Czechoslovak Mathematical Journal

Jan Kučera

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Czechoslovak Mathematical Journal, Vol. 17 (1967), No. 1, 91–96

Persistent URL: http://dml.cz/dmlcz/100763

SOLUTION IN LARGE OF CONTROL PROBLEM $\dot{x} = (\mathrm{A}u + \mathrm{B}v)\, x$

JAN KUČERA, Praha

(Received November 19, 1965)

Let us have an equation

$$(1) \qquad \dot{x} = (\mathrm{A}u + \mathrm{B}v)\, x\,, \quad x(0) = \omega\,,$$

where A, B are given n-by-n matrices, ω is a vector, written as a column, from an n-dimensional Euclidean space E_n and $u, v \in M$, which is the set of all measurable functions on $\langle 0, \infty)$ values of which lie in the interval $\langle -1, 1 \rangle$. The functions from M are called controls.

If we have two n-by-n matrices A, B, we denote by $\mathfrak{A}(\mathrm{A}, \mathrm{B})$ the smallest linear space of n-by-n matrices which has the following two properties:

1) $\mathrm{A}, \mathrm{B} \in \mathfrak{A}(\mathrm{A}, \mathrm{B})$,

2) $\mathrm{P}, \mathrm{Q} \in \mathfrak{A}(\mathrm{A}, \mathrm{B}) \Rightarrow (\mathrm{QP} - \mathrm{PQ}) \in \mathfrak{A}(\mathrm{A}, \mathrm{B})$.

Finally, for every vector $x \in \mathrm{E}_n$ we denote by $\mathrm{V}(x)$ a vector space formed by all vectors $\mathrm{P}x$, where $\mathrm{P} \in \mathfrak{A}(\mathrm{A}, \mathrm{B})$. One calls the mapping V distribution.

In the paper [1] we investigated the equation

$$(2) \qquad \dot{x} \in \mathrm{V}(x)\,, \quad x(0) = \omega\,,$$

where we considered as a solution of (2) every absolutely continuous function $x(t)$, $t \geqq 0$, with the property: if $\mathrm{d}x(t)/\mathrm{d}t$ exists, then $\mathrm{d}x(t)/\mathrm{d}t \in \mathrm{V}(x(t))$, satisfying the initial condition $x(0) = \omega$.

In [1] it was proved that all points of E_n which can be linked with ω by a solution of (2) form a manifold S_ω dimension of which is equal to dim $\mathrm{V}(\omega)$. In this paper we will prove that that every point $x \in \mathrm{S}_\omega$ lies also on a solution $x(t, u, v, \omega)$ of the equation (1), where the controls u, v are piecewise constant and acqure only the values $-1, 1$.

Notation. For $x \in E_n$ we use the norm $\|x\| = \sum |x_i|$, which induces the norm for an n-by-n matrix $A = (a_{ij})$ to be equal to $\|A\| = \max_j \sum_i |a_{ij}|$. The dimension of a (finite-dimensional) vector space V one writes dim V. The symbol $\{x_1, x_2, \ldots, x_k\}$ represents the linear hull of elements $x_1, x_2, \ldots, x_k$ of some linear space. By $O(t)$, $t \to 0$, we denote a quantity, depending on t, which can be majorised by $c|t|$, where c is a positive constant, if t tends to zero. For n-by-n matrices we use the "bracket" operation: $[A_1, A_2] = A_2 A_1 - A_1 A_2$,

$$[A_1, A_2, \ldots, A_k] = [A_1, [A_2, \ldots [A_{k-1}, A_k] \ldots]] .$$

The zero-matrix and unit-matrix are denoted by 0 and E, respectively. The A^{-1} is an inverse to a non-singular matrix A. The solution of (1) which corresponds to given controls $u, v \in M$ and satisfy the initial condition $x(0, u, v, \omega) = \omega$, one denotes by $x(t, u, v, \omega)$. Finally, we denote by $M_0 \subset M \times M$ the set of all piecewise constant functions $(u, v) \in M \times M$ values of which are only $(\pm 1, 0)$, $(0, \pm 1)$.

