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SOME ESTIMATES IN THE THEORY  
OF NON-NEGATIVE MATRICES

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*Dedicated to academician VOJTECH JARNIK on the occasion of his seventieth birthday  
on December 22, 1967*

This paper presupposes the knowledge of some results proved in [1] and [2].

We briefly recall the notions necessary in the following which have been introduced in [1] and [2].

Denote  $N = \{1, 2, \dots, n\}$  and consider the set  $S$  of all " $n \times n$  matrix units  $e_{ij}$ " together with a zero  $0$  adjoint:  $S = \{e_{ij} \mid i, j \in N\} \cup \{0\}$ . Define in  $S$  a multiplication by

$$e_{ij}e_{ml} = \begin{cases} e_{il} & \text{for } j = m, \\ 0 & \text{for } j \neq m, \end{cases}$$

the zero element  $0$  having the usual properties of a multiplicative zero. The set  $S$  becomes then a  $0$ -simple semigroup.

Let  $A = (a_{ij})$  be a non-negative  $n \times n$  matrix. By the support  $C_A$  of  $A$  we shall mean the subset of  $S$  containing  $0$  and all  $e_{ij}$  for which  $a_{ij} > 0$ .

The sequence of powers

$$(1) \quad C_A, C_A^2, C_A^3, \dots$$

contains only a finite number of different elements (subsets of  $S$ ). Let  $k = k(A) \geq 1$  be the least integer for which  $C_A^k$  appears in (1) more than once and let  $d = d(A)$  be the least integer  $\geq 1$  for which  $C_A^k = C_A^{k+d}$  holds. Then the sequence (1) is of the form

$$C_A, C_A^2, \dots, C_A^{k-1} \mid C_A^k, \dots, C_A^{k+d-1} \mid C_A^k, \dots$$

and contains exactly  $k + d - 1$  different elements. The system of sets  $\{C_A^k, \dots, C_A^{k+d-1}\}$  forms with respect to the multiplication of subsets a cyclic group of order  $d$  with the unit element  $C_A^q$ , where  $q = q(A)$  is a uniquely defined multiple of  $d$ , say  $\tau d$ , satisfying  $k \leq q = \tau d \leq k + d - 1$ .

Denote  $S_i = \{0, e_{i1}, e_{i2}, \dots, e_{im}\}$ , so that  $S_1 \cup S_2 \cup \dots \cup S_n = S$ . For a given  $A$  denote  $F_i = F_i(A) = S_i \cap C_A$ , so that  $F_i$  is the "support" of the  $i$ -th row of  $A$ .

Consider the sequence

$$(2) \quad F_i, F_i C_A, F_i C_A^2, \dots$$

and define  $F_i C_A^0 = F_i$ . Clearly  $F_i C_A^t = S_i \cap C_A^{t+1}$  for any integer  $t \geq 0$ . The sequence (2) contains again only a finite number of different elements. Denote  $k_i = k_i(A)$  the least integer  $k_i$  such that  $F_i C_A^{k_i-1}$  occurs in (2) more than once. Let further  $d_i = d_i(A)$  be the least integer  $\geq 1$  such that  $F_i C_A^{k_i-1} = F_i C_A^{k_i+d_i-1}$ . Then the sequence (2) is of the form

$$F_i, F_i C_A, \dots, F_i C_A^{k_i-2} \mid F_i C_A^{k_i-1}, \dots, F_i C_A^{k_i+d_i-2} \mid F_i C_A^{k_i-1}, \dots.$$

Clearly  $F_i C_A^{k_i-1} = F_i C_A^{k_i-1+\tau}$  ( $\tau \geq 0$ ) holds if and only if  $d_i \mid \tau$ . Since  $C_A^k = C_A^{k+d}$ , we have  $S_i \cap C_A^k = S_i \cap C_A^{k+d}$ ; hence  $F_i C_A^{k-1} = F_i C_A^{k+d-1}$ . Therefore  $k_i \leq k$  and  $d_i \leq d$ . In [2] we have proved the following more precise result:

**Lemma 1.** For any non-negative  $n \times n$  matrix  $A$  we have:

- a)  $k(A) = \max_{i=1, \dots, n} k_i(A)$ ;
- b)  $d(A) =$  the least common multiple of  $d_1, d_2, \dots, d_n$ .

