphi(\tau)\| \leqq \nu_1^{-1}K_1^2K_9K_{12}\|\tilde{c} - \tilde{b}\| +$$
$$+ [\nu_1^{-1}K_1K_{10}K_{12} + K_8]\,K_9\|\,\|\tilde{b}\|^{\mu}\,\|\tilde{\gamma} - \tilde{\beta}\|\,.$$

(65,2) and (66,2) imply that Lemma 11,2 holds, if we choose

$$(67,2)\quad K_2 = \max(K_1K_{12}, K_{10}K_{12}, \nu_1^{-1}K_1^2K_9K_{12}, [\nu_1^{-1}K_1K_{10}K_{12} + K_8]\,K_9, 1, K_1)\,,$$
$$\varkappa_5 = \min(\varrho^{1/\mu}, \varkappa_4, \tfrac{1}{2}K_2^{-1}\varkappa)\,.$$

Let us prove Theorem 1,2. Lemma 11,2 may be used, the assumptions of Theorem 1,2 being fulfilled. Choose $\nu_1 = \frac{1}{2}\nu\mu$ and let K_2 be defined by Lemma 11,2 and put $\hat{\varkappa} = K_2\varkappa_5(K_1, \varkappa, \mu, \nu, \nu_1, 1)$, $\hat{K}_2 = K_2$; according to Lemma 11,2 $\hat{\varkappa} \leqq \frac{3}{4}\varkappa$. Let $\mathscr{X}_0(\mathscr{X})$ be the set of solutions $x_0 = (c_0, \gamma_0)$ $(\varkappa = (c, \gamma)$ of (3,2) ((2,2)), which fulfil the following condition: each solution $x_0(x)$ is defined on an interval $J(x_0)\,(J(x))$ and $x_0(\tau) \in \hat{G}$ for $\tau \in J(x_0)$ $(x(\tau) \in \hat{G}$ for $\tau \in J(x))$, $\hat{G} = \mathscr{E}[(\tilde{c}, \tilde{\gamma});\ \tilde{c} \in C,\ \|\tilde{c}\| < \hat{\varkappa},\ \tilde{\gamma} \in \mathscr{C}]$. $\mathscr{X}_0$ and $\mathscr{X}$ are flows in $\hat{G}$ (i.e. they fulfil conditions (I)–(VI)).

Let us show that $\mathscr{X}_0$ fulfils conditions (Ω_1), (Ω_2), (Ω_3) of Theorem 2,2. (Ω_1) is a consequence of the first equation (7,2). Let $\tilde{c} \in C$, $\|\tilde{c}\| < K_2^{-1}\varkappa = \varkappa_5(K_1, \varkappa, \mu, \nu, \nu_1, 1)$, $\tilde{\gamma} \in \mathscr{C}$, $\tilde{\tau} \in E_1$. The existence of a solution $(c_0, \gamma_0) \in \mathscr{X}_0$, $J(c_0, \gamma_0) = \langle\tilde{\tau}, \infty)$, $c_0(\tilde{\tau}) = \tilde{c}$,

$\gamma_0(\tilde{\tau}) = \tilde{\gamma}$ is guarranted by Lemma 11,2; (13,2) is identical with (43,2). (Ω_3) is a consequence of Lemma 11,2.

Let $D, L, T, \varkappa_2, K', v', K''$ be defined by Theorem 2,2. According to Lemma 11,2, $\hat{K}_2 = K_2$ depends on $K_1, \varkappa, \mu, v, v_1$ only; therefore (cf. the choose of $\hat{\varkappa}$ and v_1) the above constants depend on $K_1, \varkappa, \mu, v$ only. $L = (6\hat{K}_2)^{-1} \leqq (6K_1)^{-1}$ as $\hat{K}_2 \geqq K_1$ (cf. Lemma 11,2). $\varkappa_2 \leqq (2\hat{K}_2)^{-1}\,\hat{\varkappa} = \frac{1}{2}\varkappa_5(K_1, \varkappa, \mu, v, v_1, 1) < \varkappa$ (cf. Theorem 2,2 and Lemma 11,2, $K_2 \geqq 1$ and $\hat{\varkappa} = \hat{K}_2\varkappa_5(K_1, \varkappa, \mu, v, v_1, 1) \leqq \frac{3}{4}\varkappa$).

Put $d = \varkappa - \hat{\varkappa}$; obviously $G_d = \hat{G}$ (G_d was introduced after the proof of Theorem 2,1). Let $\chi_1(\chi_4)$ be the function from Theorem 1,1 (Theorem 3,1) and choose $\zeta_1 > 0$ (depending on $K_1, \varkappa, \mu, v$ only) that $\chi_1(\zeta_1, T) + \chi_4(\zeta_1, T) \leqq D$. Let (10,1) be fulfilled and let $\zeta \leqq \zeta_1$. Let $x_0, y_0 \in \mathscr{X}_0$, $x, y \in \mathscr{X}$, $J(x_0) = J(y_0) = J(x) = J(y) = \langle\tilde{\tau}, \tilde{\tau}_2\rangle$, $\tilde{\tau} < \tau_2 \leqq \tilde{\tau} + T$, $x_0(\tilde{\tau}) = x(\tilde{\tau}) = \tilde{x}$, $y_0(\tilde{\tau}) = y(\tilde{\tau}) = \tilde{y} \neq \tilde{x}$. Theorems 1,1 and 3,1 imply that $\|x(\tau) - x_0(\tau)\| \leqq \chi_1(\zeta_{,1} T)$, $\|\tilde{x} - \tilde{y}\|^{-1}\,\|x(\tau) - y(\tau) - x_0(\tau) + y_0(\tau)\| \leqq$ $\leqq \chi_4(\zeta_1, T)$ for $\tau \in \langle\tilde{\tau}, \tilde{\tau}_2\rangle$. Followingly $d_T(\mathscr{X}, \mathscr{X}_0) \leqq D$. Therefore Theorem 2,2 may be applied to $\mathscr{X}, \mathscr{X}_0$. Assertions (i)–(vii) of Theorem 1,2 are simple consequences of assertions (i')–(vii') of Theorem 2,2. Theorem 1,2 is proved completely.

Note 7,2. It follows from Theorems 3,2 and 4,2 that $p(\gamma + \delta, \tau) = p(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$, if $a(\gamma + \delta, \tau) = a(\gamma, \tau)$, $\alpha(\gamma + \delta, \tau) = \alpha(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$ (in addition to the assumptions of Theorem 1,2) and that $p(\gamma, \tau + \sigma) = p(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau \in E_1$, if $a(\gamma, \tau + \sigma) = a(\gamma, \tau)$, $\alpha(\gamma, \tau + \sigma) = \alpha(\gamma, \tau)$ for $\gamma \in \mathscr{C}$, $\tau + E_1$. Conclusions on almostperiodicity may be drawn from Theorem 5,2 (cf. Note 6,2).

Note 8,2. Let

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \varepsilon\, v(r, \psi)\,, \quad \frac{\mathrm{d}\psi}{\mathrm{d}t} = \omega + \varepsilon\, w(r, \psi) \tag{68,2}$$

be given. Suppose that $v(w)$ is a continuous map from $G_1 \times E_m$ to $E_n(E_m)$, G_1 being an open subset of E_n, $v(r, \psi + e_i) = v(r, \psi)$, $w(r, \psi + e_i) = w(r, \psi)$, $r \in G_1$, $\psi \in E_m$, $e_i = (\delta_{i1}, \ldots, \delta_{\tau m})$, $\delta_{i,i} = 1$, $\delta_{i,}j = 0$ for $i \neq j$, $i, j = 1, 2, \ldots, m$, the derivatives $\partial v/\partial r$, $\partial v/\partial \psi$, $\partial w/\partial r$, $\partial w/\partial \psi$ fulfil a Hölder's condition, $\omega = (\omega_1, \ldots, \omega_m)$, the numbers ω_τ, $i = 1, 2, \ldots, m$ are rationally independent (i.e. if $\alpha_1, \ldots, \alpha_m$ are integers, $|\alpha_1| +$ $+ \ldots + |\alpha_m| \neq 0$, then $\alpha_1\omega_1 + \ldots + \alpha_m\omega_m \neq 0$). Put

$$v_0(r) = \int_0^1 \ldots \int_0^1 v(r, \psi)\,\mathrm{d}\psi_1 \ldots \mathrm{d}\psi_m\,, \quad w_0(r) = \int_0^1 \ldots \int_0^1 w(r, \psi)\,\mathrm{d}\psi_1 \ldots \mathrm{d}\psi_m\,.$$