Definition. The matrix $P \in \mathfrak{A}(A, B)$ which can be represented as $P = [P_1, P_2, \ldots, P_p]$, where $P_i = \pm A$ or $P_i = \pm B$, $i = 1, 2, \ldots, p$; one calls elementary of grade p.

It was proved in [1] that the space $\mathfrak{A}(A, B)$ is the linear hull of all elementary matrices.

Lemma 1. *Let* $P \in \mathfrak{A}(A, B)$ *be an elementary matrix of grade* p. *Then it exists a sequence* $P_1, P_2, \ldots, P_r$, *where* $r = 3 \cdot 2^{p-1} - 2$, *which is formed only by matrices* $A, -A, B, -B$, *such that it holds*

$$\prod_{i=1}^{r} e^{P_i t} = E + Pt^p + O(t^{p+1}) , \quad t \to 0 . \tag{3}$$

Proof. For $p = 1$ it is obviously $e^{Pt} = E + Pt + O(t^2)$. Let (3) hold for an integer $p > 0$, then we can write

$$\prod_{i=1}^{r} e^{P_i t} = E + Pt^p + Qt^{p+1} + O(t^{p+2}) .$$

The matrix-function $(\prod_{i=1}^{r} e^{P_i t})^{-1}$ is entire and it holds

$$(\prod_{i=1}^{r} e^{P_i t})^{-1} = E - Pt^p + Rt^{p+1} + O(t^{p+2}) ,$$

where $R = P^2 - Q$ for $p = 1$ and $R = -Q$ for $p > 1$.

We can now write

$$(\prod_{i=1}^{r} e^{P_i t})\, e^{At} (\prod_{i=1}^{r} e^{P_i t})^{-1}\, e^{-At} = (E + Pt^p + Qt^{p+1} + O(t^{p+2})) .$$

$$\cdot\left(\sum_{k\geqq 0}\frac{1}{k!}t^k A^k\right)\left(E - Pt^p + Rt^{p+1} + O(t^{p+2})\right)\sum_{k\geqq 0}\frac{1}{k!}(-t)^k A^k\right) =$$

$$E + [A, P]\, t^{p+1} + O(t^{p+2}), \quad t \to 0 .$$

The formula for the number of the multiplicators follows immediately from the construction.

Lemma 2. *Let* $P \in \mathfrak{A}(A, B)$, $P = \sum_{i=1}^{s} a_i P_i$, *where* $a_i > 0$, $P_i \in \mathfrak{A}(A, B)$ *is an elementary matrix of grade* p_i, $i = 1, 2, \ldots, s$. *Let us put* $p = \max_i p_i$ *and denote by* $F_i(t)$ *the matrix* (3) *which corresponds to the matrix* P_i, $i = 1, 2, \ldots, s$. *Then it holds*:

$$\text{(4)} \qquad F(t) = \prod_{i=1}^{s} F_i(a_i^{1/p_i} t^{p/p_i}) = E + Pt^p + O(t^{p+1}), \quad t \to 0$$

Proof. $\prod_{i=1}^{s} F_i(a_i^{1/p_i} t^{p/p_i}) = \prod_{i=1}^{s} (E + a_i P_i t^p + O(t^{(p/p_i)(p_i+1)})) =$

$$= \prod_{i=1}^{s} (E + a_i P_i t^p + O(t^{p+1})) = E + Pt^p + O(t^{p+1}) .$$

Lemma 3. *Let* $P \in \mathfrak{A}(A, B)$, $P = \sum_{i=1}^{s} a_i P_i$, *where* $a_i > 0$, *and* $P_i \in \mathfrak{A}(A, B)$ *is an elementary matrix of grade* p_i, $i = 1, 2, \ldots, s$. *Let us put* $p = \max_i p_i$. *Then there exists a constant* $K > 0$ *such that for all* $\varepsilon > 0$, $\alpha \in (0, 1\rangle$ *there exists* $(u, v) \in M_0$ *and a constant* $T \in (0, K \,.\, \varepsilon^{1-p})$ *such that for the solution* $x(t, u, v, \omega)$ *of* (1) *it holds*

$$\text{(5)} \qquad \|x(T, u, v, \omega) - e^{\alpha P}\omega\| < \varepsilon .$$