The main purpose of this paper is to find estimates for the number  $k(A)$ , whereby we restrict ourselves to the case of an *irreducible*  $n \times n$  matrix.

For an irreducible matrix  $A$  we have proved in [1] that  $d$  is the index of imprimitivity of  $A$  and we always have  $1 \leq d \leq n$ . Also  $A$  is irreducible if and only if

$$(3) \quad C_A^k \cup C_A^{k+1} \cup \dots \cup C_A^{k+d-1} = S.$$

Here the summands on the left are quasisdisjoint, i.e. the intersection of any two of them contains only the zero element 0. More generally: Any  $d$  consecutive powers  $C_A^t, C_A^{t+1}, C_A^{t+2}, \dots, C_A^{t+d-1}$ ,  $t \geq 1$ , are quasisdisjoint. This implies that for an irreducible matrix  $A$  any  $d$  consecutive members in the sequence (2) are quasisdisjoint.

If  $A$  is irreducible, then (3) implies

$$(C_A^k \cap S_i) \cup \dots \cup (C_A^{k+d-1} \cap S_i) = S \cap S_i,$$

i.e.

$$(4) \quad F_i C_A^{k-1} \cup F_i C_A^k \cup \dots \cup F_i C_A^{k+d-2} = S_i.$$

It should be noted, by the way, that for an irreducible matrix  $A$  each member in the sequence (2) contains at least one non-zero element  $\in S_i$ . (For if there were  $F_i C_A^\tau = \{0\}$  for some  $\tau \geq 0$ , we would have  $F_i C_A^t = \{0\}$  for all  $t \geq \tau$ , a contradiction with (4).)

The relation (4) implies  $\bigcup_{j=0}^{\infty} F_i C_A^j = S_i$  or, which is the same,

$$F_i \cup F_i C_A \cup \dots \cup F_i C_A^{k_i-1} \cup \dots \cup F_i C_A^{k_i+d_i-2} = S_i.$$

Since an irreducible matrix contains in each column at least one non-zero element, we have  $S_i C_A = S_i$ . Hence, multiplying the last relation by  $C_A^{k_i-1}$  from the right we

get

$$(5) \quad F_i C_A^{k_i-1} \cup F_i C_A^{k_i} \cup \dots \cup F_i C_A^{k_i+d_i-2} = S_i.$$

Since any set on the left has at least one non-zero element and any  $d$  consecutive members in (2) are quasidisjoint, we necessarily have  $d_i \geq d$ . Therefore, in the case of an irreducible matrix,  $d_i = d$ . We summarize:

**Lemma 2.** *Let  $A$  be an  $n \times n$  non-negative irreducible matrix. Then:*

- a)  $d_1(A) = d_2(A) = \dots = d_n(A) = d(A)$ ;
- b)  $F_i C_A^{k_i-1} \cup F_i C_A^{k_i} \cup \dots \cup F_i C_A^{k_i+d-2} = S_i$ ;
- c) *the summands on the left are quasidisjoint.*

**Lemma 3.** *If  $A$  is irreducible and there are two integers  $\kappa \geq 1$ ,  $\delta \geq 1$  such that*

$$F_i C_A^{\kappa-1} \cup F_i C_A^{\kappa} \cup \dots \cup F_i C_A^{\kappa+\delta-2} = S_i,$$

*then  $\delta \geq d$ .*

*Proof.* Suppose that  $\delta < d$ . Multiplying by a sufficiently high power  $C_A^t$  we get

$$F_i C_A^{\kappa-1+t} \cup F_i C_A^{\kappa+t} \cup \dots \cup F_i C_A^{\kappa+\delta-2+t} = S_i C_A^t = S_i.$$

The  $\delta$  summands on the left, which are equal to some of the quasidisjoint sets  $F_i C_A^{k_i-1}, \dots, F_i C_A^{k_i+d-2}$ , cannot exhaust the whole set  $S_i$ . This proves our assertion.

