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Solution in large of control problem \( \dot{x} = (Au + Bv)x \)

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Let us have an equation

\[ \dot{x} = (A u + B v) x, \quad x(0) = \omega, \]

where \( A, B \) are given \( n \)-by-\( n \) matrices, \( \omega \) is a vector, written as a column, from an \( n \)-dimensional Euclidean space \( E^n \) and \( u, v \in M \), which is the set of all measurable functions on \( (0, \infty) \) values of which lie in the interval \( (-1, 1) \). The functions from \( M \) are called controls.

If we have two \( n \)-by-\( n \) matrices \( A, B \), we denote by \( \mathcal{U}(A, B) \) the smallest linear space of \( n \)-by-\( n \) matrices which has the following two properties:

1) \( A, B \in \mathcal{U}(A, B) \),

2) \( P, Q \in \mathcal{U}(A, B) \Rightarrow (QP - PQ) \in \mathcal{U}(A, B) \).

Finally, for every vector \( x \in E^n \) we denote by \( V(x) \) a vector space formed by all vectors \( Px \), where \( P \in \mathcal{U}(A, B) \). One calls the mapping \( V \) distribution.

In the paper [1] we investigated the equation

\[ \dot{x} \in V(x), \quad x(0) = \omega, \]

where we considered as a solution of (2) every absolutely continuous function \( x(t) \), \( t \geq 0 \), with the property: if \( dx(t)/dt \) exists, then \( dx(t)/dt \in V(x(t)) \), satisfying the initial condition \( x(0) = \omega \).

In [1] it was proved that all points of \( E^n \) which can be linked with \( \omega \) by a solution of (2) form a manifold \( S_\omega \), dimension of which is equal to \( \dim V(\omega) \). In this paper we will prove that every point \( x \in S_\omega \) lies also on a solution \( x(t, u, v, \omega) \) of the equation (1), where the controls \( u, v \) are piecewise constant and acquire only the values \(-1, 1\).
Notation. For \( x \in \mathbb{E}^n \) we use the norm \( \|x\| = \sum |x_i| \), which induces the norm for an \( n \times n \) matrix \( A = (a_{ij}) \) to be equal to \( \|A\| = \max \sum |a_{ij}| \). The dimension of a (finite-dimensional) vector space \( V \) one writes \( \dim V \). The symbol \( \{x_1, x_2, \ldots, x_k\} \) represents the linear hull of elements \( x_1, x_2, \ldots, x_k \) of some linear space. By \( O(t) \), \( t \to 0 \), we denote a quantity, depending on \( t \), which can be majorised by \( c|t| \), where \( c \) is a positive constant, if \( t \) tends to zero. For \( n \times n \) matrices we use the "bracket" operation: \( [A_1, A_2, \ldots, A_k] = A_2 A_1 - A_1 A_2 \),

\[ [A_1, A_2, \ldots, A_k] = [A_1, [A_2, \ldots, [A_{k-1}, A_k] \ldots]] . \]

The zero-matrix and unit-matrix are denoted by \( 0 \) and \( E \), respectively. The \( A^{-1} \) is an inverse to a non-singular matrix \( A \). The solution of (1) which corresponds to given controls \( u, v \in M \) and satisfy the initial condition \( x(0, u, v, \omega) = \omega \), one denotes by \( x(t, u, v, \omega) \). Finally, we denote by \( M_0 \subset M \times M \) the set of all piecewise constant functions \( (u, v) \in M \times M \) values of which are only \((\pm 1, 0), (0, \pm 1)\).

Definition. The matrix \( P \in \mathfrak{M}(A, B) \) which can be represented as \( P = [P_1, P_2, \ldots, P_p] \), where \( P_i = \pm A \) or \( P_i = \pm B \), \( i = 1, 2, \ldots, p \); one calls elementary of grade \( p \).

It was proved in [1] that the space \( \mathfrak{M}(A, B) \) is the linear hull of all elementary matrices.

Lemma 1. Let \( P \in \mathfrak{M}(A, B) \) be an elementary matrix of grade \( p \). Then it exists a sequence \( P_1, P_2, \ldots, P_r \), where \( r = 3 \cdot 2^{p-1} - 2 \), which is formed only by matrices \( A, -A, B, -B \), such that it holds

\[ \prod_{i=1}^{r} e^{Pt} = E + Pt^p + O(t^{p+1}), \quad t \to 0 . \]

Proof. For \( p = 1 \) it is obviously \( e^{Pt} = E + Pt + O(t^2) \). Let (3) hold for an integer \( p > 0 \), then we can write

\[ \prod_{i=1}^{r} e^{Pt} = E + Pt^p + Qt^{p+1} + O(t^{p+2}) . \]

The matrix-function \((\prod_{i=1}^{r} e^{Pt})^{-1}\) is entire and it holds

\[ (\prod_{i=1}^{r} e^{Pt})^{-1} = E - Pt^p + Rt^{p+1} + O(t^{p+2}) , \]

where \( R = P^2 - Q \) for \( p = 1 \) and \( R = -Q \) for \( p > 1 \).

