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APPROXIMATE DIFFERENTIAL AND FEDERER NORMAL

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1. Introduction. This paper was initiated by Professor J. Mařík who investigated in a not published treatise the connection of the approximate derivative of a function of one variable with the Federer normal of the set of those points in E_2, which are lying below the graph of the function in question. In this paper Mařík's results are extended to the m-dimensional case.

I want to express my thanks to Professor Mařík for his very valuable advice and above all, for placing the above mentioned treatise at my disposal.

2. Notation. The set operations are denoted by the usual manner, by $A \triangle B$ we denote the symmetrical difference $(A - B) \cup (B - A)$ of sets A, B.

E_p (p natural) is the p-dimensional Euclidean space. For $x = [x_1, \ldots, x_p] \in E_p$, $y = [y_1, \ldots, y_q] \in E_q$, $\lambda \in E_1$, $M \subset E_{p+q}$ we write

$$[x, y] = [x_1, \ldots, x_p, y_1, \ldots, y_q], \quad M_x^* = \{y \in E_q, [x, y] \in M\},$$

$$\lambda x = [\lambda x_1, \ldots, \lambda x_p],$$

$$x + y = [x_1 + y_1, \ldots, x_p + y_p], \quad x \cdot y = \sum_{i=1}^{p} x_i y_i \quad \text{if} \quad p = q, \quad |x| = (x \cdot x)^{\frac{1}{2}}.$$

We shall also use the term "vectors" for points of E_p.

The outer Lebesgue measure of the set $M \subset E_p$ is denoted by the symbol $|M|$. (The meaning of the symbol $|\ldots|$ is different for sets and for vectors or numbers, but there is no danger of misunderstanding.) The terms outer measure, measure, measurable and so on are related to the Lebesgue measure.

For $a \in E_p$, $r > 0$ let us put

$$\Omega(a, r) = \{x \in E_p, |x - a| \leqq r\}.$$

Obviously $|\Omega(a, r)| = \gamma_p r^p$ with $\gamma_p = |\Omega(0, 1)|$.

For $a \in E_p$, $v \in E_p$ let us put

$$P(a, v) = \{x \in E_p, v \cdot (x - a) \leqq 0\}.$$

3. Measurable cover. A measurable set $Z \subset E_p$ is said to be a measurable cover of a set $M \subset E_p$, if $M \subset Z$ and if each measurable subset of $Z - M$ has measure zero. For the measurable cover of M the following assertions hold:

a) If Z_1 and Z_2 are measurable covers of M, then $|Z_1 \triangle Z_2| = 0$.

b) If Z is a measurable cover of M and if $N \subset E_p$ is measurable, then $Z \cap N$ is a measurable cover of $M \cap N$ and $|M \cap N| = |Z \cap N|$.

c) If $M \subset Z$, Z is measurable and if each compact subset of $Z - M$ has measure zero, then Z is a measurable cover of M.

d) For each set $M \subset E_p$ there exists a measurable cover Z which is of type G_δ.

4. Lemma. *If Z is a measurable cover of a set $M \subset E_p$, then $Z \times E_q$ is a measurable cover of the set $M \times E_q$.*

Proof. Let K be a compact subset of $Z \times E_q - M \times E_q$. Then the projection of K into E_p is a compact subset of $Z - M$ and so it has measure zero. Therefore $|K| = 0$ and this implies by 3c) our lemma.

5. Corollary. a) *Let M be a subset of E_p, let Z be a measurable cover of M and let A be a measurable subset of $E_p \times E_q$. Then*

$$|(M \times E_q) \cap A| = \int_Z |A_x^*| \, dx .$$

b) *Let M be a subset of E_p and N a measurable subset of E_q. Then*

$$|M \times N| = |M| \, |N| .$$

Proof. a) By 4, $Z \times E_q$ is a measurable cover of $M \times E_q$, therefore

$$|(M \times E_q) \cap A| = |(Z \times E_q) \cap A| = \int_Z |A_x^*| \, dx .$$

b) If we put $A = E_p \times N$ in a), we get

$$|M \times N| = \int_Z |N| \, dx = |N| \, |Z| = |M| \, |N| .$$

6. Points of dispersion. A point $a \in E_p$ is said to be a point of symmetrical dispersion of a set $M \subset E_p$, if

$$\lim_{r \to 0+} r^{-p} |\Omega(a, r) \cap M| = 0 .$$

The set of all such points for given M is denoted by $\mathscr{R}(M)$.

For two subsets M_1, M_2 of E_p it is easy to prove the following statements:

a) If $M_1 \subset M_2$ then $\mathscr{R}(M_2) \subset \mathscr{R}(M_1)$.

b) $\mathscr{R}(M_1 \cup M_2) = \mathscr{R}(M_1) \cap \mathscr{R}(M_2)$.

Remark. Put $\Phi(X) = |X \cap M|$ for $X \subset E_p$. Using the terminology in [1] the points of symmetrical dispersion are the points at which the symmetrical derivative (see p. 149 in [1]) of Φ is zero. It is easy to prove that we obtain the same concept using the general or ordinary derivative (see p. 106) of Φ. But the concept of points of dispersion in [1] p. 128 obtained using the strong derivative is different in general.