Let there exist a solution $r_0 \in G_1$ of $v_0(r) = 0$ and let the real parts of the characteristic values of $\left.\frac{\partial v_0}{\partial r}\right|_{r=r_0}$ be negative. Choose $\varkappa > 0$ in such a way that $r \in E_n$, $\|r - r_0\| \leqq \varkappa$ implies that $r \in G_1$. It may be proved that to every $\zeta > 0$ there exists an $\varepsilon_0 > 0$ that

$$\left\|\int_{\tau_1}^{\tau_2} [v(r, \psi + \omega\tau/\varepsilon) - v_0(r)]\,\mathrm{d}\tau\right\| \leqq \tfrac{1}{2}\varepsilon\,, \quad \left\|\int_{\tau_1}^{\tau_2} [w(r, \psi + \omega\tau/\varepsilon) - w_0(r)]\,\mathrm{d}\tau\right\| \leqq \tfrac{1}{2}\zeta$$

for $r \in G_1$, $\|r - r_0\| \leqq \varkappa$, $\psi \in E_m$, $0 < \varepsilon \leqq \varepsilon_0$ (for $m = 2$ cf. [1], section 2,3). Therefore the assumptions of Theorem 1,2 are fulfilled, if we put $C = E_n$, $\mathscr{C} = E_m$, $c = r - r_0$, $\gamma = \psi - \omega t$, $\tau = \varepsilon t$, $a(c, \gamma, \tau) = v(c + r_0, \gamma + \omega\tau/\varepsilon)$, $\alpha(c, \gamma, \tau) = w(c + r_0, \gamma + \omega\tau/\varepsilon)$, $a_0(c) = v_0(c + r_0)$, $\alpha_0(c) = w_0(c + r_0)$ and there exists a unique integral manifold $c = p(\gamma, \tau)$ of

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = a(c, \gamma, \tau), \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \alpha(c, \gamma, \tau), \tag{69,2}$$

$p(\gamma + e_i, \tau) = p(\gamma, \tau)$, $i = 1, 2, \ldots, m$ (cf. Note 7,2). $r = r_0 + p(\psi - \omega t, \varepsilon t)$ is an integral manifold of (68,2). As (68,2) is an autonomous system, $r = r_0 + p(\psi - \omega(t + \sigma), \varepsilon(t + \sigma))$ is an integral manifold of (68,2) for every $\sigma \in E$; the uniqueness of the integral manifold $c = p(\gamma, \tau)$ of (69,2) implies that $p(\psi - \omega t, \varepsilon t) = p^*(\psi)$ does not depend on t so that the integral manifold of (68,2) is found in the form $r = r_0 + p^*(\psi)$, $p^*(\psi + e_i) = p^*(\psi)$, $i = 1, 2, \ldots, m$ and the stability properties follow from Theorem 1,2.

Let us apply Theorem 1,1 and Lemma 2,1 from [5] to (68,2). Put (using the notations of section 1, [5])

$$h = r - r_0, \ \varphi = M\psi, \ M = \operatorname{diag}(\pi\omega_i^{-1}), \quad i = 1, 2, \ldots, m,$$

$$\Phi_1(t, \varphi, h, \varepsilon) = \varepsilon M w_0(h + r_0) = \Phi_1^*(h, \varepsilon),$$

$$\Phi_2(t, \varphi, h, \varepsilon) = M[w(h + r_0, M^{-1}\varphi) - w_0(h + r_0)].$$

Φ_1^* has to satisfy the Lipschitz condition

$$\|\Phi_1^*(h_2, \varepsilon) - \Phi_1^*(h_1, \varepsilon)\| \leqq \eta(\varrho, \varepsilon)\, \|h_2 - h_1\|$$

for $\|h_1\|, \|h_2\| \leqq \varrho$, $\varepsilon^{-1}\,\eta(\varrho, \varepsilon) \to 0$ with $\varepsilon \to 0$, $\varrho \to 0$; it is necessary to suppose in addition that $\left.\dfrac{\partial w_0}{\partial r}\right|_{r=r_0} = 0$.

Note 9,2. Theorem 1,2 may be extended to such classes of $f, f_0(F, F_0)$ that Theorems 4,1 and 6,1 (Theorems 7,1 and 9,1) are to be used in the proof instead of Theorems 1,1 and 3,1. We shall formulate a Theorem of this type, which will be needed in Notes 3,5 and 5,5.

Replace the system (2,2) by the system of generalized equations

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = \mathrm{D}_\tau A(c, \gamma, \tau), \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \mathscr{A}(c, \gamma, \tau) \tag{70,2}$$

and put $F = (A, \mathscr{A})$, $F_0(x, \tau) = \int_0^\tau f_0(x, \sigma)\,\mathrm{d}\sigma$, $f_0 = (a_0, \alpha_0)$.

Theorem 6,2. *Let f_0 fulfil* (4,2), (5,2), (6,2), (7,2) *and let* (9,2) *hold. Let F be defined for $x \in G$, $\tau \in E_1$ and fulfil the following conditions*

$$\|\Delta_\tau^\sigma F(x, \tau)\| \leqq K_1 \sigma, \ x \in G, \ \tau \in E_1, \ 0 \leqq \sigma \leqq 1,$$

$$\|\Delta_\tau^\sigma \Delta_x^z F(x, \tau)\| \leqq K_1 \|z\| \sigma, \ x, x + z \in G, \ \tau \in E_1, \ 0 \leqq \sigma \leqq 1,$$

$$\|\Delta_\tau^\sigma \Delta_x^{z_1} \Delta_x^{z_2} F(x, \tau)\| \leqq \omega(\|z_1\|) \|z_2\| \sigma, \ x, x + z, \ x + z_2, \ x + z_1 + z_2 \in G,$$

$$\tau \in E_1, \ 0 \leqq \sigma \leqq 1 .$$

Let in addition

$$\|\Delta_\tau^\sigma [F(x, \tau) - F_2(x, \tau)]\| \leqq \zeta, \quad x \in G, \ \tau \in E_1, \quad 0 \leqq \sigma \leqq 1 .$$

Then there exist positive constants $\zeta_1, \varkappa_2, L, K', \nu', K''$ depending on $K_1, \varkappa, \mu, \nu$ only, $\varkappa_2 < \varkappa$, $L \leqq (6K_1)^{-1}$ in such a way that $0 < \zeta \leqq \zeta_1$ implies that there exists a map p from $\mathscr{C} \times E_1$ to C and assertions (i)–(vii) *of Theorem* 1,2 *hold* ((2,2) *being replaced by* (70,2)).

In order to prove Theorem 6,2 the proof of Theorem 1,2 may be used with the only change that references to Theorems 1,1 and 3,1 are replaced by the ones to Theorems 7,1 and 9,1. Note 7,2 may be extended in a similar manner.

3. FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section Theorem 2,2 and Lemma 11,2 are used in order to establish the existence of integral manifolds for functional differential equations. As Lemma 11,2 is proved for differential equations with no time lag, the existence of integral manifolds will be proved for functional differential equations which are near to differential equations without time lag.

Let Y, R, Φ be Banach spaces, $Y = R \times \Phi$, $\|y\| = \|r\| + \|\varphi\|$ for $y = (r, \varphi) \in Y$, $\|y\|, \|r\|, \|\varphi\|$ denoting the norms of y, r, φ in the respective space. Let $\varkappa > 0$ and let $G = E[(r, \varphi) \in Y; \ r \in R, \ \|r\| < \varkappa, \ \varphi \in \Phi]$; let $U(G)$ ($U(Y)$, $U(R)$, $U(\Phi)$) be the set of continuous maps from $\langle -1, 0 \rangle$ to $G(Y, R, \Phi)$. Let h be a map from $G \times E_1$ to Y, let g be a map from $U(G) \times E_1$ to Y and put $h = (h_R, h_\Phi)$, $g = (g_R, g_\Phi)$, h_R, g_R being maps to R h_Φ, g_Φ being maps to Φ. The maps from intervals in E_1 to G will be denoted by $y = (r, \varphi)$, $z = (s, \psi)$, r, s being the corresponding maps to R and φ, ψ being the corresponding maps to Φ. By $y(\tau), r(\tau), \varphi(\tau)$ will be denoted the values of y, r, φ. If $y = (r, \varphi)$, $z = (s, \psi) \in U(Y)$ ($r, s \in U(R)$, $\varphi, \psi \in U(\Phi)$) and if $\lambda \in E_1$, let $\|y\|, \|r\|, \|\varphi\|, \lambda y, \lambda r, \lambda \varphi, y + z, r + s, \varphi + \psi$ have its usual meaning (i.e. $\|r\| = \sup_{\sigma \in \langle -1, 0 \rangle} \|r(\sigma)\|$ etc.). Obviously $\|y\| = \|r\| + \|\varphi\|$.