For further purposes we introduce a positive integer  $h_i$  associated with the “row  $F_i$ ” in the following way:

By  $h_i$  we shall denote the least integer  $\geq 1$  such that  $F_i \subset F_i C_A^{h_i}$ .

In [2] we have proved that  $1 \leq h_i \leq n$ .

**Lemma 4.** *For  $i = 1, 2, \dots, n$ , we have  $d|h_i$ .*

*Proof.* We have  $F_i C_A^{k_i-1} \subset F_i C_A^{h_i} C_A^{k_i-1} = F_i C_A^{k_i+h_i-1}$  and

$$S_i = F_i C_A^{k_i-1} \cup \dots \cup F_i C_A^{k_i+d-2} \subset F_i C_A^{k_i+h_i-1} \cup \dots \cup F_i C_A^{h_i+k_i+d-2}.$$

Hence,

$$S_i = F_i C_A^{k_i+h_i-1} \cup \dots \cup F_i C_A^{h_i+k_i+d-2}.$$

Since the sets on the right are quasidisjoint, we have  $F_i C_A^{k_i-1} = F_i C_A^{-1+h_i}$ . This proves  $d|h_i$ .

**Lemma 5.** *Let  $A$  be irreducible. If for some  $t \geq 1$  we have*

$$(6) \quad F_i C_A^{t-1} \cup F_i C_A^t \cup \dots \cup F_i C_A^{t+d-2} = S_i,$$

*then  $t \geq k_i$ .*

Proof. As in the proof of Lemma 4 we have

$$F_i C_A^{t-1} \subset F_i C_A^{h_i} C_A^{t-1} = F_i C_A^{t+h_i-1},$$

and consequently

$$F_i C_A^{t-1+u} \subset F_i C_A^{t+h_i+u-1}$$

for  $u = 0, 1, \dots, d-1$ . This implies

$$S_i = F_i C_A^{t-1} \cup \dots \cup F_i C_A^{t+d-2} \subset F_i C_A^{t+h_i-1} \cup \dots \cup F_i C_A^{t+d+h_i-2} = S_i.$$

Since the summands are quasisdisjoint, we have  $F_i C_A^{t-1} = F_i C_A^{t+h_i-1}$ . Hence,  $F_i C_A^{t-1}$  appears in the sequence (2) more than once. Since  $k_i$  is the least exponent with this property, we have  $t \geq k_i$ , q.e.d.

We shall also need the following

**Lemma 6.** *Let  $A$  be irreducible. If for some  $t \geq 1$  we have*

$$C_A^t \cup C_A^{t+1} \cup \dots \cup C_A^{t+d-1} = S,$$

then  $t \geq k(A)$ .

Proof. In [1] we have proved that there exists a power  $C_A^{\varrho_1}$ ,  $\varrho_1 \leq \varrho(A)$ , such that  $E = \{e_{11}, e_{22}, \dots, e_{nn}\} \subset C_A^{\varrho_1}$ . We have  $C_A^t = C_A^t E \subset C_A^{t+\varrho_1}$ , and consequently  $C_A^{t+1} \subset C_A^{t+\varrho_1+1}, \dots$ . Now

$$S = C_A^t \cup \dots \cup C_A^{t+d-1} \subset C_A^{t+\varrho_1} \cup \dots \cup C_A^{t+d-1+\varrho_1} = S.$$

Since the  $d$  summands on the right are quasisdisjoint, we have  $C_A^t = C_A^{t+\varrho_1}, \dots, C_A^{t+d-1} = C_A^{t+d-1+\varrho_1}$ . In particular, the first of these equalities implies  $k(A) \leq t$ , q.e.d.

