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MAXIMAL IDEALS IN THE DIRECT PRODUCT OF TWO SEMIGROUPS

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Let $S \times T$ denote the direct product of semigroups S and T. Several papers [3], [4], [5], [6], [7] have been written to investigate the forms of various types of ideals in $S \times T$, in terms of similar ideals in the factor semigroups S and T. Particular emphasis has been placed on determining necessary and sufficient conditions on the form of an ideal in order that it be of a special type, such as minimal, 0-minimal, prime or semi-prime. This note is concerned with maximal ideals in $S \times T$. All ideals in this paper are two-sided.

A proper ideal of a semigroup is called a maximal ideal provided that it is not properly contained in any other proper ideal of that semigroup. The problem of determining the form of maximal ideals in $S \times T$ is solved in Theorem 1. It is shown that the form of a maximal ideal is quite similar to that of a prime ideal, as determined in [5]. Theorem 2 gives necessary and sufficient conditions on a proper ideal I of $S \times T$, with the property that $(S \times T)^2 \nsubseteq I$, in order that I be a maximal ideal. These results are easily generalized to the case of the direct product of any number of semigroups.

The proofs of the theorems make use of a specific characterization of a maximal ideal, which can also be obtained using corollary 2.38 of [1].

1. Characterization of a maximal ideal. Let a be an element of some arbitrary semigroup S. Let $J(a)$ denote the principal ideal generated by a and let J_a denote the set of elements in S that generate the same principal ideal as that generated by a. Notice that S is simple (has no proper ideal) if and only if $J_a = S$ for some $a \in S$, where $a \in SaS$.

Suppose that S has a maximal ideal M. Let $A = S \backslash M$ denote the complement of M in S and let $a \in A$. If $S^2 \subseteq M$ then $A = \{a\}$ where $a^2 \in M$, and if $S^2 \nsubseteq M$ then $A = J_a$ where $a \in SaS$. This yields the following lemma, which characterizes a maximal ideal in a semigroup S.

Lemma 1. *Let I be a proper ideal of S, let $A = S \backslash I$ and let $a \in A$. Then I is a maximal ideal if and only if either A is the set $\{a\}$ where $a^2 \in I$, or else $A = J_a$ where $a \in SaS$.*

2. Main results. Let S and T denote arbitrary semigroups. Notice that if (a, b) is an element of $S \times T$ and is contained in $(S \times T)(a, b)(S \times T)$, then $J_{(a,b)} = J_a \times J_b$ where $a \in SaS$ and $b \in TbT$. This fact is used in the proof of the following theorem.

Theorem 1. *Assume that $S \times T$ has a maximal ideal M. If $(S \times T)^2 \subseteq M$ then M has the form $M = (S \times B) \cup (C \times T)$, where B and C are non-empty subsets of T and S respectively, at least one of which is a maximal ideal. If $(S \times T)^2 \nsubseteq M$ then M has the form $M = (S \times I) \cup (J \times T)$ where either J is the empty set, in which case S is a simple semigroup, or else J is a maximal ideal of S; and where either I is the empty set, in which case T is simple, or else I is a maximal ideal of T.*

Proof. Let $A = (S \times T) \backslash M$ and let $(a, b) \in A$.

If $(S \times T)^2 \subseteq M$ then $A = \{(a, b)\}$ where $(a, b)^2 \in M$ by Lemma 1. Let $B = T \backslash \{b\}$ and let $C = S \backslash \{a\}$. Then

$$M = (S \times T) \backslash A = (S \times T) \backslash \{(a, b)\} = (S \times T \backslash \{b\}) \cup (S \backslash \{a\} \times T) = \\ = (S \times B) \cup (C \times T)$$

At least one of B and C must be an ideal for otherwise (a, b) could be expressed as a product of elements in $S \times T$, contradicting the assumption that $(S \times T)^2 \subseteq M$. Thus at least one of B and C is a maximal ideal.

Now assume that $(S \times T)^2 \nsubseteq M$. Then by Lemma 1, $(a, b) \in (S \times T)(a, b)(S \times T)$ and $A = J_{(a,b)}$. Thus $A = J_a \times J_b$ where $a \in SaS$ and $b \in TbT$. Let $I = T \backslash J_b$ and $J = S \backslash J_a$. Then

$$M = (S \times T) \backslash (J_a \times J_b) = (S \times T \backslash J_b) \cup (S \backslash J_a \times T) = (S \times I) \cup (J \times T)$$

If J is the empty set then $J_a = S$ and S is a simple semigroup. Suppose that J is non-empty. It will be shown that J is an ideal of S. Let $x \in J$ and $s \in S$. Then if xs were in J_a, this would imply that $SxsS = J(xs) = J(a) = SaS$ so that $a \in SxsS \subseteq SxS$. This would mean that

$$(a, b) \in SxS \times TbT = (S \times T)(x, b)(S \times T),$$

contradicting the assumption that M is an ideal of $S \times T$, since $(x, b) \in M$. Thus $sx \in J$ and dually $xs \in J$ for all $x \in J$ and $s \in S$. Thus J is an ideal of S. Since $S \backslash J = J_a$ where $a \in SaS$, J is a maximal ideal by Lemma 1. Similarly if I is empty, T is simple if I is non-empty, then it must be a maximal ideal.

From this theorem it is evident that if $S \times T$ has a maximal ideal M, then at least one of S and T must also have a maximal ideal. Also, M can be expressed in the form $M = P_1(M) \times P_2(M)$ (where $P_i(M)$ is the projection of M onto its ith components) if and only if one of S and T is simple.

The converse of Theorem 1 is not generally true. Examples can be constructed of semigroups S and T with maximal ideals J and I respectively, so that $M = (S \times I) \cup \cup (J \times T)$ is not a maximal ideal of $S \times T$. However, the next two lemmas show that the converse is true in the case where $(S \times T)^2 \nsubseteq M$. They can readily be obtained by applying Lemma 1.

Lemma 2. *Let S and T be semigroups such that S has a maximal ideal M, and T has more then one element. Then if $S^2 \subseteq M$, $M \times T$ is not a maximal ideal of $S \times T$, and if $S^2 \nsubseteq M$, $M \times T$ is a maximal ideal if and only if T is a simple semigroup.*

Lemma 3. *Let S and T be semigroups with maximal ideals J and I respectively, and let $M = (S \times I) \cup (J \times T)$. Then*

(1) *if $S^2 \subseteq J$ and $T^2 \subseteq I$ then M is a maximal ideal of $S \times T$ where $(S \times T)^2 \subseteq \subseteq M$, and*

(2) *if $S^2 \subseteq J$ and $T^2 \nsubseteq I$ then M is a maximal ideal if and only if $T \setminus I$ consists of exactly one element, and*

(3) *if $S^2 \nsubseteq J$ and $T^2 \nsubseteq I$ then M is a maximal ideal where $(S \times T)^2 \nsubseteq M$.*

The following theorem is a combination of parts of Theorem 1 and the preceding lemmas. It gives both necessary and sufficient conditions in order that a proper ideal M of $S \times T$ be a maximal ideal in case $(S \times T)^2 \nsubseteq M$.

Theorem 2. *Let S and T be semigroups such that $S \times T$ has a proper ideal M where $(S \times T)^2 \nsubseteq M$. Then M is a maximal ideal if and only if it has the form $M = (S \times I) \cup (J \times T)$ where either J is the emply set, in which case S is simple, or else J is a maximal ideal of S; and where either I is the empty set, in which case T is simple, or else I is a maximal ideal of T.*

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