If $y = (r, \varphi)$ is a continuous map from $\langle \tau_1 - 1, \tau_2 \rangle$ ($\langle \tau_1 - 1, \tau_2)$, $\langle \tau_1 - 1, \infty)$, E_1) to G, $\tau_1 < \tau_2$ and if $\tau \in \langle \tau_1, \tau_2 \rangle$ ($\langle \tau_1, \tau_2)$, $\langle \tau_1, \infty)$, E_1) let $y_\tau = (r_\tau, \varphi_\tau)$ be the element of $U(G)$, which is defined by $y_\tau(\sigma) = y(\tau + \sigma)$ for $\sigma \in \langle -1, 0 \rangle$, r_τ and φ_τ being the corresponding maps to $U(R)$ and $U(\Phi)$. A map $y = (r, \varphi)$ from $\langle \tau_1 - 1, \tau_2 \rangle$ ($\langle \tau_1 - 1, \tau_2)$, $\langle \tau_1 - 1, \infty)$, E_1) to G will be called a solution of

$$(1,3) \qquad \frac{dr}{d\tau} = h_R(r, \varphi, \tau) + g_R(r_\tau, \varphi_\tau, \tau)\,, \quad \frac{d\varphi}{d\tau} = h_\Phi(r, \varphi, \tau) + g_\Phi(r_\tau, \varphi_\tau, \tau)\,,$$

if y is a continuous map and if the derivatives $dr/d\tau$, $d\varphi/d\tau$ exist and fulfil (1,3) for $\tau \in \langle \tau_1, \tau_2 \rangle$ ($\langle \tau_1, \tau_2)$, $\langle \tau_1, \infty)$, E_1).

It will be supposed that h and g fulfil the following conditions:

$$(2,3) \qquad h \text{ is continuous and } \|h(y, \tau)\| \leqq K_1 \text{ for } y \in G,\ \tau \in E_1,$$

$$(3,3) \qquad \frac{\partial h}{\partial y} \text{ exists and } \left\|\frac{\partial h}{\partial y}(y, \tau)\right\| \leqq K_1 \text{ for } y \in G,\ \tau \in E_1$$

(the norm of $\partial h/\partial y(y, \tau)$ being defined in the usual way),

$$(4,3) \qquad \left\|\frac{\partial h}{\partial y}(z, \tau) - \frac{\partial h}{\partial y}(y, \tau)\right\| \leqq K_1 \|z - y\|^\mu,\ 0 < \mu \leqq 1 \text{ for } y, z \in G,\ \tau \in E_1,$$

$$(5,3) \qquad h_R(0, \tilde{\varphi}, \tau) = 0 \text{ for } \tilde{\varphi} \in \Phi,\ \tau \in E_1,$$

$$(6,3) \qquad h_\Phi(0, \tilde{\varphi}, \tau) = h^*_\Phi(\tau) \text{ does not depend on } \tilde{\varphi},\ \tilde{\varphi} \in \Phi,\ \tau \in E_1,$$

$$(7,3) \qquad g \text{ is continuous, } \|g(\bar{y}, \tau)\| \leqq \zeta \text{ for } \bar{y} \in U(G),\ \tau \in E_1,$$

$$(8,3) \qquad \|g(\bar{z}, \tau) - g(\bar{y}, \tau)\| \leqq \zeta \|\bar{z} - \bar{y}\| \text{ for } \bar{z}, \bar{y} \in U(G),\ \tau \in E_1.$$

Denote by w the map from $\Phi \times E_1$ to $U(\Phi)$ defined by $w(\tilde{\varphi}, \tilde{\tau})(\sigma) = \tilde{\varphi} + \int_{\tilde{\tau}}^{\tilde{\tau}+\sigma} h^*_\Phi(\lambda)\, d\lambda$, $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$, $\sigma \in \langle -1, 0 \rangle$. Let $U_0(\Phi)$ be the set of such $\bar{y} \in U(\Phi)$ that $\bar{y}(0) = 0$.

Theorem 1,3. *Let h fulfil (2,3)–(6,3) and let g fulfil (7,3) and (8,3). Put $H(\tilde{\varphi}, \tau) = \partial h_R/\partial r(0, \tilde{\varphi}, \tau)$, $\tilde{\phi} \in \Phi$, $\tau \in E_1$ and suppose that the solutions of the linear equation*

$$\frac{dr}{d\tau} = H\left(\tilde{\varphi} + \int_{\tilde{\tau}}^{\tau} h^*_\Phi(\lambda)\, d\lambda, \tau\right) r\,, \quad r(\tilde{\tau}) = \tilde{r}$$

may be estimated by

$$(9,3) \qquad \|r(\tau)\| \leqq K_1 e^{-\nu(\tau - \tilde{\tau})} \|\tilde{r}\|\,, \quad \tau \geqq \tilde{\tau}\,,$$

$K_1 > 0$ and $\nu > 0$ being independent of $\tilde{r} \in R$, $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$.

Then there exist positive constants $\zeta_1, \varkappa_2, L, K', \nu', K''$ depending on $K_1, \varkappa, \mu, \nu$ only, $\varkappa_2 < \varkappa$, $L \leqq (6K_1)^{-1}$ in such a way that $0 < \zeta \leqq \zeta_1$ implies that there exist *a map u from $\Phi \times E_1$ to $U(R)$ and a map v from $\Phi \times E_1$ to $U_0(\Phi)$ and the following assertions hold:*

(i″) $\|u(\tilde{\varphi}, \tau)\| \leqq \varkappa_2$, $\|v(\tilde{\varphi}, \tau)\| \leqq \varkappa_2$ *for* $\tilde{\varphi} \in \Phi$, $\tau \in E_1$.

(ii″) $\|u(\tilde{\varphi}, \tau) - u(\tilde{\psi}, \tau)\| \leqq L\|\tilde{\varphi} - \tilde{\psi}\|$, $\|v(\tilde{\varphi}, \tau) - v(\tilde{\psi}, \tau)\| \leqq L\|\tilde{\varphi} - \tilde{\psi}\|$ *for* $\tilde{\varphi}$, $\tilde{\psi} \in \Phi$, $\tau \in E_1$.

(iii″) *For* $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$ *put* $\bar{r} = u(\tilde{\varphi}, \tilde{\tau})$, $\bar{\varphi} = v(\tilde{\varphi}, \tilde{\tau}) + w(\tilde{\varphi}, \tilde{\tau})$. *Then there exists a solution* r, φ *of* (1,3) *on* E_1 *and* $r_{\tilde{\tau}} = \bar{r}$, $\varphi_{\tilde{\tau}} = \bar{\varphi}$, $r_\tau = u(\varphi(\tau), \tau)$, $\varphi_\tau = w(\varphi(\tau), \tau) + v(\varphi(\tau), \tau)$ *for* $\tau \in E_1$.

(iv″) *If* $\bar{r} \in U(R)$, $\bar{\vartheta} \in U_0(\Phi)$, $\|\bar{r}\| + \|\bar{\vartheta}\| \leqq \varkappa_2$, $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$, *then the solution* (r, φ) *of* (1,3) $r_{\tilde{\tau}} = \bar{r}$, $\varphi_{\tilde{\tau}} = w(\tilde{\varphi}, \tilde{\tau}) + \bar{\vartheta}$ *exists on* $\langle\tilde{\tau}, \infty)$ *and*

$$\|r_\tau\| + \|\varphi_\tau - w(\varphi(\tau), \tau)\| \leqq \tfrac{3}{4}\varkappa\,,$$

$$\|r_\tau - u(\varphi(\tau), \tau)\| + \|\varphi_\tau - w(\varphi(\tau), \tau) - v(\varphi(\tau), \tau)\| \leqq$$

$$\leqq K' e^{-\nu'(\tau - \tilde{\tau})}[\|\bar{r} - u(\varphi(\tilde{\tau}), \tilde{\tau})\| + \|\bar{\vartheta} - v(\varphi(\tilde{\tau}), \tilde{\tau})\|] \quad \text{for} \quad \tau \geqq \tilde{\tau}\,.$$

(v″) *If* u', v' *are maps from* $\Phi \times E_1$ *to* $U(R)$, $\Phi \times E_1$ *to* $U_0(\Phi)$, *which fulfil* (i″) *and* (iii″), *then* $u' = u$, $v' = v$.