7. Federer exterior normal. A vector $v \in E_p$, $v \neq 0$, is said to be a Federer exterior normal (briefly an F-normal) of a set $M \subset E_p$ at a point $a \in E_p$, if $a \in \mathcal{R}(M \triangle P(a, v))$.

By the direction of a vector $v \in E_p$, $v \neq 0$, we mean the set of all vectors λv for $\lambda > 0$. Since $P(a, v) = P(a, \lambda v)$ for $\lambda > 0$, the following assertion obviously holds: If a vector of some direction is an F-normal of M at a, then each vector of this direction is an F-normal of M at a.

In the case $p = 1$ the number $v > 0$ is an F-normal of M at a if and only if a is a righthand point of dispersion of M and a lefthand point of dispersion of $E_1 - M$ (with obvious meaning of these terms). Similarly for $v < 0$.

8. Lemma. *Let M be a subset of E_p. Then for each $a \in E_p$ there exists at most one F-normal v of M at a such that $|v| = 1$.*

Proof. Let v_1 and v_2 be two F-normals of M at a such that $|v_1| = |v_2| = 1$. By definition we have $a \in \mathcal{R}(M \triangle P(a, v_1)) \cap \mathcal{R}(M \triangle P(a, v_2)) = \mathcal{R}((M \triangle P(a, v_1)) \cup \cup (M \triangle P(a, v_2)))$. From the relation $P(a, v_1) \triangle P(a, v_2) \subset (M \triangle P(a, v_1)) \cup \cup (M \triangle P(a, v_2))$ it follows that $a \in \mathcal{R}(P(a, v_1) \triangle P(a, v_2))$. This means that the halfspaces $P(a, v_1)$, $P(a, v_2)$ must be identical. Since $|v_1| = |v_2|$ we have $v_1 = v_2$.

9. Notation. In the following sections m denotes a fixed natural number. We shall now introduce some sets which appear in the text below. Given $a \in E_m$, $v \in E_m$, $b \in E_1$, $r > 0$, $\varepsilon > 0$, $c = [a, b] \in E_{m+1}$ let us put

$$K(a, v, r, \varepsilon) = \{x \in E_m, |x - a| \leqq r, |v \,.\, (x - a)| \leqq \varepsilon |x - a|\} \quad \text{for} \quad v \neq 0 ,$$

$$L(c, v, r, \varepsilon) = \{z = [x, y], x \in E_m, y \in E_1, |z - c| \leqq r,$$
$$|y - b - v \,.\, (x - a)| \leqq \varepsilon |x - a|\} ,$$

$$H(c, v, r, \varepsilon) = \Big\{z = [x, y], x \in E_m, y \in E_1, |z - c| \leqq r ;$$
$$\frac{y - b}{v \,.\, (x - a)} \geqq \frac{1}{\varepsilon} \text{ or } v \,.\, (x - a) = 0\Big\} \quad \text{for} \quad v \neq 0 .$$

All these sets are measurable and

$$|K(a, v, r, \varepsilon)| = \varkappa(v, \varepsilon)\, r^m , \quad |L(c, v, r, \varepsilon)| = \lambda(v, \varepsilon)\, r^{m+1} ,$$
$$|H(c, v, r, \varepsilon)| = \iota(v, \varepsilon)\, r^{m+1}$$

with

$$\varkappa(v, \varepsilon) = |K(0, v, 1, \varepsilon)|\,, \quad \lambda(v, \varepsilon) = |L(0, v, 1, \varepsilon)|\,, \quad \iota(v, \varepsilon) = |H(0, v, 1, \varepsilon)|\,.$$

Since

$$\bigcap_{\varepsilon>0} K(0, v, 1, \varepsilon) = \{x \in E_m, |x| \leqq 1, v\,.\,x = 0\}\,,$$

$$\bigcap_{\varepsilon>0} L(0, v, 1, \varepsilon) = \{z = [x, y], x \in E_m, y \in E_1, |z| \leqq 1, y = v\,.\,x\}\,,$$

$$\bigcap_{\varepsilon>0} H(0, v, 1, \varepsilon) = \{z = [x, y], x \in E_m, y \in E_1, |z| \leqq 1, v\,.\,x = 0\}$$

we have $\lim_{\varepsilon\to 0+} \varkappa(v, \varepsilon) = \lim_{\varepsilon\to 0+} \lambda(v, \varepsilon) = \lim_{\varepsilon\to 0+} \iota(v, \varepsilon) = 0$. (All expressions with $K, H, \varkappa, \iota$ have a meaning only for $v \neq 0$.)

