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STOCHASTIC APPROXIMATIONS IN THE PRESENCE OF TREND

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1. Summary. Two basic stochastic approximation methods deal with solving an equation (the Robbins-Monro method) or with seeking the point of a maximum (the Kiefer-Wolfowitz method), when the function values are determined with an experimental error. In the present paper both methods are adapted to the case, when the root or the point of a maximum move in a specified manner during the approximation process. As compared to the author's previous paper on this theme [1], the conditions, under which the approximations converge, are generalized in several directions.

2. The Robbins-Monro case. Denote by N the set of all positive integers, and by R the real line. For $n \in N$ let $M_n(x)$, $x \in R$, be Borel-measurable functions, let θ_n be the unique root of the equation $M_n(x) = 0$. Both M_n and θ_n are unknown to the experimenter, but it is supposed, that he can choose two sequences $a_n > 0$ and q_n such that

$$(1) \quad \sum_1^\infty a_n = +\infty, \quad \sum_1^\infty a_n^2 < +\infty;$$

$$(2) \quad \theta_{n+1} = q_n \theta_n + o(a_n);$$

$$(3) \quad (|q_n| - 1)^+ = o(a_n);$$

(z^+ denotes $(z + |z|)/2$).

Let x_1 be an arbitrary random variable; for $n \in N$ define

$$(4) \quad x_{n+1} = x_n^* - a_n y_n^*$$

where $x_n^* = q_n x_n$, and y_n^* is a random variable such that

$$(5) \quad E(y_n^* | x_1, x_2, \dots, x_n) = M_{n+1}(x_n^*)$$

$$(6) \quad \text{Var}(y_n^* | x_1, x_2, \dots, x_n) \leq \text{const (say } \sigma^2 \text{)}.$$

Theorem 1. Let $M_n(x)$ satisfy the conditions:

$$(7) \quad |M_n(x)| \leq A|x - \Theta_n| + B, \text{ for every } x \in \mathbb{R}, n \in \mathbb{N} \text{ and suitable } A, B;$$

$$(8) \quad \inf_{n \in \mathbb{N}} \inf_{|x - \Theta_n| > \delta} \frac{M_n(x)}{x - \Theta_n} > 0 \text{ for every } \delta > 0.$$

Then $x_n - \Theta_n \rightarrow 0$ with probability one; if $E(x_1^2) < +\infty$, then also $E[(x_n - \Theta_n)^2] \rightarrow 0$.

Remark 1. In particular, (7) and (8) are satisfied if they are satisfied for $n = 1$ and if $M_n(x) = M_1(x - \Theta_n + \Theta_1)$, $n \in \mathbb{N}$.

Remark 2. The condition (8) cannot be replaced by

$$(8') \quad \inf_{n \in \mathbb{N}} \inf_{|x - \Theta_n| > \delta} |M_n(x)| > 0, \quad \frac{M_n(x)}{x - \Theta_n} > 0, \quad x \neq \Theta_n, \quad n \in \mathbb{N}$$

as the following counterexample shows (it satisfies (1)–(7) and (8'), but not (8)):

$$M_n(x) = \varepsilon \operatorname{sgn} x \text{ for all } x \in \mathbb{R}, n \in \mathbb{N} \text{ and for some } 0 < \varepsilon < \frac{1}{2}$$

(hence $\Theta_n = 0$, $n \in \mathbb{N}$);

$$x_1 = 1; \quad \sigma^2 = 0; \quad q_n = 1 + \frac{1}{n}; \quad a_n = \frac{1}{n^\alpha}, \quad \frac{1}{2} < \alpha < 1.$$

It follows

$$x_{n+1} = \left(1 + \frac{1}{n}\right)x_n - \varepsilon \frac{1}{n^\alpha} \operatorname{sgn} x_n,$$

hence

$$x_n = n \left(1 - \varepsilon \sum_{i=1}^{n-1} \frac{1}{i^\alpha(i+1)}\right) \rightarrow +\infty.$$