(vi″) *If* (r, φ) *has the same meaning as in* (iv″), *then there exists a solution* (s, ψ) *of* (1,3) *on* E_1, $s_\tau = u(\psi(\tau), \tau)$, $\psi_\tau = w(\psi(\tau), \tau) + v(\psi(\tau), \tau)$ *for* $\tau \in E_1$ *and* $\|r_\tau - s_\tau\| + \|\varphi_\tau - w(\psi(\tau), \tau) - v(\psi(\tau), \tau)\| + \|\varphi(\tau) - \psi(\tau)\| \leqq 2K'' e^{-\nu'(\tau - \tilde{\tau})}[\|\bar{r} - u(\varphi(\tilde{\tau}), \tilde{\tau})\| + \|\bar{\vartheta} - v(\varphi(\tilde{\tau}), \tilde{\tau})\|]$, $\tau \geqq \tilde{\tau}$.

(vii″) *The maps* u, v *are uniformly continuous on* $\Phi \times E_1$.

Note 1,3. According to (iii″) the set of such $(\bar{r}, \bar{\varphi}, \tau) \in U(R) \times U(\Phi) \times E_1$ that $\bar{r} = u(\bar{\varphi})(0), \tau)$, $\bar{\varphi} = w(\bar{\varphi}(0), \tau) + v(\bar{\varphi}(0), \tau)$ may be interpreted as an integral manifold of (1,3) and solutions (r, φ) from (iii″) are said to lie on this manifold. These solutions fulfil obviously

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = h_\Phi(u(\varphi(\tau), \tau)(0), \;\; \varphi(\tau), \tau) + g_\Phi(u(\varphi(\tau), \tau), \;\; w(\varphi(\tau), \tau) + v(\varphi(\tau), \tau), \tau), \;\; \tau \in E_1\,,$$

which is a differential equation without time lag.

Note 2,3. Let K_1, $\varkappa$, μ, ν be given. Similarly as in Note 4,2 there exist positive functions $\bar{\bar{L}}(\zeta)$, $\bar{\bar{\varkappa}}_2(\zeta)$ on $(0, \zeta_1\rangle$ such that $\lim_{\zeta \to 0+} \bar{\bar{L}}(\zeta) = 0 = \lim_{\zeta \to 0+} \bar{\bar{\varkappa}}_2(\zeta)$ and the following assertion holds: if $0 < \zeta \leqq \zeta_1$ and if all assumptions of Theorem 1,3 are fulfilled, then the maps u, v fulfil (i″), (ii″) with $\varkappa_2$, Lreplaced by $\bar{\bar{\varkappa}}_2(\zeta)$, $\bar{\bar{L}}_2(\zeta)$.

Note 3,3. As $r(\tau) = r_\tau(0)$, $s(\tau) = s_\tau(0)$, it follows from (vi″) that $\|r(\tau) - s(\tau)\| + \|\varphi(\tau) - \psi(\tau)\|$ tends to zero exponentially.

In order to prove Theorem 1,3 the following Lemmas will be needed:

Lemma 1,3. *Let $y = (r, \varphi)$ be a solution of (1,3) on $\langle \tau_1, \tau_2 \rangle$, $\tau_1 < \tau_2$, $r_{\tau_1} = \bar{r}$, $\varphi_{\tau_1} = w(\tilde{\varphi}, \tau_1) + \vartheta$, $\bar{r} \in U(R)$, $\vartheta \in U_0(\Phi)$, $\tilde{\varphi} \in \Phi$. Let $y_0 = (r_0, \varphi_0)$ be a solution of*

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = h_R(r, \varphi, \tau)\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = h_\Phi(r, \varphi, \tau) \tag{10,3}$$

on $\langle \tau_1, \tau_2 \rangle$, $r_0(\tau_1) = \bar{r}(0)$, $\varphi_0(\tau_1) = \tilde{\varphi}$. Then

$$\|r(\tau) - r_0(\tau)\| + \|\varphi(\tau) - \varphi_0(\tau)\| \leqq \zeta(\tau_2 - \tau_1)\, e^{K_1(\tau_2 - \tau_1)} \text{ for } \tau_1 \leqq \tau \leqq \tau_2\,. \tag{11,3}$$

Proof. Rewrite equations (1,3) and (10,3) in the form

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = h(y, \tau) + g(y_\tau, \tau)\,, \tag{12,3}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = h(y, \tau)\,. \tag{13,3}$$

As $y(\tau_1) = y_0(\tau_1)$, it follows from (3,3) and (7,3) that

$$\|y(\tau) - y_0(\tau)\| \leqq K_1 \int_{\tau_1}^{\tau} \|y(\sigma) - y_0(\sigma)\|\, \mathrm{d}\sigma + \zeta(\tau - \tau_1) \quad \text{for} \quad \tau \in \langle \tau_1, \tau_2 \rangle$$

and hence (11,3) follows.

Lemma 2,3. *Let $\zeta \leqq 1$. There exists an $M_1 > 0$ depending on K_1, μ and $\tau_2 - \tau_1$ only that the following assertion holds:*

Let (r, φ), (s, ψ) be solutions of (1,3) on $\langle \tau_1, \tau_2 \rangle$, $\tau_1 < \tau_2$, $r_{\tau_1} = \bar{r}$, $\varphi_{\tau_1} = w(\tilde{\varphi}, \tau_1) + \vartheta$, $s_{\tau_1} = \bar{s}$, $\psi_{\tau_1} = w(\tilde{\psi}, \tau_1) + \delta$, $\bar{r}, \bar{s} \in U(R)$, $\vartheta, \delta \in U_0(\Phi)$, $\tilde{\varphi}, \tilde{\psi} \in \Phi$; let (r_0, φ_0), (s_0, ψ_0) be solutions of (10,3) on $\langle \tau_1, \tau_2 \rangle$, $r_0(\tau_1) = \bar{r}(0)$, $s_0(\tau_1) = \bar{s}(0)$, $\varphi_0(\tau_1) = \tilde{\varphi}$, $\psi_0(\tau_1) = \tilde{\psi}$. Then

$$\begin{aligned} &\|r(\tau) - s(\tau) - r_0(\tau) + s_0(\tau)\| + \|\varphi(\tau) - \psi(\tau) - \varphi_0(\tau) + \psi_0(\tau)\| \leqq \\ &\leqq \zeta^\mu M_1 [\|\bar{r} - \bar{s}\| + \|\vartheta - \delta\| + \|\tilde{\varphi} - \tilde{\psi}\|]\,, \quad \tau \in \langle \tau_1, \tau_2 \rangle\,. \end{aligned} \tag{14,3}$$

Proof. Putting $y = (r, \varphi)$, $z = (s, \psi)$, $y_0 = (r_0, \varphi_0)$, $z_0 = (s_0, \psi_0)$ we find that

$$\begin{aligned} &y(\tau) - z(\tau) - y_0(\tau) + z_0(\tau) = \\ &\int_{\tau_1}^{\tau} [h(y(\sigma), \sigma) - h(y_0(\sigma) - z_0(\sigma) + z(\sigma), \sigma)]\, \mathrm{d}\sigma + \\ &+ \int_{\tau_1}^{\tau} [h(y_0(\sigma) - z_0(\sigma) + z(\sigma), \sigma) - h(z(\sigma), \sigma) - h(y_0(\sigma), \sigma) + h(z_0(\sigma), \sigma)]\, \mathrm{d}\sigma + \\ &+ \int_{\tau_1}^{\tau} [g(y_\sigma, \sigma) - g(z_\sigma, \sigma)]\, \mathrm{d}\sigma = I_1 + I_2 + I_3\,. \end{aligned} \tag{15,3}$$

Denoting $\bar{y} = y_{\tau_1}$, $\bar{z} = z_{\tau_1}$ one obtains from

$$y(\tau) - z(\tau) = y(\tau_1) - z(\tau_1) + \int_{\tau_1}^{\tau} [h(y(\sigma), \sigma) - h(z(\sigma), \sigma)] \, d\sigma + \\ + \int_{\tau_1}^{\tau} [g(y_\sigma, \sigma) - g(z_\sigma, \sigma)] \, d\sigma$$

by standard methods (cf. (3,3), (8,3) and $\zeta \leqq 1$) that

$$(16,3) \qquad \|y(\tau) - z(\tau)\| \leqq e^{(K_1+1)(\tau_2-\tau_1)} \|\bar{y} - \bar{z}\| .$$