Proof. For $P = 0$ lemma is trivial. Let us further assume $P \neq 0$. Let us take a positive integer m and put

$$x_i = e^{(i\alpha/m)P}\omega, \quad i = 0, 1, \ldots, m,$$

$$y_0 = \omega, \quad y_{i+1} = \left(E + \frac{\alpha}{m}P\right) y_i, \quad i = 0, 1, \ldots, (m-1)$$

Then it holds: $\|y_0 - x_0\| = 0$, $\|y_{i+1} - x_{i+1}\| \leqq \|(E + (\alpha/m) P) y_i - e^{(\alpha/m)P} x_i\| \leqq$ $\leqq (1 + (\alpha/m)\|P\|) \|y_i - x_i\| + (\alpha^2/m^2)\|P\|^2 e^{(\alpha/m)\|P\|}\|x_i\|$. If we put $\varkappa = \max_{\tau\in\langle 0,1\rangle} \|e^{\tau P}\omega\|$, then $\|x_i\| \leqq \varkappa$, $i = 1, 2, \ldots, m$,

(6)

$$\|y_m - x_m\| \leqq \varkappa \frac{\alpha^2}{m^2} \|P\|^2 e^{(\alpha/m)\|P\|} \frac{(1 + \alpha\|P\|/m)^m - 1}{\alpha\|P\|/m} < \varkappa\, \alpha\|P\|/m\, e^{(\alpha+\alpha/m)\|P\|} .$$

Now we put $z_0 = \omega$. Let us have already defined the points $z_0, z_1, \ldots, z_i$, $i < m$. In lemma 2 the matrix-function (4) was constructed so that it exists a constant $K_1 > 0$, dependent only on the matrix P, such that it holds

$$\|\mathrm{F}(t) - (\mathrm{E} + \mathrm{P}t^p)\| \leqq K_1 \,.\, t^{p+1}\,, \quad t \in \langle 0, 1 \rangle\,.$$

Furthermore according to lemma 2 for every $t \in \langle 0, 1\rangle$ there exists $(u, v) \in M_0$ such that $\mathrm{F}(t)\, z_i = x(\vartheta(t), u, v, z_i)$, where $\vartheta(t) = \sum_{i=1}^{s} (3 \,.\, 2^{p_i - 1} - 2)\, a_i^{1/p_i} t^{p/p_i}$. If we put $t = (\alpha/m)^{1/p}$ and $z_{i+1} = x(\vartheta(t), u, v, z_i)$, we get

$$\left\| z_{i+1} - \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) z_i \right\| \leqq K_1 \left(\frac{\alpha}{m}\right)^{1+(1/p)} \|z_i\|\,.$$

Thus we have defined all points $z_0, z_1, \ldots, z_m$ by mathematical induction.

It holds:

$$\|z_{i+1}\| \leqq \left\| z_{i+1} - \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) z_i \right\| + \left\| \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) z_i \right\| \leqq$$

$$\leqq \left(K_1 \left(\frac{\alpha}{m}\right)^{1+(1/p)} + 1 + \frac{\alpha}{m}\|\mathrm{P}\| \right) \|z_i\| < \left(1 + \frac{\alpha}{m}(K_1 + \|\mathrm{P}\|)\right) \|z_i\| <$$

$$< \left(1 + \frac{\alpha}{m}(K_1 + \|\mathrm{P}\|)\right)^m \|z_0\| < e^{\alpha(K_1 + \|\mathrm{P}\|)} \|\omega\|\,.$$

So all points z_i, $i = 0, 1, \ldots, m$, are contained in the sphere $\|z\| < e^{\alpha(K_1 + \|\mathrm{P}\|)}\|\omega\|$.