We can now write

\[ (\prod_{i=1}^{r} e^{Pt}) e^{At}(\prod_{i=1}^{r} e^{Pt})^{-1} e^{-At} = (E + Pt^p + Qt^{p+1} + O(t^{p+2})) . \]
The formula for the number of the multiplicators follows immediately from the construction.

**Lemma 2.** Let \( P \in \mathcal{U}(A, B) \), \( P = \sum_{i=1}^{s} a_i P_i \), where \( a_i > 0 \), \( P_i \in \mathcal{U}(A, B) \) is an elementary matrix of grade \( p_i \), \( i = 1, 2, \ldots, s \). Let us put \( p = \max p_i \) and denote by \( F_i(t) \) the matrix (3) which corresponds to the matrix \( P_i \), \( i = 1, 2, \ldots, s \). Then it holds:

\[
F(t) = \sum_{i=1}^{s} F_i(a_i^{1/(p_i + p_i)}) = E + Pt^p + O(t^{p+1}), \quad t \to 0
\]

**Proof.** \( \prod_{i=1}^{s} F_i(a_i^{1/(p_i + p_i)}) = \prod_{i=1}^{s} (E + a_i P_i t^p + O(t^{p+1})) = \sum_{i=1}^{s} (E + a_i P_i t^p + O(t^{p+1})) = E + Pt^p + O(t^{p+1}). \)

**Lemma 3.** Let \( P \in \mathcal{U}(A, B) \), \( P = \sum_{i=1}^{s} a_i P_i \), where \( a_i > 0 \), and \( P_i \in \mathcal{U}(A, B) \) is an elementary matrix of grade \( p_i \), \( i = 1, 2, \ldots, s \). Let us put \( p = \max p_i \). Then there exists a constant \( K > 0 \) such that for all \( \varepsilon > 0 \), \( \alpha \in (0, 1) \) there exists \( (u, v) \in M_0 \) and a constant \( T \in (0, K \cdot \varepsilon^{-1} - p) \) such that for the solution \( x(t, u, v, \omega) \) of (1) it holds:

\[
\|x(T, u, v, \omega) - e^{tP} \omega\| < \varepsilon.
\]

**Proof.** For \( P = 0 \) lemma is trivial. Let us further assume \( P \neq 0 \). Let us take a positive integer \( m \) and put

\[
x_i = e^{(ix/m)p} \omega, \quad i = 0, 1, \ldots, m, \quad y_0 = \omega, \quad y_{i+1} = \left( E + \frac{x}{m} P \right) y_i, \quad i = 0, 1, \ldots, (m - 1)
\]

Then it holds:

\[
\|y_0 - x_0\| = 0, \quad \|y_{i+1} - x_{i+1}\| \leq \|(E + (x/m) P) y_i - e^{(x/m)p} x_i\| \leq (1 + (x/m)\|P\|) \|y_i - x_i\| + (x^2/m^2)\|P\|^2 e^{(x/m)p}\|x_i\|.\]

If we put \( \varepsilon = \max_{\omega \in (0,1)} \|e^{tP} \omega\|, \) then \( \|x_i\| \leq \varepsilon, \ i = 1, 2, \ldots, m, \)

\[
\|y_m - x_m\| \leq \varepsilon \frac{x^2}{m^2} \|P\|^2 e^{(x/m)p}\left(1 + \frac{\|P\|/m^m - 1}{\varepsilon \|P\|/m}\right) \leq \varepsilon \frac{\|P\|/m e^{(x/m)p}}{\|P\|/m}.
\]
Now we put $z_0 = \omega$. Let us have already defined the points $z_0, z_1, \ldots, z_i, i < m$. In lemma 2 the matrix-function (4) was constructed so that it exists a constant $K_1 > 0$, dependent only on the matrix $P$, such that it holds
\[ \| F(t) - (E + Pt) \| \leq K_1 \cdot t^{p+1}, \quad t \in \langle 0, 1 \rangle. \]