Therefore

$$(17,3) \qquad \|I_3\| \leqq \zeta(\tau_2 - \tau_1) \, e^{(K_1+1)(\tau_2-\tau_1)} \|\bar{y} - \bar{z}\| .$$

Similarly as (16,3) one obtains (cf. $y_{0\tau_1} = \bar{y}$, $z_{0\tau_1} = \bar{z}$)

$$\|y_0(\tau) - z_0(\tau)\| \leqq e^{K_1(\tau_2-\tau_1)} \|\bar{y} - \bar{z}\| , \quad \tau \in \langle \tau_1, \tau_2 \rangle .$$

As

$$h(y_0(\sigma) - z_0(\sigma) + z(\sigma), \sigma) - h(z(\sigma), \sigma) - h(y_0(\sigma), \sigma) + h(z_0(\sigma), \sigma) = \\ = \int_0^1 \left[\frac{\partial h}{\partial y} (\lambda(y_0(\sigma) - z_0(\sigma)) + z(\sigma), \sigma) - \right. \\ \left. - \frac{\partial h}{\partial y} (\lambda(y_0(\sigma) - z_0(\sigma)) + z_0(\sigma), \sigma) \right] d\sigma [y_0(\sigma) - z_0(\sigma)] ,$$

it follows that (cf. (4,3) and Lemma 1,3)

$$(18,3) \qquad \|I_2\| \leqq K_1 \int_{\tau_1}^{\tau} \|z(\tau) - z_0(\tau)\|^\mu . \|y_0(\sigma) - z_0(\sigma)\| \, d\sigma \leqq \\ \leqq \zeta^\mu K_1(\tau_2 - \tau_1)^{1+\mu} \, e^{K_1(\tau_2-\tau_1)(1+\mu)} \|\bar{y} - \bar{z}\| .$$

(15,3), (3,3), (17,3) and (18,3) imply that

$$\|y(\tau) - z(\tau) - z_0(\tau) + y_0(\tau)\| \leqq K_1 \int_{\tau_1}^{\tau} \|y(\sigma) - z(\sigma) - y_0(\sigma) + z_0(\sigma)\| \, d\sigma + \\ + [\zeta(\tau_2 - \tau_1) \, e^{(K_1+1)(\tau_2-\tau_1)} + \zeta^\mu K_1(\tau_2 - \tau_1)^{1+\mu} \, e^{K_1(\tau_2-\tau_1)(1+\mu)}] \, \|\bar{y} - \bar{z}\| .$$

Consequently there exists an $M_1 > 0$ depending on K_1, μ and $\tau_2 - \tau_1$ only that

$$\|y(\tau) - z(\tau) - y_0(\tau) + z_0(\tau)\| \leqq \zeta^\mu M_1 \|\bar{y} - \bar{z}\| .$$

Taking into account that $\|w(\tilde{\varphi}, \tau_1) - w(\tilde{\psi}, \tau_1)\| = \|\tilde{\varphi} - \tilde{\psi}\|$, $\|\bar{y} - \bar{z}\| = \|\bar{r} - \bar{s}\| + \|\varphi_{\tau_1} - \psi_{\tau_1}\| \leqq \|\bar{r} - \bar{s}\| + \|\bar{\vartheta} - \bar{\delta}\| + \|\tilde{\varphi} - \tilde{\psi}\|$ and that $\|\hat{y}\| = \|\hat{r}\| + \|\hat{\varphi}\|$ for $\hat{y} = (\hat{r}, \hat{\varphi}) \in Y$, one concludes that Lemma 2,3 holds.

Proof of Theorem 1,3. In order to prove Theorem 1,3 Theorem 2,2 will be applied. Let us put $C = U(R) \times U_0(\Phi)$, $\mathscr{C} = \Phi$, $X = C \times \mathscr{C}$, the norms being defined by $\|c\| = \|\bar{r}\| + \|\bar{\vartheta}\|$ for $c = (\bar{r}, \bar{\vartheta})$, $\bar{r} \in U(R)$, $\bar{\vartheta} \in U_0(\Phi)$,

$$\|x\| = \|c\| + \|\gamma\| \quad \text{for} \quad x = (c, \gamma),\ c \in C,\ \gamma \in \mathscr{C}\,.$$

Put $\hat{\varkappa} = \varkappa$, $\hat{G} = \mathscr{E}[(c, \gamma);\ c \in C,\ \|c\| < \hat{\varkappa},\ \gamma \in \mathscr{C}]$. To every solution $y = (r, \varphi)$ of (1,3) on $J = \langle \tau_1, \tau_2 \rangle$ $(\langle \tau_1, \tau_2), \langle \tau_1, \infty), E_1)$, $-\infty < \tau_1 < \tau_2 < \infty$ there corresponds a map $x = (c, \gamma)$ from J to X defined in the following way:

$$c(\tau) = (r_\tau, \varphi_\tau - w(\varphi(\tau), \tau)),\ \gamma(\tau) = \varphi(\tau) \quad \text{for} \quad \tau \in J\,; \tag{19,3}$$

let $\mathscr{X}$ be the set of such maps $x = (c, \gamma)$ corresponding in the above way to the solutions y of (1,3) that $\|c(\tau)\| < \hat{\varkappa}$ for $\tau \in J$, $J(x) = J$. Let $y_0 = (r_0, \varphi_0)$ be a continuous map from $J_1 = \langle \tau_1 - 1, \tau_2 \rangle$ $(\langle \tau_1 - 1, \tau_2), \langle \tau_1 - 1, \infty), E_1)$, $-\infty < \tau_1 < \tau_2 < \infty$ to G, let the derivative $\mathrm{d}y_0/\mathrm{d}\tau$ exist and fulfil (10,3) for $\tau \in (\tau_1, \tau_2\rangle$, $((\tau_1, \tau_2), (\tau_1, \infty), E_1)$; to every such map y_0 there corresponds a map $x_0 = (c_0, \gamma_0)$ from $J = \langle \tau_1, \tau_2 \rangle$ $(\langle \tau_1, \tau_2), \langle \tau_1, \infty), E_1)$ to X defined in the following way:

$$c_0(\tau) = (r_{0\tau}, \varphi_{0\tau} - w(\varphi_0(\tau), \tau)),\ \gamma_0(\tau) = \varphi_0(\tau) \quad \text{for} \quad \tau \in J\,; \tag{20,3}$$

let $\mathscr{X}_0$ be the set of all such maps $x_0 = (c_0, \gamma_0)$ corresponding in the above way to a map y_0 that $\|c_0(\tau)\| < \hat{\varkappa}$ for $\tau \in J = J(x_0)$.

It is a simple consequence of elementary properties of differential equations and functional differential equations that $\mathscr{X}_0$ and $\mathscr{X}$ are flows in $\hat{G}$. Let us verify that $\mathscr{X}_0$ fulfils conditions (Ω_1), (Ω_2) and (Ω_3) from Theorem 2,2.

Let $\tilde{\gamma} = \tilde{\varphi} \in \mathscr{C} = \Phi$, $\tilde{\tau} \in E_1$; put $\varphi_0(\tau) = \tilde{\varphi} + \int_{\tilde{\tau}}^{\tau} h_\Phi^*(\lambda)\,\mathrm{d}\lambda$, $r_0(\tau) = 0$ for $\tau \geqq \tilde{\tau} - 1$. According to (5,3) and (6,3) $y_0 = (r_0, \varphi_0)$ is a solution of (10,3) on $\langle \tau_1 - 1, \infty)$; according to the definition of $\varphi_{0\tau}$ and w it follows that $\varphi_{0\tau} = w(\varphi_0(\tau), \tau)$ for $\tau \geqq \tilde{\tau}$ and $x_0 = (c_0, \gamma_0)$, $c_0(\tau) = 0 \in C = U(R) \times U_0(\Phi)$, $\gamma_0(\tau) = \varphi_0(\tau) \in \mathscr{C} = \Phi$ for $\tau \geqq \tilde{\tau}$, is an element of $\mathscr{X}_0$ and (Ω_1) is satisfied.