Remark 3. Let $\Theta_n = gn^\beta + h$, where $\beta > 0$ is known, g and h unknown; then the conditions (1), (2), (3) are satisfied by the choice

$$(9) \quad q_n = 1 + \binom{\beta}{1} \frac{1}{n} + \binom{\beta}{2} \frac{1}{n^2} + \dots + \binom{\beta}{r} \frac{1}{n^r}$$

and

$$(10) \quad a_n = \frac{a}{n^\alpha}, \quad a > 0, \quad \frac{1}{2} < \alpha < 1$$

with $r = [\alpha + \beta]$.

So we can choose $q_n = 1 + 1/n$, $a_n = a/n^\alpha$, $\frac{1}{2} < \alpha < 1$ for the linear trend, i.e. for $\beta = 1$; or $q_n = 1$, $a_n = a/n^\alpha$, $\frac{1}{2} < \alpha < 1 - \beta$ for $\beta < \frac{1}{2}$.

Proof of Remark 3: We shall only verify (2), everything else is obvious; we have

$$\Theta_{n+1} = g(n+1)^\beta + h = gn^\beta \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{1}{n^k} + h,$$

$$q_n \Theta_n = gn^\beta \sum_{k=0}^r \binom{\beta}{k} \frac{1}{n^k} + h \sum_{k=0}^r \binom{\beta}{k} \frac{1}{n^k},$$

hence

$$\Theta_{n+1} - q_n \Theta_n = O\left(\frac{1}{n^{r+1-\beta}} + \frac{1}{n}\right) = o(a_n),$$

since $[\alpha + \beta] + 1 - \beta > \alpha$.

Proof of Theorem 1: Rewrite the scheme (4) in the form

$$(11) \quad x_{n+1} = x_n^* - a_n M_{n+1}(x_n^*) + \varepsilon_n,$$

where $\varepsilon_n = -a_n(y_n^* - M_{n+1}(x_n^*))$. Subtract Θ_{n+1} on both sides of (11) and denote $\omega_n = \Theta_{n+1} - q_n \Theta_n$, $z_n = x_n - \Theta_n$; so that

$$x_n^* = q_n x_n = q_n z_n + q_n \Theta_n = q_n z_n - \omega_n + \Theta_{n+1}.$$

We get

$$(12) \quad z_{n+1} = T_n(z_n) + \varepsilon_n$$

where

$$(13) \quad T_n(r) = q_n r - \omega_n - a_n M_{n+1}(q_n r - \omega_n + \Theta_{n+1}).$$

We shall show, that the scheme (12) satisfies the conditions of Dvoretzky's theorem [2] for the convergence $z_n \rightarrow 0$ with probability one and in mean-square. The fulfilment of the conditions

$$(14) \quad \mathbb{E}(\varepsilon_n | z_1, \dots, z_n) = 0, \quad \sum_{n=1}^{\infty} \mathbb{E}(\varepsilon_n^2) < +\infty$$

is obvious from (5) and (6), so that it suffices to prove the inequality

$$(15) \quad |T_n(r)| \leq \max \{\alpha_n, |r| - \gamma_n\}, \quad n \in N, \quad r \in R,$$

for some positive α_n, γ_n such that $\alpha_n \rightarrow 0, \sum_1^{\infty} \gamma_n = +\infty$.

The next proposition follows easily from (8): For every sequence $\varrho_n > 0, \varrho_n \rightarrow 0$, bounded by a sufficiently small constant, there exists a sequence $\eta_n > 0, \eta_n \rightarrow 0$, such that

$$(16) \quad |M_m(x)| > \varrho_n |x - \Theta_m| \quad \text{for all } |x - \Theta_m| > \eta_n, \quad m \in N, \quad n \in N.$$

Let us choose ϱ_n such that

$$(17) \quad \sum_1^{\infty} a_n \varrho_n^2 = +\infty, \quad \omega_n = o(a_n \varrho_n^2), \quad (|q_n| - 1)^+ = o(a_n \varrho_n);$$

let the corresponding η_n be such that

$$(18) \quad a_n = o(\eta_n);$$

this can always be done.