Further it holds:

$$\|z_{i+1} - y_{i+1}\| \leqq \left\| z_{i+1} - \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) z_i \right\| + \left\| \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) z_i - \left(\mathrm{E} + \frac{\alpha}{m}\mathrm{P}\right) y_i \right\| \leqq$$

$$\leqq K_1 \left(\frac{\alpha}{m}\right)^{1+(1/p)} e^{\alpha(K_1 + \|\mathrm{P}\|)}\|\omega\| + \left(1 + \frac{\alpha}{m}\|\mathrm{P}\|\right) \|z_i - y_i\|\,,$$

(7) $$\|z_m - y_m\| \leqq K_1 \left(\frac{\alpha}{m}\right)^{1+(1/p)} e^{\alpha(K_1 + \|\mathrm{P}\|)}\|\omega\| \frac{(1 + (\alpha/m)\,\|\mathrm{P}\|)^m - 1}{(\alpha/m)\,\|\mathrm{P}\|} <$$

$$< K_1 \left(\frac{\alpha}{m}\right)^{1/p} \frac{1}{\|\mathrm{P}\|} e^{\alpha(K_1 + 2\|\mathrm{P}\|)} \|\omega\|\,.$$

If we now put together the estimates (6), (7), we get $\|z_m - x_m\| < K_2(\alpha/m)^{1/p}$.

Now let us choose m so that $K_2(\alpha/m)^{1/p} < \varepsilon \leqq K_2(\alpha/(m-1))^{1/p}$. Then (5) holds and we get the estimate for T:

$$T \leqq m \sum_{i=1}^{s} (3 \,.\, 2^{p_i - 1} - 2) a_i^{1/p_i} \left(\frac{\alpha}{m}\right)^{1/p_i} \leqq m K_3 \left(\frac{\alpha}{m}\right)^{1/p} < m K_3 \,.\, \frac{\varepsilon}{K_2} \leqq$$

$$\leqq \left(1 + \alpha \left(\frac{K_2}{\varepsilon}\right)^p\right) K_3 \,.\, \frac{\varepsilon}{K_2} < K \,.\, \varepsilon^{1-p}\,.$$

Lemma is proved.

Theorem. *Let* R_ω, *resp.* S_ω, *be the set of all points* $x \in E_n$ *which can be linked with* ω *by a solution of the equation* (1), *resp.* (2). *Then* $R_\omega = S_\omega$.

Moreover, each point from R_ω *can be linked with* ω *by a solution of* (1) *which corresponds to some piecewise constant controls* $u, v \in M$, *values of which are only* $-1, 1$.

Proof. Evidently every solution of (1) is also a solution of (2), hence $R_\omega \subset S_\omega$. We prove the inverse inclusion in two steps: 1) Let us choose $x \in S_\omega$, then there exists a solution $x(t)$, $t \geqq 0$, of (2) and a number $T \geqq 0$ such that $x = x(T)$.

Let $\dim V(\omega) = r$, then according to [1] every point $y \in S_\omega$ is contained in an r-dimensional manifold S, given by a mapping

$$\varphi(t) = e^{P_1 t_1} e^{P_2 t_2} \dots e^{P_r t_r} y \,, \quad t \in G \,,$$

where $G \subset E_r$ is some neighbourhood of the origin and matrices $P_i \in \mathfrak{A}(A, B,)$ $i = 1, 2, \dots, r$, are such that $V(y) = \{P_1 y, P_2 y, \dots, P_r y\}$. The set $\varphi(G)$ is open in S_ω.

Thus the compact set $E(x(t), t \in \langle 0, T \rangle)$ can be covered by a finite number of such manifolds. If we choose two points $x_{1,2} \in \varphi(G)$, then according to lemma 3 for every $\varepsilon > 0$ there exists $(u, v) \in M_0$ and a number $t_0 > 0$ so that for the solution $x(t, u, v, x_1)$ of (1) it holds: $\|x_2 - x(t_0, u, v, x_1)\| < \varepsilon$.