In order to verify (Ω_2) and (Ω_3), observe that Lemma 11,2 may be applied to (10,3). Choose $\nu_1' = \frac{1}{2}\mu\nu$; according to Lemma 11,2 there exists a $K_2 = K_2(K_1, \varkappa, \mu, \nu, \frac{1}{2}\mu\nu)$ and to every ϱ, $0 < \varrho \leqq 1$ there exists a $\varkappa_5(\varrho) > 0$ – $\varkappa_5$ depends on $K_1, \varkappa, \mu, \nu, \nu_1' = \frac{1}{2}\mu\nu$ and ϱ; by writing $\varkappa_5(\varrho)$ let us underline the dependence on ϱ while $K_1, \varkappa, \mu, \nu$ are kept fixed – that the solutions (r_0, φ_0), (s_0, ψ_0) of (10,3), $r_0(\tilde{\tau}) = \tilde{r}$, $\varphi_0(\tilde{\tau}) = \tilde{\varphi}$, $s_0(\tilde{\tau}) = \tilde{s}$, $\psi_0(\tilde{\tau}) = \tilde{\psi}$, $\tilde{r}, \tilde{s} \in R$, $\|\tilde{r}\|, \|\tilde{s}\| \leqq \varkappa_5(\varrho)$, $\tilde{\varphi}, \tilde{\psi} \in \Phi$, $\tilde{\tau} \in E_\tau$ exist on $\langle \tilde{\tau}, \infty)$ and the following inequality takes place

$$\|r_0(\tau) - s_0(\tau)\| \leqq K_2 e^{-\nu_1'(\tau - \tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|]\,, \quad \tau \geqq \tilde{\tau}\,, \tag{21,3}$$

$$\|\varphi_0(\tau) - \psi_0(\tau) - \tilde{\varphi} + \tilde{\psi}\| \leqq K_2[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|]\,, \quad \tau \geqq \tilde{\tau}\,. \tag{22,3}$$

If $\tilde{s} = 0$, $\tilde{\psi} = \tilde{\varphi}$ then it follows from (21,3) (cf. (5,3)) that

$$\|r_0(\tau)\| \leqq K_2 e^{-\nu_1'(\tau - \tilde{\tau})}\|\tilde{r}\|\,. \tag{23,3}$$

Let $\tilde{\tau} \in E_1$, $\tilde{c} \in C$, $\tilde{c} = (\bar{r}, \bar{\vartheta})$, $\bar{r} \in U(R)$, $\bar{\vartheta} \in U_0(\Phi)$, $\|\tilde{c}\| = \|\bar{r}\| + \|\bar{\vartheta}\| \leqq \varkappa_5(1)$, $\tilde{\gamma} = \tilde{\varphi} \in \mathscr{C} = \Phi$. Put $\tilde{r} = \bar{r}(0)$ and let $y_0 = (r_0, \varphi_0)$ be the solution of (10,3), $r_0(\tilde{\tau}) = \tilde{r}$, $\varphi_0(\tilde{\tau}) = \tilde{\varphi}$; this solution is defined for $\tau \geqq \tilde{\tau}$ and fulfils (23,3). Let us extend the map $y_0 = (r_0, \varphi_0)$ as follows: for $\tau \in \langle \tilde{\tau} - 1, \tilde{\tau})$ put $r_0(\tau) = \bar{r}(\tau - \tilde{\tau})$, $\varphi_0(\tau) = w(\tilde{\varphi}, \tilde{\tau})(\tau - \tilde{\tau}) + \bar{\vartheta}(\tau - \tilde{\tau})$. As $\|\tilde{c}\| = \|\bar{r}\| + \|\bar{\vartheta}\| \geqq \|\bar{r}\|$, $K_2 > 1$, it follows from (21,3) that the element $x_0 = (c_0, \gamma_0)$ that corresponds to the map $y_0 = (r_0, \varphi_0)$ according to the definition of fulfils the inequalities

$$\|c_0(\tau)\| \leqq K_2 \|\tilde{c}\| \qquad \text{for} \quad \tau \in \langle \tilde{\tau}, \tilde{\tau} + 1 \rangle$$

$$\|c_0(\tau)\| \leqq K_2 e^{-\nu_1'(\tau - 1 - \tau)} \|\tilde{c}\| \quad \text{for} \quad \tau \in \langle \tilde{\tau} + 1, \infty)$$

and (13,2) is satisfied, if $\hat{K}_2 \geqq K_2 e^{\nu_1'}$, $\nu_1 \leqq \nu_1'$.

Let $0 < \varrho \leqq 1$, suppose in addition that $\|\tilde{c}\| \leqq \varkappa_5(\varrho)$ and let $\tilde{\tau}, \tilde{c}, \bar{r}, \bar{\vartheta}, \tilde{\gamma} = \tilde{\varphi}$, $y_0 = (r_0, \varphi_0)$, $x_0 = (c_0, \gamma_0)$ have the same meaning as above, let $\tilde{b} \in C$, $\tilde{b} = (\bar{s}, \bar{\delta})$, $\bar{s} \in U(R)$, $\bar{\delta} \in U_0(\Phi)$, $\|\tilde{b}\| = \|\bar{s}\| + \|\bar{\delta}\| \leqq \varkappa_5(\varrho)$, $\tilde{\beta} = \tilde{\psi} + \mathscr{C} = \Phi$. Put $\tilde{s} = \bar{s}(0)$ and let $z_0 = (s_0, \psi_0)$ be the solution of (10,3), $s_0(\tilde{\tau}) = \tilde{s}$, $\psi_0(\tilde{\tau}) = \tilde{\psi}$; this solution is defined for $\tau \geqq \tilde{\tau}$. Let us extend the map $z_0 = (s_0, \psi_0)$ as follows: for $\tau \in \langle \tilde{\tau} - 1, \tilde{\tau})$ put $s_0(\tau) = \bar{s}(\tau - \tilde{\tau})$, $\psi_0(\tau) = w(\tilde{\psi}, \tilde{\tau})(\tau - \tilde{\tau}) + \bar{\delta}(\tau - \tilde{\tau})$ and let $x_0^* = (b_0, \beta_0)$ be the element of $\mathscr{X}_0$ that corresponds to the map $y_0 = (s_0, \psi_0)$ according to the definition of $\mathscr{X}_0$. It follows from (21,3) that

$$\|r_{0\tau}(\sigma) - s_{0\tau}(\sigma)\| = \|r_0(\tau + \sigma) - s_0(\tau + \sigma)\| \leqq$$
$$\leqq K_2 e^{-\nu_1'(\tau - 1 - \tilde{\tau})} [\|\tilde{r} - \tilde{s}\| + \varrho \|\tilde{\varphi} - \tilde{\psi}\|] \quad \text{for} \quad \tau + \sigma \geqq \tilde{\tau}, \ \sigma \in \langle -1, 0 \rangle .$$

Obviously

$$\|r_{0\tau}(\sigma) - s_{0\tau}(\sigma)\| \leqq \|\bar{r} - \bar{s}\| \quad \text{for} \quad \tau + \sigma \leqq \tilde{\tau}, \ \sigma \in \langle -1, 0 \rangle, \ \tau \geqq \tilde{\tau} .$$

Therefore

(*)
$$\|r_{0,} - s_{0,}\| \leqq K_2 e^{-\nu_1'(\tau - 1 - \tilde{\tau})} [\|\tilde{r} - \tilde{s}\| + \varrho \|\tilde{\varphi} - \tilde{\psi}\|] + $$
$$+ \|\bar{r} - \bar{s}\| \,.\, S(\tau - 1 - \tilde{\tau}) ,$$

$$\tau \geqq \tilde{\tau}, \ S(\lambda) = 1 \quad \text{for} \quad \lambda \leqq 0 , \quad S(\lambda) = 0 \quad \text{for} \quad \lambda > 0 .$$

If $\tau \geqq \tilde{\tau}$, $\sigma \in \langle -1, 0 \rangle$, then

(**)
$$\varphi_{0\tau}(\sigma) - w(\varphi_0(\tau), \tau)(\sigma) - \psi_{0\tau}(\sigma) + w(\psi_0(\tau), \tau)(\sigma) =$$

$$= \varphi_0(\tau + \sigma) - \left[\varphi_0(\tau) + \int_\tau^{\tau + \sigma} h_\Phi^*(\lambda) \, \mathrm{d}\lambda \right] - \psi_0(\tau + \sigma) + \left[\psi_0(\tau) + \int_\tau^{\tau + \sigma} h_\Phi^*(\lambda) \, \mathrm{d}\lambda \right] =$$

$$= \varphi_0(\tau + \sigma) - \varphi_0(\tau) - \psi_0(\tau + \sigma) + \psi_0(\tau) .$$