We now give a series of theorems concerning  $k = k(A)$  all being consequences of the following Theorem 1.

**Theorem 1.** *If  $A$  is an  $n \times n$  non-negative irreducible matrix, then*

$$k(A) \leq n-1 + \min_{i=1, \dots, n} k_i(A).$$

Proof. Let  $e_{ix}$  be any element  $\in S_i$ . Take  $j \neq i$  and write  $e_{ix} = e_{ij} e_{jx}$ . By Lemma 2 in [2] we have  $e_{ij} \in F_i C_A^t$  with  $t = t(i, j)$  satisfying  $0 \leq t \leq n-2$ . By Lemma 2 we have (for any  $\alpha$ )

$$e_{jx} \in S_j = F_j C_A^{k_j-1} \cup \dots \cup F_j C_A^{k_j+d-2}.$$

Hence,

$$\begin{aligned} S_i &= \{0, e_{i1}, e_{i2}, \dots, e_{in}\} \subset F_i C_A^t \{F_j C_A^{k_j-1} \cup \dots \cup F_j C_A^{k_j+d-2}\} \subset \\ &\subset F_i C_A^{t+k_j} \cup \dots \cup F_i C_A^{t+k_j+d-1}. \end{aligned}$$

Therefore

$$(7) \quad S_i = F_i C_A^{t+k_j} \cup F_i C_A^{t+k_j+1} \cup \dots \cup F_i C_A^{t+k_j+d-1}.$$

By Lemma 5 we have  $k_i \leq t + k_j + 1$ . Since  $j$  is arbitrary, we have  $k_i \leq n - 2 + \min_j k_j + 1$ . By Lemma 1 we finally obtain  $k(A) \leq n - 1 + \min_j k_j$ , q.e.d.

In [2] we have proved: If  $F_i$  contains  $g_i$  non-zero elements, then  $k_i \leq (n - g_i)^2 + (n - g_i) + 1 = 1 + (n - g_i)(n - g_i + 1)$ . This implies:

**Theorem 2.** *If  $A$  is irreducible and the  $i$ -th row of  $A$  contains  $g_i \geq 1$  non-zero elements, then*

$$k(A) \leq n + \min_i (n - g_i)(n - g_i + 1).$$

**Remark.** If  $A$  is primitive and  $n \geq 2$ ,  $A$  contains at least one row with  $g_i \geq 2$ , so that  $k(A) \leq n + (n - 2)^2 + (n - 2) = n^2 - 2n + 2$ . It is known that this result is sharp in the class of matrices with  $C_A = \{0, e_{12}, e_{23}, \dots, e_{n-1,n}, e_{n1}, e_{n2}\}$ .

In [2] (see Theorem 1) we have proved: If  $A$  is irreducible, then  $k_i \leq (n - g_i) h_i + 1$ . This implies by Theorem 1:

**Theorem 3.** *If  $A$  is irreducible,  $g_i$  and  $h_i$  have the meaning introduced above, then*

$$k(A) \leq n + \min_i (n - g_i) h_i.$$

**Corollary 1.** *If  $A$  is irreducible and if it contains a row with  $F_i \subset F_i C_A$ , then  $A$  is primitive and  $k(A) \leq 2n - 1$ .*

This follows from  $h_i = 1, d|h_i$  and  $g_i \geq 1$ .

**Corollary 2.** *If  $A$  is irreducible and if it contains a non-zero element in the main diagonal, then  $A$  is primitive and  $k(A) \leq 2n - 2$ .*

**Proof.** The row containing  $e_{ii}$  contains at least one other non-zero element (since otherwise  $A$  would not be irreducible). Hence  $g_i \geq 2$ . Further  $F_i = e_{ii} C_A \subset F_i C_A$ , hence  $h_i = 1$ . Therefore  $d = 1$ , and our statement follows from Theorem 3.

Theorems 4 through 6 below give information concerning the relations between  $k(A)$  and  $d(A)$ .