If $|q_n r - \omega_n| \leq \eta_n$, then

$$(19) \quad |T_n(r)| \leq (1 + Aa_n) |q_n r - \omega_n| + Ba_n < 2\eta_n$$

for sufficiently large n , according to (7) and (18).

The case $|q_n r - \omega_n| > \eta_n$ is a little more complicated: The terms $q_n r - \omega_n$ and $a_n M_{n+1}(q_n r - \omega_n + \Theta_{n+1})$ are of the same sign, the first being larger than or equal to the second one in absolute value, for large n , so that we have

$$(20) \quad |T_n(r)| = |q_n r - \omega_n - a_n M_{n+1}(q_n r - \omega_n + \Theta_{n+1})|.$$

Setting $m = n + 1$, $x = q_n r - \omega_n + \Theta_{n+1}$ in (16) and using it in (20), we get

$$(21) \quad |T_n(r)| \leq (1 - a_n \varrho_n) |q_n r - \omega_n| \leq \\ \leq (1 - a_n \varrho_n) (1 + (|q_n| - 1)^+) |r| + |\omega_n| = \tau_{n,r} \quad (\text{say}).$$

Now, if $|r| \leq \varrho_n$, then $\tau_{n,r} < 2\varrho_n$; if $|r| > \varrho_n$, then

$$(22) \quad \tau_{n,r} < (1 - \frac{2}{3} a_n \varrho_n) |r| + |\omega_n| < |r| - \frac{1}{3} a_n \varrho_n^2,$$

both inequalities in (22) being consequences of (17). So it is proved, that (15) holds with $\alpha_n = \max(2\eta_n, 2\varrho_n)$, $\gamma_n = a_n \varrho_n^2 / 3$.

3. The multidimensional Kiefer-Wolfowitz case. Denote by R^p the p -dimensional Euclidean space. For $n \in N$ let $M_n(x)$, $x \in R^p$, be Borel-measurable functions, let $\Theta_n \in R^p$ be the point at which $M_n(x)$ has the unique maximum. It is supposed, that the experimenter can choose two sequences of positive constants a_n , c_n and a sequence of real matrices $Q_n(p \times p)$ such that

$$(23) \quad c_n \rightarrow 0, \quad \sum_1^{\infty} a_n = +\infty, \quad \sum_1^{\infty} (a_n^2 / c_n^2) < +\infty, \quad a_n / c_n^2 \rightarrow 0;$$

$$(24) \quad \|\Theta_{n+1} - Q_n \Theta_n\| = o(a_n);$$

$$(25) \quad (\|Q_n\| - 1)^+ = o(a_n).$$

Let x_1 be an arbitrary p -dimensional random vector; for $n \in N$ define

$$(26) \quad x_{n+1} = x_n^* + a_n \frac{y_{2n}^* - y_{2n-1}^*}{c_n}$$

where $x_n^* = Q_n x_n$ and y_{2n}^*, y_{2n-1}^* are random vectors such that their coordinates $y_{2n,i}^*, y_{2n-1,i}^*, i = 1, 2, \dots, p$, are all conditionally independent given x_1, x_2, \dots, x_n and satisfy

$$(27) \quad \begin{aligned} E(y_{2n,i}^* \mid x_1, \dots, x_n) &= M_{n+1}(x_{n,i}^* + c_n e_i), \\ E(y_{2n-1,i}^* \mid x_1, \dots, x_n) &= M_{n+1}(x_n^* - c_n e_i), \quad i = 1, 2, \dots, p, \end{aligned}$$

$$(28) \quad \text{Var}(y_{v,i}^* \mid x_1, \dots, x_n) \leq \text{const.}, \quad v = 2n, 2n-1, \quad i = 1, 2, \dots, p$$