If we repeat this procedure we get that for every $\varepsilon > 0$ it exists $(u_\varepsilon, v_\varepsilon) \in M_0$ and a number t_ε so that for the solution $x(t, u_\varepsilon, v_\varepsilon, \omega)$ of (1) it holds: $\|x(t_\varepsilon, u_\varepsilon, v_\varepsilon, \omega) - x\| < \varepsilon$.

2) Let us choose elementary matrices $Q_i \in \mathfrak{A}(A, B)$ with grades q_i, $i = 1, 2, \dots, r$, so that $V(x) = \{Q_1 x, Q_2 x, \dots, Q_r x\}$. To every matrix Q_i it corresponds the matrix-function (3), let us denote it by $F_i(t)$, $i = 1, 2, \dots, r$. Now we take the mapping

$$\psi(t_1, t_2, \dots, t_r) = F_1(t_1^{1/p_1}) \,.\, F_2(t_2^{1/p_2}) \dots F_r(t_r^{1/p_r})\, x \,, \qquad (8)$$

$$t = (t_1, t_2, \dots, t_r)^* \in E_r \,.$$

Then the functional matrix $\partial\psi/\partial t|_{t=0}$ exists and has the vectors $Q_i x$, $i = 1, 2, \dots, r$, as columns. So the rank of $\partial\psi/\partial t|_{t=0}$ is equal to r.

We choose so small open environ $G \subset E_r$ of the origin that the rank of $\partial\psi(t)/\partial t$ is equal to r for all $t \in G$. Then the set $\psi(G)$ is an open environ of x in S_ω. From the step 1) it is obvious that it exists $t_0 \in G$ and $\varepsilon > 0$ so that $x(t_\varepsilon, u_\varepsilon, v_\varepsilon, \omega) = \psi(t_0)$. And from (8) immediately follows that there exists $(\tilde{u}, \tilde{v}) \in M_0$ and $\tilde{t} > 0$ such that the solution $x(t, \tilde{u}, \tilde{v}, \psi(t_0))$ of (1) passes through the point x.

3) Let us take the matrices $A_1 = A + B$, $B_1 = A - B$, instead of the matrices A, B. The matrices A_1, B_1 create the same space $\mathfrak{A}(A_1, B_1) = \mathfrak{A}(A, B)$ and hence the same distribution V and the same manifold S_ω.

$$Au + Bv = A_1(u + v) \,.\, \tfrac{1}{2} + B_1(u - v) \,.\, \tfrac{1}{2} = A_1 u_1 + B_1 v_1 \,.$$

In the first two steps we have proved that for every point $x \in S_\omega$ there exists $(u_1, v_1) \in M_0$ and a number $t_1 > 0$ such that if we denote by $y(t, u_1, v_1, \omega)$ the solution of the equation

$$\dot{y} = (A_1 u_1 + B_1 v_1)\, y\,, \quad y(0, u_1, v_1, \omega) = \omega\,,$$

it is $x = y(t_1, u_1, v_1, \omega)$.

If we now put $u = u_1 + v_1$, $v = u_1 - v_1$, then u, v are piecewise constant, have only the values $-1, 1$ and it holds $x = y(t_1, u_1, v_1, \omega) = x(t_1, u, v, \omega)$.

This completes the proof.

References

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Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).

Резюме

РЕШЕНИЕ В ЦЕЛОМ УРАВНЕНИЯ УПРАВЛЕНИЯ $\dot{x} = (Au + Bv)\,x$

ЯН КУЧЕРА, (Jan Kučera), Прага

В работе показано что равны множества R_ω или S_ω всех точек, в которые возможно попасть из данной начальной точки ω по некотором решению уравнения (1) или (2). В каждую точку из R_ω возможно попасть при помощи по частьях постоянных управлений u, v, которые имеют только величины $1, -1$.