Furthermore according to lemma 2 for every $t \in \langle 0, 1 \rangle$ there exists $(u, v) \in M_0$ such that $F(t) z_i = x(\mathcal{G}(t), u, v, z_i)$, where $\mathcal{G}(t) = \sum_{i=1}^{s} (3 \cdot 2^{p_{i}-1} - 2) a_{i}^{1/p} t^{p/p_i}$. If we put $t = (\alpha/m)^{1/p}$ and $z_{i+1} = x(\mathcal{G}(t), u, v, z_i)$, we get
\[ \| z_{i+1} - \left( E + \frac{\alpha}{m} P \right) z_i \| \leq K_1 \left( \frac{\alpha}{m} \right)^{1 + (1/p)} \| z_i \|. \]

Thus we have defined all points $z_0, z_1, \ldots, z_m$ by mathematical induction. It holds:
\[ \| z_{i+1} \| \leq \| z_{i+1} - \left( E + \frac{\alpha}{m} P \right) z_i \| + \left\| \left( E + \frac{\alpha}{m} P \right) z_i \right\| \leq \]
\[ \leq \left( K_1 \left( \frac{\alpha}{m} \right)^{1 + (1/p)} \right) \| z_i \| + \left( 1 + \frac{\alpha}{m} (K_1 + \| P \|) \right) \| z_i \| < \]
\[ < \left( 1 + \frac{\alpha}{m} (K_1 + \| P \|) \right)^m \| z_0 \| < e^{\alpha(K_1 + \| P \|) \| \omega \|}. \]

So all points $z_i$, $i = 0, 1, \ldots, m$, are contained in the sphere $\| z \| < e^{\alpha(K_1 + \| P \|) \| \omega \|}$.

Further it holds:
\[ \| z_{i+1} - y_{i+1} \| \leq \| z_{i+1} - \left( E + \frac{\alpha}{m} P \right) z_i \| + \left\| \left( E + \frac{\alpha}{m} P \right) z_i - \left( E + \frac{\alpha}{m} P \right) y_i \right\| \leq \]
\[ \leq K_1 \left( \frac{\alpha}{m} \right)^{1 + (1/p)} e^{\alpha(K_1 + \| P \|) \| \omega \|} + \left( 1 + \frac{\alpha}{m} \| P \| \right) \| z_i - y_i \|, \]
\[ (7) \quad \| z_m - y_m \| \leq K_1 \left( \frac{\alpha}{m} \right)^{1 + (1/p)} e^{\alpha(K_1 + \| P \|) \| \omega \|} \left( 1 + \frac{\alpha}{m} \| P \| \right)^m - 1 < \]
\[ < K_1 \left( \frac{\alpha}{m} \right)^{1/p} \frac{1}{\| P \|} e^{\alpha(K_1 + 2 \| P \|) \| \omega \|}. \]

If we now put together the estimates (6), (7), we get $\| z_m - x_m \| < K_2 (\alpha/m)^{1/p}$.

Now let us choose $m$ so that $K_2 (\alpha/m)^{1/p} < \varepsilon \leq K_2 (\alpha(m - 1))^{1/p}$. Then (5) holds and we get the estimate for $T$:
\[ T \leq m \sum_{i=1}^{s} (3 \cdot 2^{p_{i}-1} - 2) a_{i}^{1/p_i} \left( \frac{\alpha}{m} \right)^{1/p_i} \leq mK_3 \left( \frac{\alpha}{m} \right)^{1/p} < mK_3 \cdot \frac{\varepsilon}{K_2} \leq \]
\[ \leq \left( 1 + \frac{\alpha}{m} \left( \frac{K_2}{\varepsilon} \right) \right) K_3 \cdot \frac{\varepsilon}{K_2} < K \cdot \varepsilon^{1-p}. \]

Lemma is proved.
Theorem. Let $R^\omega$, resp. $S^\omega$, be the set of all points $x \in \mathbb{E}^n$ which can be linked with $\omega$ by a solution of the equation (1), resp. (2). Then $R^\omega = S^\omega$.

Moreover, each point from $R^\omega$ can be linked with $\omega$ by a solution of (1) which corresponds to some piecewise constant controls $u, v \in M$, values of which are only $-1, 1$.

Proof. Evidently every solution of (1) is also a solution of (2), hence $R^\omega \subset S^\omega$.

We prove the inverse inclusion in two steps: 1) Let us choose $x \in S^\omega$, then there exists a solution $x(t), t \geq 0$, of (2) and a number $T \geq 0$ such that $x = x(T)$.