($e_i, i = 1, 2, \dots, p$ are elements of the usual orthonormal set in R^p ; the constant in (28) is independent of n). Denote by $D_\varepsilon M_n(x)$ the vector with coordinates

$$\frac{M_n(x + \varepsilon e_i) - M_n(x - \varepsilon e_i)}{\varepsilon}, \quad i = 1, 2, \dots, p.$$

Theorem 2. Let $M_n(x)$ satisfy the conditions

$$(29) \quad |M_n(x + \varepsilon e_i) - M_n(x)| \leq A \|x - \Theta_n\| + B$$

for all $0 < \varepsilon < 1, i = 1, 2, \dots, p, x \in R^p, n \in N$ and suitable A, B ;

$$(30) \quad \sup_{n \in N} \sup_{\|x - \Theta_n\| > \delta} \sup_{0 < \varepsilon < \delta} \frac{(D_\varepsilon M_n(x), x - \Theta_n)}{\|x - \Theta_n\|^2} < 0 \quad \text{for each } 0 < \delta < \delta_0.$$

Then $x_n - \Theta_n \rightarrow 0$ with probability one.

Remark 4. Let $\Theta_n = n^B g + h$, where B is a known matrix, g and h unknown vectors; let

$$(31) \quad \|n^B\| = O(n^\beta) \quad \text{for some } \beta > 0.$$

Then the conditions (23), (24), (25) are satisfied by the choice

$$(32) \quad Q_n = E + \binom{B}{1} \frac{1}{n} + \binom{B}{2} \frac{1}{n^2} + \dots + \binom{B}{r} \frac{1}{n^r},$$

$$(33) \quad a_n = \frac{a}{n^\alpha}, \quad a > 0, \quad \frac{1}{2} < \alpha < 1,$$

$$(34) \quad c_n = \frac{c}{n^\gamma}, \quad c > 0, \quad 0 < \gamma < \alpha - \frac{1}{2}$$

where $\binom{B}{k}$ denotes $[B(B - E)(B - 2E) \dots (B - (k - 1)E)]/k!$, and $r = [\alpha + \beta]$.

So we can choose $Q_n = (1 + 1/n)E$ for the case of a linear trend in each coordinate, i.e. for $\Theta_{n,i} = g_i n + h_i$, $i = 1, 2, \dots, p$. If $\beta < \frac{1}{2}$, B may be unknown; (23)–(25) are then satisfied by $Q_n = E$, and a_n, c_n given by (33), (34) with the additional restriction $\alpha < 1 - \beta$.

Proof of Remark 4 is formally analogous to that of Remark 3. We note only that n^B is well defined as $e^{B \lg n}$, that β satisfying (31) always exist, since $\|n^B\| \leq n^{\|B\|}$, and that the expansion

$$(35) \quad (1 + a)^B = \sum_{k=0}^{\infty} \binom{B}{k} a^k,$$

is valid for every $|a| < 1$ and for every matrix B , as follows from [3, Section 5.4, Theorem 1'] and from the Weierstrass Theorem (on uniformly convergent series of analytic functions).

Also Remarks 1 and 2 (of Section 2) can be repeated with obvious changes, the counterexample being the one-dimensional $M_n(x) = -\varepsilon|x|$, etc.