It is known: If  $A$  is irreducible with index of imprimitivity  $d \geq 1$ , then there is a permutation matrix  $P$  such that  $P^{-1}AP$  is of the form

$$P^{-1}AP = \begin{pmatrix} 0 & M_1 & 0 & \dots & 0 \\ 0 & 0 & M_2 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & M_{d-1} \\ M_d & 0 & 0 & \dots & 0 \end{pmatrix},$$

where in the diagonal there are square blocks of orders  $n_1, n_2, \dots, n_d$  such that  $n_1 + n_2 + \dots + n_d = n$ . Also,  $A^d$  is completely reducible into primitive matrices

and

$$(8) \quad P^{-1}A^dP = \text{diag}(A_1, A_2, \dots, A_d),$$

where  $A_i = M_i M_{i+1} \dots M_d M_1 \dots M_{i-1}$  is of order  $n_i$ .

Clearly,  $k(P^{-1}AP) = k(A)$  and  $d(P^{-1}AP) = d(A)$ .

We now prove:

**Theorem 4.** *Let  $A$  be irreducible with the index of imprimitivity  $d \geq 1$ . Write  $n = rd + s$  with the integers  $r, s$  satisfying  $r \geq 1, 0 \leq s \leq d - 1$ . Then*

$$k(A) \leq \begin{cases} (n^2/d) - 2n + 3d - s - 1, & \text{if } r \geq 2, \\ s + 1, & \text{if } r = 1. \end{cases}$$

Proof. Consider the matrix  $A_1$ . Since it is primitive, it contains in the case  $n_1 \geq 2$  at least one row with two non-zero elements. Hence

$$(9) \quad \min_{i=1,2,\dots,n_1} k_i(A_1) \leq (n_1 - 2)^2 + (n_1 - 2) + 1 = n_1^2 - 3n_1 + 3.$$

This result holds also in the case  $n_1 = 1$ . This implies that

$$\min_{i=1,2,\dots,n_1} k_i(P^{-1}AP) \leq d(n_1^2 - 3n_1 + 3).$$

Since the same holds for  $A_2, A_3, \dots, A_d$ , we have with respect to Theorem 1,

$$(10) \quad k(A) \leq n - 1 + d \min_{j=1,2,\dots,d} (n_j^2 - 3n_j + 3).$$

Note for further purposes that at least one of the numbers  $n_1, n_2, \dots, n_d$  is  $\leq r$ .<sup>1)</sup> We shall consider two cases:

a) If  $r \geq 2$ , we have

$$\begin{aligned} k(A) &\leq n - 1 + d(r^2 - 3r + 3) = dr + s - 1 + d(r^2 - 3r + 3) = \\ &= d(r^2 - 2r + 3) + s - 1. \end{aligned}$$

To get a result in terms of  $n$  we write

$$\begin{aligned} k(A) &\leq d \left[ \left( \frac{n-s}{d} \right)^2 - 2 \cdot \frac{n-s}{d} + 3 \right] + s - 1 = \\ &= \frac{n^2}{d} - 2n + 3d - 1 - \frac{s}{d} (2n - s - 3d). \end{aligned}$$

<sup>1)</sup> For if there were  $n_j \geq r + 1$  for all  $j$ , we would have  $\sum_j n_j \geq d(r + 1) > dr + s = n$ , a contradiction.

Now

$$\frac{s}{d}(2n - s - 3d) = \frac{s}{d}[2(dr + s) - s - 3d] = s(2r - 3) + \frac{s^2}{d} \geq s + \frac{s^2}{d} \geq s.$$