Let $\dim V(\omega) = r$, then according to [1] every point $y \in S^\omega$ is contained in an $r$-dimensional manifold $S$, given by a mapping

$$
\varphi(t) = e^{P_1 t_1} e^{P_2 t_2} \cdots e^{P_r t_r} y, \quad t \in G,
$$

where $G \subset \mathbb{E}_r$ is some neighbourhood of the origin and matrices $P_i \in \mathfrak{U}(A, B), i = 1, 2, \ldots, r$, are such that $V(y) = \{P_1 y, P_2 y, \ldots, P_r y\}$. The set $\varphi(G)$ is open in $S^\omega$.

Thus the compact set $E(x(t), t \in [0, T])$ can be covered by a finite number of such manifolds. If we choose two points $x_{1,2} \in \varphi(G)$, then according to lemma 3 for every $\varepsilon > 0$ there exists $(u, v) \in M_0$ and a number $t_0 > 0$ so that for the solution $x(t, u, v, x_1)$ of (1) it holds: $\|x_2 - x(t_0, u, v, x_1)\| < \varepsilon$.

If we repeat this procedure we get that for every $\varepsilon > 0$ it exists $(u, v, \varphi) \in M_0$ and a number $t_0$ so that for the solution $x(t, u, v, \varphi)$ of (1) it holds: $\|x(t_0, u, v, \varphi) - x\| < \varepsilon$.

2) Let us choose elementary matrices $Q_i \in \mathfrak{U}(A, B)$ with grades $q_i, i = 1, 2, \ldots, r$, so that $V(x) = \{Q_1 x, Q_2 x, \ldots, Q_r x\}$. To every matrix $Q_i$ it corresponds the matrix-function (3), let us denote it by $F_i(t), i = 1, 2, \ldots, r$. Now we take the mapping

$$
(8) \quad \psi(t_1, t_2, \ldots, t_r) = F_1(t_1^{1/p_1}) F_2(t_2^{1/p_2}) \cdots F_r(t_r^{1/p_r}) x,
$$

$$
t = (t_1, t_2, \ldots, t_r) \in E_r.
$$

Then the functional matrix $\partial \psi/\partial t|_{t=0}$ exists and has the vectors $Q_i x, i = 1, 2, \ldots, r$, as columns. So the rank of $\partial \psi/\partial t|_{t=0}$ is equal to $r$.

We choose so small open environ $G \subset E_r$ of the origin that the rank of $\partial \psi(t)/\partial t$ is equal to $r$ for all $t \in G$. Then the set $\psi(G)$ is an open environ of $x$ in $S^\omega$. From the step 1) it is obvious that it exists $t_0 \in G$ and $\varepsilon > 0$ so that $x(t_0, u, v, \varphi) = \psi(t_0)$.

And from (8) immediately follows that there exists $(u, v) \in M_0$ and $\tilde{t} > 0$ such that the solution $x(t, u, v, \varphi)$ of (1) passes through the point $x$.

3) Let us take the matrices $A_1 = A + B, B_1 = A - B$, instead of the matrices $A, B$. The matrices $A_1, B_1$ create the same space $\mathfrak{U}(A_1, B_1) = \mathfrak{U}(A, B)$ and hence the same distribution $V$ and the same manifold $S^\omega$.

$$
Au + Bv = A_1(u + v) \cdot \frac{1}{2} + B_1(u - v) \cdot \frac{1}{2} = A_1u_1 + B_1v_1.
$$
In the first two steps we have proved that for every point \( x \in S^\omega \) there exists \((u_1, v_1) \in M_0\) and a number \( t_1 > 0\) such that if we denote by \( y(t, u_1, v_1, \omega)\) the solution of the equation

\[
\dot{y} = (A_1 u_1 + B_1 v_1) y, \quad y(0, u_1, v_1, \omega) = \omega,
\]
it is \( x = y(t_1, u_1, v_1, \omega) \).

If we now put \( u = u_1 + v_1, \quad v = u_1 - v_1, \) then \( u, v \) are piecewise constant, have only the values \(-1, 1\) and it holds \( x = y(t_1, u_1, v_1, \omega) = x(t_1, u, v, \omega) \).

This completes the proof.

References


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Резюме

РЕШЕНИЕ В ЦЕЛОМ УРАВНЕНИЯ УПРАВЛЕНИЯ

\[ \dot{x} = (Au + Bv) x \]

ЯН КУЧЕРА, (Jan Kučera), Прага

В работе показано что равны множества \( R^\omega \) или \( S^\omega \) всех точек, в которые возможно попасть из данной начальной точки \( \omega \) по некотором решению уравнения (1) или (2). В каждую точку из \( R^\omega \) возможно попасть при помощи по частях постоянных управлений \( u, v \), которые имеют только величины 1, \(-1\).