If in addition $\tau + \sigma \geqq \tilde{\tau}$, then

$$\varphi_{0\tau}(\sigma) - w(\varphi_0(\tau)\,\tau)(\sigma) - \psi_{0\tau}(\sigma) + w(\psi_0(\tau), \tau)(\sigma) =$$
$$= \int_\tau^{\tau+\sigma} [h_\Phi(r_0(\lambda), \varphi_0(\lambda), \lambda) - h_\Phi(s_0(\lambda), \varphi_0(\lambda), \lambda)]\, d\lambda +$$
$$+ \int_\tau^{\tau+\sigma} [h_\Phi(s_0(\lambda), \varphi_0(\lambda), \lambda) - h_\Phi(s_0(\lambda), \psi_0(\lambda), \lambda)]\, d\lambda\,.$$

Therefore (cf. (3,3), (4,3), (6,3), (21,3), (22,3), (23,3))

$$(\overset{**}{*}) \qquad \|\varphi_{0\tau}(\sigma) - w(\varphi_0(\tau), \tau(\sigma) - \psi_{0\tau}(\sigma) + w(\psi_0(\tau), \tau)(\sigma)\| \leqq$$
$$\leqq K_1 \int_{\tau+\sigma}^{\tau} \|r_0(\lambda) - s_0(\lambda)\|\, d\lambda + K_1 \int_{\tau+\sigma}^{\tau} \|s_0(\lambda)\|^\mu \|\psi_0(\lambda) - \varphi_0(\lambda)\|\, d\lambda \leqq$$
$$\leqq K_1 K_2 e^{-\nu_1'(\tau-1-\tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|] +$$
$$+ K_1 K_2^\mu e^{-\nu_1'\mu(\tau-1-\tilde{\tau})} \|\tilde{s}\|^\mu K_2[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|] \leqq$$
$$\leqq K_3 e^{\nu_1\mu(\tau-1-\tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|]\,, \quad K_3 = K_1 K_2 + K_1 K_2^{1+\mu} \varkappa^\mu\,.$$

If $\sigma + \tau < \tilde{\tau}$, then (cf. $(\overset{*}{*})$, $(\overset{**}{*})$)

$$\|\varphi_{0\tau}(\sigma) - w(\varphi_0(\tau), \tau)(\sigma) - \psi_{0\tau}(\sigma) + w(\psi_0(\tau), \tau)(\sigma)\| \leqq$$
$$\leqq \|\varphi_0(\tilde{\tau}) - \varphi_0(\tau) - \psi_0(\tilde{\tau}) + \psi_0(\tau)\| +$$
$$+ \|\varphi_0(\tau + \sigma) - \varphi_0(\tilde{\tau}) - \psi_0(\tau + \sigma) + \psi_0(\tilde{\tau})\| \leqq$$
$$\leqq K_3 e^{-\nu_1'\mu(\tau-1-\tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|] + \|\bar{\vartheta} - \bar{\delta}\|$$

(as $\varphi_0(\tau + \sigma) - \varphi_0(\tilde{\tau}) - \psi_0(\tau + \sigma) + \psi_0(\tilde{\tau}) = \bar{\vartheta}(\tau + \sigma) - \bar{\delta}(\tau + \sigma)$). Hence

$$(\dagger) \qquad \|\varphi_{0\tau} - w(\varphi_0(\tau), \tau) - \psi_{0\tau} + w(\psi_0(\tau), \tau)\| \leqq$$
$$\leqq K_3 e^{-\nu_1'\mu(\tau-1-\tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|] + \|\bar{\vartheta} - \bar{\delta}\|\,.\, S(\tau - 1 - \tilde{\tau})\,.$$

(20,3), (*) and (†) imply that

$$\|c_0(\tau) - b_0(\tau)\| = \|r_{0\tau} - s_{0,}\| + \|\varphi_{0\tau} - w(\varphi_0(\tau), \tau) - \psi_{0,} + w(\psi_0(\tau), \tau)\| \leqq$$
$$\leqq (K_2 + K_3)\, e^{-\nu_1'\mu(\tau-1-\tilde{\tau})}[\|\tilde{r} - \tilde{s}\| + \varrho\|\tilde{\varphi} - \tilde{\psi}\|] +$$
$$+ [\|\bar{r} - \bar{s}\| + \|\bar{\vartheta} - \bar{\delta}\|]\, S(\tau - 1 - \tilde{\tau})\,.$$

As $\|\tilde{c} - \tilde{b}\| = \|\bar{r} - \bar{s}\| + \|\bar{\vartheta} - \bar{\delta}\|$, $S(\tau - 1 - \tilde{\tau}) \leqq e^{-\nu_1'\mu(\tau-1-\tilde{\tau})}$ it follows that (14,2) holds for $\hat{K}_2 = (K_2 + K_3 + 1)\, e^{\nu_1'\mu}$, $\nu_1 = \nu_1'\mu$.

As $\gamma_0(\tau) = \varphi_0(\tau)$, $\beta_0(\tau) = \psi_0(\tau)$ for $\tau \geqq \tilde{\tau}$, $\tilde{\gamma} = \tilde{\varphi}$, $\tilde{\beta} = \tilde{\psi}$, it follows from (22,3) that (15,2) holds too.

Let $D, T, \varkappa_2, K', v', K''$ be defined in Theorem 2,2. Let $x = (c, \gamma)$, $x^* = (b, \beta) \in \mathscr{X}$, $x_0 = (c_0, \gamma_0)$, $x_0^* = (b_0, \beta_0) \in \mathscr{X}_0$,

$$J(x) = J(x^*) = J(x_0) = J(x_0^*) = \langle \tau_1, \tau_2 \rangle\,, \quad 0 < \tau_2 - \tau_1 \leqq T\,,$$

$$x(\tau_1) = x_0(\tau_1) = \tilde{x}\,, \quad x^*(\tau_1) = x_0^*(\tau_1) = \tilde{x}^*\,,$$

let $y = (r, \varphi)$, $z = (s, \psi)$ be solutions of (1,3), which correspond to x, x^* according to the definition of $\mathscr{X}$ and let $y_0 = (r_0, \varphi_0)$, $z_0 = (s_0, \psi_0)$ be the maps, which correspond to x_0 and x_0^* according to the definition of $\mathscr{X}_0$. Let $\tau \in \langle \tau_1, \tau_2 \rangle$. As

$$w(\varphi(\tau), \tau)(\sigma) - w(\varphi_0(\tau), \tau)(\sigma) = \varphi(\tau) - \varphi_0(\tau)\,,$$

$$w(\psi(\tau), \tau)(\sigma) - w(\psi_0(\tau), \tau)(\sigma) = \psi(\tau) - \psi_0(\tau)\,, \quad \sigma \in \langle -1, 0 \rangle\,,$$

it follows from Lemmas 1,3 and 2,3 and from the definition of (cf. (20,3)) that

$$(24,3) \qquad \|x(\tau) - x_0(\tau)\| = \|c(\tau) - c_0(\tau)\| + \|\gamma(\tau) - \gamma_0(\tau)\| =$$
$$= \|r_\tau - r_{0\tau}\| + \|\varphi_\tau - w(\varphi(\tau), \tau) - \varphi_{0\tau} + w(\varphi_0(\tau), \tau)\| + \|\varphi(\tau) - \varphi_0(\tau)\| \leqq$$
$$\leqq \sup_{\sigma \in \langle -1, 0 \rangle} \|r(\tau + \sigma) - r_0(\tau + \sigma)\| + \sup_{\sigma \in \langle -1, 0 \rangle} \|\varphi(\tau + \sigma) - \varphi_0(\tau + \sigma)\| +$$
$$+ 2\|\varphi(\tau) - \varphi_0(\tau)\| \leqq \zeta 4 T e^{K_1 T}\,,$$

$$(25,3) \qquad \|x(\tau) - x^*(\tau) - x_0(\tau) + x_0^*(\tau)\| =$$
$$= \|c(\tau) - b(\tau) - c_0(\tau) + b_0(\tau)\| + \|\gamma(\tau) - \beta(\tau) - \gamma_0(\tau) + \beta_0(\tau)\| =$$
$$= \|r_\tau - s_\tau - r_{0\tau} + s_{0\tau}\| + \|\varphi_\tau - w(\varphi(\tau), \tau) - \psi_\tau + w(\psi(\tau), \tau) -$$
$$- \varphi_{0\tau} + w(\varphi_0(\tau), \tau) + \psi_{0\tau} - w(\psi_0(\tau), \tau)\| + \|\varphi(\tau) - \psi(\tau) - \varphi_0(\tau) + \psi_0(\tau)\| \leqq$$
$$\leqq \zeta^\mu 4 M_1 [\|r_{\tau_1} - s_{\tau_1}\| + \|\varphi_{\tau_1} - w(\varphi(\tau_1), \tau_1) - \psi_{\tau_1} + w(\psi(\tau_1), \tau_1)\| +$$
$$+ \|\varphi(\tau_1) - \psi(\tau_1)\|] = \zeta^\mu 4 M_1 \|\tilde{x} - \tilde{x}^*\|\,.$$