Therefore

$$k(A) \leq \frac{n^2}{d} - 2n + 3d - s - 1.$$

(This is slightly better than a corresponding result in [3].)<sup>2)</sup>

b) If  $r = 1$ , i.e.  $n = d + s$ ,  $0 \leq s \leq d - 1$ , we have from (10),  $k(A) \leq n - 1 + d$ . This result can be strengthened. Since  $A$  is irreducible, we have  $C_A \cup C_A^2 \cup \dots \cup C_A^n = S$ . Now it follows from the results in [1] that the non-zero idempotents  $\in S$ , i.e. the elements of the set  $E = \{e_{11}, e_{22}, \dots, e_{nn}\}$  can be contained only in the powers  $C_A^d, C_A^{2d}, C_A^{3d}, \dots$ . Hence (in our case) we necessarily have  $E \subset C_A^d$ . This implies  $C_A = C_A E \subset C_A^{d+1}$ ,  $C_A^2 \subset C_A^{d+2}$ , ...,  $C_A^{n-d} \subset C_A^n$ ; hence  $C_A^{n-d+1} \cup C_A^{n-d+2} \cup \dots \cup C_A^n = S$ . By Lemma 6 we have  $k(A) \leq n - d + 1 = s + 1$ . This completes the proof of Theorem 4.

In the "extreme case"  $s = d - 1$  we have  $k(A) \leq n^2/d - 2n + 2d$  for  $r \geq 2$ , and  $k(A) \leq d$  for  $r = 1$ . Since  $d \leq (n - d)^2/d + d = n^2/d - 2n + 2d$ , we have  $k(A) \leq n^2/d - 2n + 2d$  in both cases. We next show by modifying our argument that the same is true in the second "extreme case", namely  $s = 0$ .

**Theorem 5.** *If  $A$  is irreducible and  $d|n$ , then*

$$k(A) \leq \frac{n^2}{d} - 2n + 2d.$$

**Proof.** There are two possibilities. Either all matrices  $A_1, A_2, \dots, A_d$  are of order  $r = n/d$ , or there is at least one matrix, say  $A_1$ , such that  $n_1 \leq r - 1$ .

A) The first case. Since  $A_1, \dots, A_d$  are primitive of order  $r$ , we have  $k(A_j) \leq r^2 - 2r + 2$  for  $j = 1, 2, \dots, d$ , and  $k_i(A) \leq d(r^2 - 2r + 2)$  for  $i = 1, 2, \dots, n$ . By Lemma 1

$$k(A) = \max_i k_i(A) \leq d(r^2 - 2r + 2) = \frac{n^2}{d} - 2n + 2d.$$

B) In the second case we have to distinguish the following possibilities.

<sup>2)</sup> In the meantime the paper [4] appeared in which even a slightly better result than our is proved, namely  $k(A) \leq d(r^2 - 2r + 2) + 2s$ .



a)  $2 \leq n_1 \leq r - 1$ . Then  $r \geq 3$ . We have from (10):

$$\begin{aligned} k(A) &\leq n - 1 + d[(r - 1)^2 - 3(r - 1) + 3] = \frac{n^2}{d} - 4n + 7d - 1 = \\ &= \frac{n^2}{d} - 2n + 2d + (5d - 2n - 1). \end{aligned}$$

But for  $r \geq 3$  we have  $5d - 2n - 1 = 5d - 2rd - 1 = (5 - 2r)d - 1 \leq -d - 1$ , so that  $k(A) \leq n^2/d - 2n + d - 1 < n^2/d - 2n + 2d$ .

$\beta$ )  $1 = n_1 \leq r - 1$ . Here  $r \geq 2$ . By (10) we have  $k(A) \leq n - 1 + d = (r + 1)d - 1$ .

$\alpha$ ) If  $r > 2$ , then  $(r + 1)d - 1 < d(r^2 - 2r + 2)$ , so that our statement holds.

$\beta$ ) If  $r = 2$  (i.e.  $n = 2d$ ) our result  $k(A) \leq 3d - 1$  is not sufficient for the proof of our statement. It can be strengthened in the following way.