10. Notation. In the rest of this paper let f be a function with the domain of definition $D \subset E_m$. We shall use the following notations:

$$S(f) = \{z = [x, y], x \in D, y \in E_1, y \leqq f(x)\}\,,$$

$$Q^+(f, k) = \{x \in D, f(x) \geqq k\}, \quad Q^-(f, k) = \{x \in D, f(x) \leqq k\} \quad \text{for} \quad k \in E_1\,.$$

Given $a \in D$, $v \in E_m$, $\varepsilon > 0$, we put further:

$$T(f, a, v, \varepsilon) = \{x \in D, |f(x) - f(a) - v\,.\,(x - a)| > \varepsilon|x - a|\}\,,$$

$$U(f, a, v, \varepsilon) = \left\{x \in D, \frac{f(x) - f(a)}{v\,.\,(x - a)} < \frac{1}{\varepsilon}\right\} \quad \text{for} \quad v \neq 0\,.$$

11. Approximate gradient and approximate differential. a) A vector $v \in E_m$ is said to be an approximate gradient of the function f at a point $a \in D$ with respect to a set $M \subset E_m$ if and only if

$$a \in \mathscr{R}((M - D) \cup (M \cap T(f, a, v, \varepsilon))) \quad \text{for} \quad \text{each} \quad \varepsilon > 0\,.$$

If v is an approximate gradient of f at a with respect to M, then the linear function $v\,.\,(x - a)$ is termed an approximate differential of f at a with respect to M.

In the case $m = 1$ we say also approximate derivative instead of approximate gradient.

b) The function f is said to possess an improper approximate gradient in the direction of a vector $v \in E_m$, $v \neq 0$, at a point $a \in D$ with respect to $M \subset E_m$ if and only if

$$a \in \mathscr{R}((M - D) \cup (M \cap U(f, a, v, \varepsilon))) \quad \text{for each} \quad \varepsilon > 0\,.$$

(Clearly, the last relation depends upon the direction of v only.)

In the case $m = 1$, $v > 0$ $[v < 0]$, f is also said to possess an (improper) approximate derivative $+\infty[-\infty]$ at a with respect to M.

Remark. In the concepts just defined we omit the phrase "with respect to M" for $M = E_m$.

12. Theorem. a) *If v_1 and v_2 are two approximate gradients of the function f at a with respect to M, then either $v_1 = v_2$ or $a \in \mathscr{R}(M)$.*

b) *If a vector v_1 is an approximate gradient of f at a with respect to M and if f possesses an improper approximate gradient in the direction of a vector $v_2 \neq 0$ at a with respect to M, then $a \in \mathscr{R}(M)$.*

c) *If the function f possesses improper approximate gradients in the directions of vectors $v_1 \neq 0$ and $v_2 \neq 0$ at a with respect to M, then $a \in \mathscr{R}(M \cap \cap (P(a, v_1) \triangle P(a, v_2)))$.* (See example 13.)

Proof. a) Suppose $v_1 \neq v_2$. Put $v = v_1 - v_2$ and

$$A(\varepsilon) = (M - D) \cup (M \cap T(f, a, v_1, \varepsilon)) \cup (M \cap T(f, a, v_2, \varepsilon)) \quad \text{for} \quad \varepsilon > 0 .$$

If $x \in M - A(\varepsilon)$, then $x \in D$, $|f(x) - f(a) - v_1 . (x - a)| \leqq \varepsilon|x - a|$, $|f(x) - f(a) - v_2 . (x - a)| \leqq \varepsilon|x - a|$, whence $|(v_1 - v_2) . (x - a)| \leqq 2\varepsilon|x - a|$. Hence it follows that $M \cap \Omega(a, r) \subset (A(\varepsilon) \cap \Omega(a, r)) \cup K(a, v, r, 2\varepsilon)$ and $r^{-m}|M \cap \Omega(a, r)| \leqq r^{-m}|A(\varepsilon) \cap \Omega(a, r)| + \varkappa(v, 2\varepsilon)$ for each $\varepsilon > 0$, $r > 0$. By assumption $a \in \mathscr{R}(A(\varepsilon))$ for each $\varepsilon > 0$. Making first of all $r \to 0+$ and then $\varepsilon \to 0+$, we obtain $\lim_{r \to 0+} r^{-m}|M \cap \Omega(a, r)| = 0$, i.e. $a \in \mathscr{R}(M)$.

b) Put $B(\varepsilon) = (M \cap T(f, a, v_1, \varepsilon)) \cup (M \cap U(f, a, v_2, \varepsilon)) \cup (M - D)$ for $\varepsilon > 0$. If $x \in M - B(\varepsilon)$, then $x \in D$, $|f(x) - f(a) - v_1 . (x - a)| \leqq \varepsilon|x - a|$, and either $(f(x) - f(a)) : (v_2 . (x - a)) \geqq \varepsilon^{-1}$ or $v_2 . (x - a) = 0$, whence further $|v_2 . (x - a)| \leqq \varepsilon|f(x) - f(a)|$, $|f(x) - f(a)| \leqq |v_1|\,|x - a| + \varepsilon|x - a|$, i.e. $|v_2 . (x - a)| \leqq \varepsilon(|v_1| + \varepsilon)\,|x - a|$. Hence it follows that $M \cap \Omega(a, r) \subset (B(\varepsilon) \cap \Omega(a, r)) \cup K(a, v_2, r, \varepsilon(|v_1| + \varepsilon))$ and $r^{-m}|M \cap \Omega(a, r)| \leqq r^{-m}|B(\varepsilon) \cap \Omega(a, r)| + \varkappa(v_2, \varepsilon(|v_1| + \varepsilon))$ for each $\varepsilon > 0$, $r > 0$. Since $a \in \mathscr{R}(B(\varepsilon))$ for each $\varepsilon > 0$, we obtain $\lim_{r \to 0+} r^{-m}|M \cap \Omega(a, r)| = 0$, i.e. $a \in \mathscr{R}(M)$.