Proof of Theorem 2: Similarly as in the proof of Theorem 1, the scheme (26) can be rewritten in the form

$$(36) \quad z_{n+1} = T_n(z_n) + \varepsilon_n,$$

where

$$(37) \quad T_n(r) = Q_n r - \omega_n + a_n D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1}), \quad r \in R^p,$$

and $\omega_n = \Theta_{n+1} - Q_n \Theta_n$, $z_n = x_n - \Theta_n$, ε_n being the vector with coordinates ($i = 1, 2, \dots, p$)

$$\varepsilon_{n,i} = \frac{a_n}{c_n} [y_{2n,i}^* - M_{n+1}(x_{n,i}^* + c_n e_i) - (y_{2n-1,i}^* - M_{n+1}(x_n^* - c_n e_i))].$$

We shall verify, that the scheme (36) satisfies the conditions of the multidimensional version of Dvoretzky's theorem [4, Theor. 2] for the convergence $z_n \rightarrow 0$ with probability one. The fulfilment of

$$(38) \quad \mathbf{E}(\varepsilon_n \mid z_1, \dots, z_n) = 0, \quad \sum_{n=1}^{\infty} \mathbf{E}(\|\varepsilon_n\|^2) < +\infty$$

is obvious from (27), (28), so that it again suffices to prove

$$(39) \quad \|T_n(r)\| \leq \max \{\alpha_n, \|r\| - \gamma_n\}, \quad n \in N, \quad r \in R^p$$

for some positive α_n, γ_n such that $\alpha_n \rightarrow 0$, $\sum_1^{\infty} \gamma_n = +\infty$.

We shall first estimate $\|D_\varepsilon M_n(x)\|$ with help of (29):

$$(40) \quad \begin{aligned} \|D_\varepsilon M_n(x)\| &\leq p^{\frac{1}{2}} \operatorname{Max}_{1 \leq i \leq p} \frac{|M_n(x + \varepsilon e_i) - M_n(x - \varepsilon e_i)|}{\varepsilon} \leq \\ &\leq \frac{p^{1/2}}{\varepsilon} (A \|x - \Theta_n\| + A\varepsilon + B) \leq \frac{A_1 \|x - \Theta_n\| + B_1}{\varepsilon}, \quad x \in R^p, \\ &n \in N, \quad 0 < \varepsilon < \frac{1}{2}. \end{aligned}$$

Setting this into (37), we get

$$(41) \quad \|T_n(r)\| \leq \left(1 + A_1 \frac{a_n}{c_n}\right) \|Q_n r - \omega_n\| + B_1 \frac{a_n}{c_n}, \quad r \in R^p, \quad c_n < \frac{1}{2}.$$

Further, we shall use the expression for $\|T_n(r)\|^2$, calculated from (37):

$$(42) \quad \|T_n(r)\|^2 = \|Q_n r - \omega_n\|^2 \{1 + A + A'\}$$

where

$$\begin{aligned} A &= 2a_n(D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1}), Q_n r - \omega_n) / \|Q_n r - \omega_n\|^2, \\ A' &= a_n^2 \|D_{c_n} M_{n+1}(Q_n r - \omega_n + \Theta_{n+1})\|^2 / \|Q_n r - \omega_n\|^2. \end{aligned}$$

The next proposition follows from (30): For every sequence $\varrho_n > 0$, $\varrho_n \rightarrow 0$, bounded by a sufficiently small number, there exists a sequence $\eta_n > 0$, $\eta_n \rightarrow 0$, such that

$$(43) \quad \frac{(D_\varepsilon M_m(x), x - \Theta_m)}{\|x - \Theta_m\|^2} < -\varrho_n$$

for all $0 < \varepsilon < \eta_n$, $\|x - \Theta_m\| > \eta_n$, $m \in N$, $n \in N$.

Let us choose ϱ_n such that it holds

$$(44) \quad \begin{aligned} \sum_1^\infty a_n \varrho_n^2 &= +\infty, \quad \|\omega_n\| = o(a_n \varrho_n^2), \quad (\|Q_n\| - 1)^+ = o(a_n \varrho_n), \\ (a_n / c_n^2) &= o(\varrho_n); \end{aligned}$$

let the corresponding sequence η_n be chosen in such a way that it satisfies the conditions

$$(45) \quad \frac{a_n}{c_n} = o(\eta_n), \quad c_n = o(\eta_n), \quad \frac{a_n}{c_n^2 \varrho_n} = o(\eta_n^2).$$

Let $\|Q_n r - \omega_n\| \leq \eta_n$, let n be sufficiently large; then $\|T_n(r)\| < 2\eta_n$, according to (41) and (45).