Since  $n_1 = 1$ , we have in (8)  $A_1 = I_1\alpha$  ( $I_1 =$  the unit matrix of order 1,  $\alpha$  is a positive number) and  $I_1\alpha = M_1M_2 \dots M_d$ . Therefore  $A_i^2 = (M_i \dots M_d M_1 \dots M_{i-1}) \cdot (M_i \dots M_d M_1 \dots M_{i-1}) = (M_i \dots M_d) I_1 \alpha (M_1 \dots M_{i-1})$ . Now  $M_i M_{i+1} \dots M_d I_1$  is an  $n_i \times 1$  positive matrix, since the existence of a zero row in  $M_i M_{i+1} \dots M_d$  would imply the existence of a zero row in  $A_i^2$ , contrary to the fact that  $A_i$  is primitive. Analogously,  $I_1 M_1 M_2 \dots M_{i-1}$  is a  $1 \times n_i$  positive matrix. Hence  $A_i^2$  is positive. Therefore  $k_i(A) \leq 2d$  for  $i = 1, 2, \dots, d$ , so that  $k(A) \leq 2d = n^2/d - 2n + 2d$ . This completes the proof of Theorem 5.

**Remark 1.** The result of Theorem 5 is sharp in the sense that to any  $n$  and  $d$ ,  $d|n$ , there exists a matrix  $A$  for which  $k(A) = n^2/d - 2n + 2d$  holds. (See [3].)

**Remark 2.** The fact that in the two extreme cases, i.e.  $s = d - 1$  and  $s = 0$ , we have  $k(A) \leq n^2/d - 2n + 2d$  leads to the *conjecture* that the last inequality holds in all cases. However, at this time I am unable to prove or disprove this conjecture.

The next Theorem 6 gives a result which in the special case  $d = 1$  goes back to Frobenius.

**Lemma 7.** Suppose that  $e_{ii} \in F_i C_A^{d-1}$ . Then

$$k_i \begin{cases} = 1, & \text{if } d = n, \\ \leq n - d, & \text{if } d < n. \end{cases}$$

Proof. a) If  $d = n$ , then

$$(11) \quad F_i \cup F_i C_A \cup \dots \cup F_i C_A^{n-1} = S_i$$

implies (with respect to Lemma 5)  $k_i \leq 1$ ; hence  $k_i = 1$ .

b) If  $d < n$ , we have (see [2], Lemma 2)

$$(12) \quad F_i \cup F_i C_A \cup \dots \cup F_i C_A^{n-2} = S_i.$$

Further

$$F_i = e_{ii} C_A \subset F_i C_A^d, \quad F_i C_A \subset F_i C_A^{d+1}, \dots, F_i C_A^{n-d-2} \subset F_i C_A^{n-2}.$$

Hence, (12) can be written in the form

$$F_i C_A^{n-d-1} \cup \dots \cup F_i C_A^{n-2} = S_i.$$

Lemma 5 implies  $k_i \leq n - d$ , q.e.d.

Suppose now that  $C_A^d$  contains at least one element  $\in E = \{e_{11}, e_{22}, \dots, e_{nn}\}$ . Then, if  $d = n$ ,  $C_A \cup \dots \cup C_A^d = S$  implies (by Lemma 6)  $k(A) = 1$ . If  $d < n$  we may use Theorem 1 and Lemma 7 by which

$$k(A) \leq n - 1 + \min_i k_i(A) \leq n - 1 + (n - d) = 2n - d - 1.$$

We have proved:

**Theorem 6.** *Let  $A$  be irreducible with the index of imprimitivity  $d \geq 1$ . Then if  $A^d$  contains at least one non-zero element in the main diagonal, we have  $k(A) \leq 2n - d - 1$ .*

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#### Резюме

### НЕКОТОРЫЕ ОЦЕНКИ В ТЕОРИИ НЕОТРИЦАТЕЛЬНЫХ МАТРИЦ

ШТЕФАН ШВАРЦ (Štefan Schwarz), Братислава

Пусть  $C_A$  — носитель неотрицательной матрицы  $A$  (в смысле работы [1]). Пусть  $C_A^k$  — самая низкая степень, которая встречается в последовательности (1) более чем один раз. Цель статьи — доказательство некоторых теорем, касающихся оценки числа  $k = k(A)$ .