c) If $\operatorname{sign}(v_1 . (x - a)) = -\operatorname{sign}(v_2 . (x - a)) \neq 0$, $x \in D$, then either $(f(x) - f(a)) : (v_1 . (x - a)) \leqq 0$ or $(f(x) - f(a) : (v_2 . (x - a)) \leqq 0$. Hence it follows that $M \cap (P(a, v_1) \triangle P(a, v_2)) \subset (M - D) \cup (M \cap U(f, a, v_1, \varepsilon)) \cup (M \cap U(f, a, v_2, \varepsilon)) \cup \{x \in E_m, v_1 . (x - a) = 0 \text{ or } v_2 . (x - a) = 0\}$ for each $\varepsilon > 0$ and $a \in \mathscr{R}(M \cap (P(a, v_1) \triangle P(a, v_2)))$.

13. Example. Let v_1, v_2 be two linearly independent vectors of E_2. Put $D = M = \{x \in E_2, \operatorname{sign}(v_1 . x) = \operatorname{sign}(v_2 . x)\}$, $f(x) = (\operatorname{sign}(v_1 . x))\,|x|^{\frac{1}{2}}$ for $x \in D$. Then f is continuous at $0 \in E_2$ with respect to M and possesses an improper approximate gradient at 0 with respect to M in every direction $v \neq 0$ such that $v = \lambda v_1 + \mu v_2$ with $\lambda \geqq 0$, $\mu \geqq 0$.

14. Theorem. *Let v be an approximate gradient of f at $a \in D$. Then the following statements hold:*

a) $w = [-v, 1]$ *is the F-normal of* $S(f)$ *at* $c = [a, f(a)]$.

b) If $v \neq 0$, *then* v *is the F-normal of* $Q^-(f, f(a))$ *at* a *and* $-v$ *is the F-normal of* $Q^+(f, f(a))$ *at* a.

Proof. For $\varepsilon > 0$, $r > 0$ put

$$A(\varepsilon) = (E_m - D) \cup T(f, a, v, \varepsilon),$$
$$B(r, \varepsilon) = (A(\varepsilon) \cap \Omega(a, r)) \times \langle f(a) - r, f(a) + r \rangle,$$
$$V(r) = (P(c, w) \bigtriangleup S(f)) \cap \Omega(c, r),$$
$$W(r) = (P(a, v) \bigtriangleup Q^-(f, f(a))) \cap \Omega(a, r) \quad \text{if} \quad v \neq 0.$$

a) Let us consider the following three cases (for given $\varepsilon > 0$, $r > 0$):

α) $z = [x, y] \in V(r)$, $x \in A(\varepsilon)$. Then $x \in A(\varepsilon) \cap \Omega(a, r)$, $|y - f(a)| \leqq r$, whence $z \in B(r, \varepsilon)$.

β) $z = [x, y] \in P(c, w) - S(f)$, $z \in \Omega(c, r)$, $x \notin A(\varepsilon)$. Then $x \in D$, $y - f(a) - v \, . \, (x - a) = (z - c) \, . \, w \leqq 0$, $y > f(x)$, $|f(x) - f(a) - v \, . \, (x - a)| \leqq \varepsilon |x - a|$. It follows that $0 \geqq y - f(a) - v \, . \, (x - a) > -\varepsilon |x - a|$, i.e. $z \in L(c, v, r, \varepsilon)$.

γ) Similarly the relations $z = [x, y] \in S(f) - P(c, w)$, $z \in \Omega(c, r)$, $x \notin A(\varepsilon)$ imply $z \in L(c, v, r, \varepsilon)$.

Combining α), β), γ) we obtain the inclusion

$$V(r) \subset B(r, \varepsilon) \cup L(c, v, r, \varepsilon).$$

According to 5b) and 9 we have $|B(r, \varepsilon)| = 2r |A(\varepsilon) \cap \Omega(a, r)|$ and $|L(c, v, r, \varepsilon)| = r^{m+1} \lambda(v, \varepsilon)$. Therefore

$$r^{-(m+1)} |V(r)| \leqq 2r^{-m} |A(\varepsilon) \cap \Omega(a, r)| + \lambda(v, \varepsilon) \quad \text{for each} \quad \varepsilon > 0, \quad r > 0.$$

Hence

$$\lim_{r \to 0+} r^{-(m+1)} |V(r)| = 0, \quad \text{i.e.} \quad c \in \mathcal{R}(P(c, w) \bigtriangleup S(f)).$$

b) Suppose $v \neq 0$. Let us consider the following two cases:

α) $x \in P(a, v) - Q^-(f, f(a))$, $x \in \Omega(a, r)$, $x \notin A(\varepsilon)$. Then $x \in D$, $(x - a) \, . \, v \leqq 0$, $f(x) > f(a)$, $|f(x) - f(a) - v \, . \, (x - a)| \leqq \varepsilon |x - a|$, $|x - a| \leqq r$. It follows that $0 \leqq -(x - a) \, . \, v < f(x) - f(a) - v \, . \, (x - a) \leqq \varepsilon |x - a|$, $|x - a| \leqq r$, i.e. $x \in K(a, v, r, \varepsilon)$.