Finally, let $\|Q_n r - \omega_n\| > \eta_n$, n be large. Setting $\varepsilon = c_n$, $m = n + 1$, $x = Q_n r - \omega_n + \Theta_{n+1}$ in (43) we get $\Delta < -2a_n \varrho_n$; using (40) and (45) we get further

$$\Delta' \leq 2 \frac{a_n^2}{c_n^2} \left(A_1^2 + \frac{B_1^2}{\|Q_n r - \omega_n\|^2} \right) < \frac{C a_n^2}{c_n^2 \eta_n^2} < a_n \varrho_n.$$

Setting these estimates into (42), we get

$$\|T_n(r)\|^2 \leq \|Q_n r - \omega_n\|^2 (1 - a_n \varrho_n),$$

i.e.

$$\|T_n(r)\| \leq \|Q_n r - \omega_n\| (1 - \frac{1}{2} a_n \varrho_n).$$

The rest of the proof coincides with the proof of Theorem 1 (cf. (21) and below).

Remark 5. In the one-dimensional Kiefer-Wolfowitz case, the condition $a_n/c_n^2 \rightarrow 0$ can be omitted, and the condition (29) can be weakened to

$$|M_n(x+1) - M_n(x)| < A|x - \Theta_n| + B.$$

Furthermore, $E(x_1^2) < +\infty$ implies $E((x_n - \Theta_n)^2) \rightarrow 0$. We omit the proof.

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СТОХАСТИЧЕСКИЕ АППРОКСИМАЦИИ ПРИ НАЛИЧИИ ТРЕНДА

ВАЦЛАВ ДУПАЧ (Václav Dupač), Прага¹

Стохастический аппроксимационный метод для нахождения корня уравнения или для отыскания точки максимума функции приспособляется для случая, когда корень или точка максимума изменяются в течение аппроксимационного процесса. Именно, доказывается следующий результат:

Пусть $M_n(x)$, $n = 1, 2, \dots$ — бэрвские функции, пусть Θ_n — корень уравнения $M_n(x) = 0$. Пусть M_n и Θ_n неизвестны, а известны некоторые постоянные $a_n > 0$ и q_n так, что $\sum a_n = +\infty$, $\sum a_n^2 < +\infty$; $\Theta_{n+1} = q_n \Theta_n + o(a_n)$; $(|q_n| - 1)^+ = o(a_n)$. Пусть x_1 — произвольная случайная величина; для $n \geq 1$ положим $x_{n+1} = x_n^* - a_n y_n^*$, где $x_n^* = q_n x_n$ и y_n^* — случайная величина такая, что $E(y_n^* | x_1, \dots, x_n) = M_{n+1}(x_n^*)$, $\text{Var}(y_n^* | x_1, \dots, x_n) \leq \text{const}$. Пусть $M_n(x)$ удовлетворяют условиям: $|M_n(x)| \leq A|x - \Theta_n| + B$ для всякого $-\infty < x < +\infty$ и $n = 1, 2, \dots$; $\inf_{n=1,2,\dots} \inf_{|x - \Theta_n| > \delta} M_n(x)/(x - \Theta_n) > 0$ для всякого $\delta > 0$. Тогда $x_n - \Theta_n \rightarrow 0$ п.н. Если, в частности, $\Theta_n = gn^\beta + h$, где β известно, а g и h неизвестны, то можно выбрать $a_n = an^{-\alpha}$, $q_n = 1 + \binom{\beta}{1} n^{-1} + \dots + \binom{\beta}{r} n^{-r}$, где $\frac{1}{2} < \alpha < 1$, $r = [\alpha + \beta]$.

Аналогичные результаты доказаны для отыскания точки максимума функции нескольких переменных. Статья является продолжением работы автора [1].