(24,3) and (25,3) imply that there exists a $\zeta_1 > 0$ that $d_T(\mathscr{X}, \mathscr{X}_0) \leqq D$ for $\zeta \leqq \zeta_1$. Assuming $\zeta \leqq \zeta_1$ we may apply Theorem 2,2. As $C = U(R) \times U_0(\Phi)$, let us write the map p, the existence of which is guarranteed by Theorem 2,2, in the form $p = (u, v)$, u, v being the corresponding maps to $U(R), U_0(\Phi)$. Then assertions (i″) and (ii″) of Theorem 1,3 are obvious consequences of assertions (i′) and (ii′) of Theorem 2,2. For $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$, $\bar{r} = u(\tilde{\varphi}, \tilde{\tau})$, $\bar{\varphi} = v(\tilde{\varphi}, \tilde{\tau})$ put $\tilde{c} = (\bar{r}, \bar{\varphi})$, $\tilde{\gamma} = \tilde{\varphi}$. Let $(c, \gamma) \in \mathscr{X}$ fulfil (iii′) and let the map (r, φ) correspond to (c, γ) according to the definition of $\mathscr{X}$; (r, φ) fulfils (iii″). Similarly one proves (iv″). (v″) and (vii″) are obvious. For $\bar{r} \in U(R)$, $\bar{\vartheta} \in U_0(\Phi)$, $\|\bar{r}\| + \|\bar{\vartheta}\| \leqq \varkappa_2$, $\tilde{\varphi} \in \Phi$, $\tilde{\tau} \in E_1$ put $\tilde{c} = (\bar{r}, \bar{\vartheta})$, $\tilde{\gamma} = \tilde{\varphi}$; let $(c, \gamma) \in \mathscr{X}$, $J(c, \gamma) = \langle \tilde{\tau}, \infty)$, $c(\tilde{\tau}) = \tilde{c}$, $\gamma(\tilde{\tau}) = \tilde{\gamma}$, let $(b, \beta) \in \mathscr{X}$ fulfil the conditions of (vi′) and let (s, ψ) correspond to (b, β) according to the definition of $\mathscr{X}$. (s, ψ) is a solution of (1,3) on E_1. (19,3) and the relation $b(\tau) = p(s(\tau), \tau)$ for $\tau \in E_1$ imply that $s_\tau = u(\psi(\tau), \tau)$, $\psi_\tau = w(\psi(\tau), \tau) + v(\psi(\tau), \tau)$ for $\tau \in E_1$ and from the inequality in (vi′) it follows (cf. (19,3)) that

$$\|r_\tau - s_\tau\| + \|\varphi_\tau - w(\varphi(\tau), \tau) - \psi_\tau + w(\psi(\tau), \tau)\| + \|\varphi(\tau) - \psi(\tau)\| \leqq$$
$$\leqq K'' e^{-v'(\tau - \tilde{\tau})} [\|\bar{r} - u(\tilde{\varphi}, \tilde{\tau})\| + \|\bar{\vartheta} - v(\tilde{\varphi}, \tilde{\tau})\|]\,.$$

As $-\psi_\tau + w(\psi(\tau), \tau) = -v(\psi(\tau), \tau)$, $w(\varphi(\tau), \tau)(\sigma) = w(\psi(\tau), \tau)(\sigma) + \varphi(\tau) - \psi(\tau)$ for $\tau \geqq \tilde{\tau}$, $\sigma \in \langle -1, 0)$, it follows that

$$\|\varphi_\tau - w(\psi(\tau), \tau) - v(\psi(\tau), \tau)\| \leqq$$
$$\leqq \|\varphi_\tau - w(\varphi(\tau), \tau) - \psi_\tau + w(\psi(\tau), \tau)\| + \|\varphi(\tau) - \psi(\tau)\|$$

and the inequality in (vi″) holds. Theorem 1,3 is proved.

Note 4,3. Let $\vartheta \in \Phi$, $\bar{\vartheta} \in U(\Phi)$, $\bar{\vartheta}(\sigma) = \vartheta$ for $\sigma \in \langle -1, 0\rangle$. If $h(r, \varphi + \vartheta, \tau) = h(r, \varphi, \tau)$ for $r \in R$, $\|r\| < \varkappa$, $\varphi \in \Phi$, $\tau \in E_1$ and if $g(\bar{r}, \bar{\varphi} + \bar{\vartheta}, \tau) = g(\bar{r}, \bar{\varphi}, \tau)$ for $\bar{r} \in U(R)$, $\|\bar{r}\| < \varkappa$, $\bar{\varphi} \in U(\Phi)$, $\tau \in E_1$, then $u(\varphi + \vartheta, \tau) = u(\varphi, \tau)$, $v(\varphi + \vartheta, \tau) = v(\varphi, \tau)$ for $\varphi \in \Phi$, $\tau \in E_1$. This situation is a consequence of Theorem 3,2, as the flow which was constructed in the proof of Theorem 1,3 is periodic in γ with the period ϑ in this case. In the same manner the periodicity in τ and the almostperiodicity may be treated.

Note 5,3. Let the right hand side of

$$\frac{d\xi}{d\tau} = W(\xi) \tag{26,3}$$

be defined and continuous in an open subset G_0 of E_n and let the derivative $\partial W/\partial \xi$ (i.e. $\partial W_i/\partial \xi_j$ if $W = (W_1, \ldots, W_n)$, $\xi = (\xi_1, \ldots, \xi_n)$) fulfil a Hölder condition. Let there exist an invariant k-dimensional torus Z of (26,3), $k = 0, 1, \ldots, n - 1$. Put $R = E_{n-k}$, $\Phi = E_k$, $G = \mathscr{E}[(r, \varphi);\ r \in R,\ \|r\| < \varkappa,\ \varphi \in \Phi]$ and assume that there exists a map h from G to E_n and a map S from G to G_0 that h and S have derivatives with respect to (r, φ) which fulfil a Hölder condition, $\partial S/\partial(r, \varphi)$ is nonsingular on G, $S(r, \varphi + e_i) = S(r, \varphi)$ for $(r, \varphi) \in G$, $e_i = (\delta_{i1}, \ldots, \delta_{ik})$, $\delta_{i,i} = 1$, $\delta_{i,j} = 0$ for $i \neq j$, $i, j = 1, 2, \ldots, k$, $h(r, \varphi + e_i) = h(r, \varphi)$ for $(r, \varphi) \in G$, $i = 1, 2, \ldots, k$, $S(r, \varphi) = S(s, \psi)$ if and only if $r = s$, $\varphi - \psi = \sum_{i=1}^{k} \lambda_i e_i$, λ_i being integers, $S(r, \varphi) \in Z$ if and only if $r = 0$, S transforms (10,3) to (26,3). Denote by $U(G_0)$ $(U(E_n))$ the set of continuous maps from $\langle -1, 0\rangle$ to $G_0(E_n)$ and for $\xi \in U(E_n)$ let $\|\xi\|$ have its usual meaning. If in addition h fulfils (6,3) ((5,3) is fulfilled, as Z is an integral manifold of (26,3) and therefore $r = 0$ is an integral manifold of (10,3)) and if (9,3) holds, then Theorem 1,3 and Note 4,3 imply that there exists $\hat{\varrho} > 0$ with the following property: if V is a continuous map from $U(G_0) \times E_1$ to E_n and $\|V(\xi, \tau)\| \leqq \hat{\varrho}$, $\|V(\xi, \tau) - V(\bar{\eta}, \tau)\| \leqq \hat{\varrho}\|\xi - \bar{\eta}\|$ for $\xi, \bar{\eta} \in U(G_0)$, $\tau \in E_1$, then there exists a $(k + 1)$-dimensional integral manifold (torus) of

$$\frac{d\xi}{d\tau} = W(\xi) + V(\xi_\tau, \tau)$$

(in the space $U(G_0) \times E_1$).

(To be continued)