β) Similarly the relations $x \in Q^-(f, f(a)) - P(a, v)$, $x \in \Omega(a, r)$, $x \notin A(\varepsilon)$ imply $x \in K(a, v, r, \varepsilon)$.

Combining α), β) we obtain the inclusion

$$W(r) \subset (A(\varepsilon) \cap \Omega(a, r)) \cup K(a, v, r, \varepsilon).$$

Therefore

$$r^{-m}|W(r)| \leqq r^{-m}|A(\varepsilon) \cap \Omega(a, r)| + \varkappa(v, \varepsilon) \quad \text{for each} \quad \varepsilon > 0,\ r > 0\,,$$

whence

$$\lim_{r \to 0+} r^{-m}|W(r)| = 0\,, \quad \text{i.e.} \quad a \in \mathscr{R}(P(a, v) \bigtriangleup Q^-(f, f(a)))\,.$$

The proof of the second part of b) is similar.

15. Theorem. *Suppose that f possesses an improper approximate gradient in the direction of a vector $v \neq 0$ at a point $a \in D$. Then $w = [-v, 0]$ is the F-normal of $S(f)$ at $c = [a, f(a)]$, v is the F-normal of $Q^-(f, f(a))$ at a and $-v$ is the F-normal of $Q^+(f, f(a))$ at a.*

Proof. For each $\varepsilon > 0$ and $r > 0$ put

$$A(\varepsilon) = (E_m - D) \cup U(f, a, v, \varepsilon)\,,$$
$$B(r, \varepsilon) = (A(\varepsilon) \cap \Omega(a, r)) \times \langle f(a) - r, f(a) + r \rangle\,,$$
$$V(r) = (P(c, w) \bigtriangleup S(f)) \cap \Omega(c, r)\,, \quad W(r) = (P(a, v) \bigtriangleup Q^-(f, f(a)) \cap \Omega(a, r)\,.$$

a) Let us consider the following three cases:

α) $z = [x, y] \in V(r)$, $x \in A(\varepsilon)$. Then $x \in A(\varepsilon) \cap \Omega(a, r)$, $|y - f(a)| \leqq r$, whence $z \in B(r, \varepsilon)$.

β) $z = [x, y] \in P(c, w) - S(f)$, $z \in \Omega(c, r)$, $x \notin A(\varepsilon)$. Then $x \in D$, $v \,.\, (x - a) = -(z - c) \,.\, w \geqq 0$, $y > f(x)$ and either $(f(x) - f(a)) : (v \,.\, (x - a)) \geqq \varepsilon^{-1}$ or $v \,.\, (x - a) = 0$. It follows that either $(y - f(a)) : (v \,.\, (x - a)) \geqq \varepsilon^{-1}$ or $v \,.\, (x - a) = 0$, i.e. $z \in H(c, v, r, \varepsilon)$.

γ) Similarly the relations $z = [x, y] \in S(f) - P(c, w)$, $z \in \Omega(c, r)$, $x \notin A(\varepsilon)$ imply $z \in H(c, v, r, \varepsilon)$.

Combining α), β), γ) we obtain the inclusion

$$V(r) \subset B(r, \varepsilon) \cup H(c, v, r, \varepsilon)\,.$$

According to 5b) and 9 we have $|B(r, \varepsilon)| = 2r|A(\varepsilon) \cap \Omega(a, r)|$ and $|H(c, v, r, \varepsilon)| = \iota(v, \varepsilon)\, r^{m+1}$. Hence

$$r^{-(m+1)}|V(r)| \leqq 2r^{-m}|A(\varepsilon) \cap \Omega(a, r)| + \iota(v, \varepsilon) \quad \text{for each} \quad \varepsilon > 0,\ r > 0\,,$$

so that $\lim_{r \to 0+} r^{-(m+1)}|V(r)| = 0$, i.e. $c \in \mathscr{R}(P(c, w) \bigtriangleup S(f))$.

b) Let us consider the following cases (for given $\varepsilon > 0$):

α) $x \in P(a, v) - Q^-(f, f(a))$, $x \in D$. Then $(x - a) \,.\, v \leqq 0$, $f(x) > f(a)$. It follows that either $(x - a) \,.\, v = 0$ or $(f(x) - f(a)) : (v \,.\, (x - a)) < 0 < \varepsilon^{-1}$.

β) Similarly the relations $x \in Q^-(f, f(a)) - P(a, v)$, $x \in D$ imply $(f(x) - f(a)) : : (v \, . \, (x - a)) \leqq 0 < \varepsilon^{-1}$.

γ) Finally the relation $x \in E_m - D$ implies $x \in A(\varepsilon)$.

Combining α), β), γ) we obtain the inclusion

$$W(r) \subset (A(\varepsilon) \cap \Omega(a, r)) \cup \{x \in E_m, (x - a) \, . \, v = 0\} \, .$$

Therefore

$$r^{-m}|W(r)| \leqq r^{-m}|A(\varepsilon) \cap \Omega(a, r)| \quad \text{for each} \quad \varepsilon > 0, \ r > 0 \, ,$$

whence

$$\lim_{r \to 0+} r^{-m}|W(r)| = 0, \quad \text{i.e.} \quad a \in \mathscr{R}(P(a, v) \triangle Q^-(f, f(a))) \, .$$

The proof of the last part of our theorem is similar.

16. Theorem. *If* $w = [v, t] \in E_{m+1}$, $t \neq 0$ *is an F-normal of the set* $S(f)$ *at a point* $c = [a, b] \in E_{m+1}$, *then the following statements hold:*

α) $t > 0$;

β) $a \in \mathscr{R}(E_m - D)$.

Proof. Let A^+ denote the set of all $x \in E_m$ such that $\{x\} \times (E_{m+1} - P(c, w))^*_x \subset \subset S(f)$ and let A^- denote the set of all $x \in E_m$ such that $\{x\} \times (P(c, w))^*_x \subset E_{m+1} - - S(f)$. Clearly $E_m - D \subset A^-$.

For each $r > 0$ put

$$A^+(r) = A^+ \cap \Omega(a, r), \ A^-(r) = A^- \cap \Omega(a, r) \, ,$$
$$B^+(r) = \{z = [x, y], x \in E_m, y \in E_1, tb - v \, . \, (x - a) < ty < tb - v \, . \, (x - a) + r\} \, ,$$
$$B^-(r) = \{z = [x, y], x \in E_m, y \in E_1, tb - v \, . \, (x - a) - r < ty < tb - v \, . \, (x - a)\} \, ,$$
$$C^+(r) = B^+(r) \cap (A^+(r) \times E_1), \ C^-(r) = B^-(r) \cap (A^-(r) \times E_1) \, ,$$
$$V(r) = (P(c, w) \triangle S(f)) \cap \Omega(c, r) \, .$$

Suppose $z = [x, y] \in C^+(r)$. Then $|z - c| = (|x - a|^2 + |y - b|^2)^{\frac{1}{2}} \leqq (|x - a|^2 + + |t|^{-2} (|v| \, |x - a| + r)^2)^{\frac{1}{2}} \leqq r(1 + |t|^{-2} (|v| + 1)^2)^{\frac{1}{2}} = \omega r$ if we put $\omega = (1 + + |t|^{-2} (|v| + 1)^2)^{\frac{1}{2}}$. Further $(z - c) \, . \, w = (x - a) \, . \, v + t(y - b) > 0$, whence $z \in \in E_{m+1} - P(c, w)$. Since $x \in A^+$, we have $z \in S(f)$. It follows that $C^+(r) \subset V(\omega r)$. Similarly $C^-(r) \subset V(\omega r)$.

The set $B^+(r)$ is measurable, whence by 5a)

$$|C^+(r)| = \int_{Z^+(r)} |(B^+(r))^*_x| \, \mathrm{d}x = |t|^{-1} \, r|Z^+(r)| = |t|^{-1} \, r|A^+(r)| \, ,$$

where $Z^+(r)$ is a measurable cover of $A^+(r)$. Similarly $|C^-(r)| = |t|^{-1} \, r|A^-(r)|$. Hence

$$r^{-m}|A^+(r) \cup A^-(r)| \leqq r^{-(m+1)}|t| \, (|C^+(r)| + |C^-(r)|) \leqq 2r^{-(m+1)}|t| \, |V(\omega r)| \, .$$

By assumption $a \in \mathscr{R}(P(c, w) \triangle S(f))$, so that $\lim_{r \to 0+} r^{-m}|A^+(r) \cap A^-(r)| = 0$, i.e. $a \in \mathscr{R}(A^+ \cup A^-)$. Since $E_m - D \subset A^-$, we have $a \in \mathscr{R}(E_m - D)$ and the proof of $\beta)$ is complete.

Since $a \in \mathscr{R}(A^+ \cup A^-)$, there exists a point $x_0 \in E_m - (A^+ \cup A^-)$. According to the definition of A^- and A^+ there exist y_1 and y_2 such that $[x_0, y_1] \in (P(c, w) \cap \cap S(f))$, $[x_0, y_2] \in (E_{m+1} - (P(c, w) \cup S(f)))$. Hence we have $x_0 \in D$, $y_1 \leqq f(x_0) < < y_2$, $(x_0 - a) \cdot v + t(y_1 - b) \leqq 0$, $(x_0 - a) \cdot v + t(y_2 - b) > 0$. It follows that $y_2 - y_1 > 0$, $t(y_2, - y_1) > 0$, whence $t > 0$.

Remark. Let w be an F-normal of $S(f)$ at $c = [a, b]$. Put $g(x) = f(x)$ for $x \in D - \{a\}$; $g(a) = b$. Clearly w is an F-normal of $S(g)$ at c. Therefore we assume in the following sections that $b = f(a)$.

17. Theorem. *If $w = [v, 1]$ is an F-normal of $S(f)$ at $c = [a, f(a)]$, then $-v$ is the approximate gradient of f at a.*

Proof. Choose $\varepsilon > 0$. For each $r > 0$, $0 < \eta < 1$ put

$$\begin{aligned}
A(r) &= T(f, a, -v, \varepsilon) \cap \Omega(a, r), \quad A(r\ \eta) = A(r) - \Omega(a, \eta r)\\
A^+(r, \eta) &= A(r, \eta) \cap \{x \in D, f(x) - f(a) + v \cdot (x - a) > \varepsilon|x - a|\},\\
A^-(r, \eta) &= A(r, \eta) \cap \{x \in D, f(x) - f(a) + v \cdot (x - a) < -\varepsilon|x - a|\},\\
B^+ &= \{z = [x, y], x \in E_m, y \in E_1, 0 < y - f(a) + v \cdot (x - a) \leqq \varepsilon|x - a|\},\\
B^- &= \{z = [x, y], x \in E_m, y \in E_1, 0 > y - f(a) + v \cdot (x-a) \geqq -\varepsilon|x-a|\},\\
C^+(r, \eta) &= B^+ \cap (A^+(r, \eta) \times E_1), \quad C^-(r, \eta) = B^- \cap (A^-(r, \eta) \times E_1),\\
V(r) &= (P(c, w) \triangle S(f)) \cap \Omega(c, r).
\end{aligned}$$

Suppose $z = [x, y] \in C^+(r, \eta)$. Then $x \in D$, $f(x) - f(a) + v \cdot (x - a) > \varepsilon|x - a|$, $0 < (z - c) \cdot w = y - f(a) + v \cdot (x - a) \leqq \varepsilon|x - a|$, $|x - a| \leqq r$. It follows that $z \in E_{m+1} - P(c, w)$, $y < f(x)$ (i.e. $z \in S(f)$), $|z - c| = (|x - a|^2 + |y - f(a)|^2)^{\frac{1}{2}} \leqq \leqq (|x - a|^2 + (\varepsilon|x - a| + |v \cdot (x - a)|)^2)^{\frac{1}{2}} \leqq |x - a| (1 + (\varepsilon + |v|)^2)^{\frac{1}{2}} \leqq \omega r$ with $\omega = (1 + (\varepsilon + |v|)^2)^{\frac{1}{2}}$. Hence $C^+(r, \eta) \subset V(\omega r)$.

Similarly $C^-(r, \eta) \subset V(\omega r)$.

The set B^+ is measurable, whence by 5a)

$$|C^+(r, \eta)| = \int_{Z^+(r,\eta)} |(B^+)^*_x| \,\mathrm{d}x = \varepsilon \int_{Z^+(r,\eta)} |x - a| \,\mathrm{d}x\,,$$

where $Z^+(r, \eta)$ is a measurable cover of $A^+(r, \eta)$ such that $Z^+(r, \eta) \subset E_m - \Omega(a, \eta r)$. Hence $|C^+(r, \eta)| \geqq \varepsilon\eta r|Z^+(r, \eta)| = \varepsilon\eta r|A^+(r, \eta)|$. Similarly $|C^-(r, \eta)| \geqq \varepsilon\eta r|A^-(r, \eta)|$. Since $A(r) \subset A^+(r, \eta) \cup A^-(r, \eta) \cup \Omega(a, \eta r)$, it follows that

$$\begin{aligned}
r^{-m}|A(r)| &\leqq r^{-m}(|A^+(r, \eta)| + |A^-(r, \eta)|) + \eta^m\gamma_m \leqq\\
&\leqq \varepsilon^{-1}\eta^{-1}r^{-(m+1)}(|C^+(r, \eta)| + |C^-(r, \eta)|) + \eta^m\gamma_m \leqq\\
&\leqq 2\varepsilon^{-1}\eta^{-1}r^{-(m+1)}|V(\omega r)| + \eta^m\gamma_m\,.
\end{aligned}$$

Making first $r \to 0+$ and then $\eta \to 0+$ we obtain

$$\lim_{r \to 0+} r^{-m}|A(r)| = 0\,, \quad \text{i.e.} \quad a \in \mathscr{R}(T(f, a, -v, \varepsilon))\,.$$

18. Theorem. *If $w = [v, 0]$ is an F-normal of $S(f)$ at $c = [a, f(a)]$, then $a \in \in \mathscr{R}(P(a, v) - D)$.*

Proof. For each $r > 0$ put

$$A(r) = (P(a, v) - D) \cap \Omega(a, r)\,, \quad B(r) = A(r) \times \langle f(a) - r, f(a) + r\rangle\,,$$
$$V(r) = (P(c, w) \triangle S(f)) \cap \Omega(c, r)\,.$$

Suppose $z = [x, y] \in B(r)$. Then $z \notin S(f)$, $(z - c) \,.\, w = (x - a) \,.\, v \leqq 0$ and $|z - c| = (|x - a|^2 + |y - f(a)|^2)^{\frac{1}{2}} \leqq 2^{\frac{1}{2}} r$. It follows that $B(r) \subset V(2^{\frac{1}{2}} r)$. According to 5b) we have $|B(r)| = 2r|A(r)|$. Hence $r^{-m}|A(r)| = 2^{-1} r^{-(m+1)}|B(r)| \leqq \leqq 2^{-1} r^{-(m+1)} |V(2^{\frac{1}{2}} r)|$ for each $r > 0$. Consequently

$$\lim_{r \to 0+} r^{-m}|A(r)| = 0\,, \quad \text{i.e.} \quad a \in \mathscr{R}(P(a, v) - D)\,.$$

19. Theorem. *If $w = [v, 0]$ is an F-normal of $S(f)$ at $c = [a, f(a)]$, then the function f possesses an improper approximate gradient in the direction of the vector $-v$ at a with respect to each set M such that $a \in \mathscr{R}(M - D)$.*

Proof. Choose $\varepsilon > 0$. For each $r > 0$ put

$$A(r) \;= (M \cap U(f, a, -v, \varepsilon)) \cap \Omega(a, r)\,,$$
$$A^+(r) = A(r) \cap P(a, v), \; A^-(r) = A(r) - P(a, v)\,,$$
$$B^+(r) = \{z = [x, y],\, x \in E_m,\, y \in E_1,\, (x - a) \,.\, v < 0,$$
$$f(a) - \varepsilon^{-1} v \,.\, (x - a) < y < f(a) - \varepsilon^{-1} v \,.\, (x - a) + r\}\,,$$
$$B^-(r) = \{z = [x, y],\, x \in E_m,\, y \in E_1,\, (x - a) \,.\, v > 0,$$
$$f(a) - \varepsilon^{-1} v \,.\, (x - a) - r < y < f(a) - \varepsilon^{-1} v \,.\, (x - a)\}\,,$$
$$C^+(r) = B^+(r) \cap (A^+(r) \times E_1), \; C^-(r) = B^-(r) \cap (A^-(r) \times E_1)\,,$$
$$V(r) \;= (P(c, w) \triangle S(f)) \cap \Omega(c, r)\,.$$

Suppose $z = [x, y] \in C^+(r)$. Then $x \in M$, $x \in D$, $(x - a) \,.\, v < 0$, $f(a) - \varepsilon^{-1} v \,.\, .\, (x - a) < y < f(a) - \varepsilon^{-1} v \,.\, (x - a) + r$, $(f(x) - f(a)) : (-v \,.\, (x - a)) < \varepsilon^{-1}$. It follows that $f(x) < f(a) - \varepsilon^{-1} v \,.\, (x - a) < y$. Further we have $|z - c| = (|x - a|^2 + + |y - f(a)|^2)^{\frac{1}{2}} \leqq (|x - a|^2 + (r + \varepsilon^{-1}|v \,.\, (x - a)|)^2)^{\frac{1}{2}} \leqq r(1 + (1 + \varepsilon^{-1}|v|)^2)^{\frac{1}{2}} = = \omega r$ with $\omega = (1 + (1 + \varepsilon^{-1}|v|)^2)^{\frac{1}{2}}$. Hence $C^+(r) \subset V(\omega r)$.

Similarly $C^-(r) \subset V(\omega r)$.

The set $B^+(r)$ is measurable, whence by 5a)

$$|C^+(r)| = \int_{Z^+(r)} |(B^+(r))^*_x| \, \mathrm{d}x = r|Z^+(r)| = r|A^+(r)| ,$$

where $Z^+(r)$ is a measurable cover of $A^+(r)$. Similarly $|C^-(r)| = r|A^-(r)|$. It follows that $r^{-m}|A(r)| \leqq r^{-m}(|A^+(r)| + |A^-(r)|) = r^{-(m+1)}(|C^+(r)| + |C^-(r)|) \leqq \leqq 2r^{-(m+1)}|V(\omega r)|$ for each $r > 0$. Hence

$$\lim_{r \to 0+} r^{-m}|A(r)| = 0 , \quad \text{i.e.} \quad a \in \mathscr{R}(M \cap U(f, a, -v, \varepsilon)) .$$

References

[1] *S. Saks:* Theory of the integral, New York 1937.

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Резюме

АППРОКСИМАТИВНЫЙ ДИФФЕРЕНЦИАЛ И НОРМАЛЬ ФЕДЕРЕРА

ЙИРЖИ МАТЫСКА (Jiří Matyska), Прага

Пусть f – функция, определенная в некоторой части пространства E_m, и пусть $S(f)$ – множество точек из E_{m+1}, которые находятся ниже графика функции f. В статье показано, что функция f имеет в точке a аппроксимативный дифференциал $v \, . \, (x - a)$ тогда и только тогда, когда множество $S(f)$ имеет в точке $[a, f(a)]$ нормаль Федерера $[-v, 1]$. Дальше показано, что множество $S(f)$ не может иметь нормаль Федерера $[v, -1]$, и исследуется поведение функции f в окрестности такой точки a, что множество $S(f)$ имеет в точке $[a, f(a)]$ нормаль Федерера $[v, 0]$.