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ON CARATHÉODORY OPERATORS

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1. This note represents a direct continuation of section 2 of [1]; definitions and notation of that paper are used here throughout. Our aim is to prove the following assertion:

**Theorem A.** *Let  $T$  be a Carathéodory operator on  $\mathbf{C}(I; G)$ . Then there exist classical Carathéodory operators  $T_i$ ,  $i \in \mathcal{N}$  such that for each  $\varphi \in \mathbf{C}(I; G)$ ,  $\lim \varrho(T_i \varphi, T \varphi) = 0$ .*

2. First we prove some auxiliary results on polynomials in  $n$  variables. Let  $\pi_n(x) = x_1 x_2 \dots x_n$ . A real polynomial  $P_n$  in  $x_1, \dots, x_n$  is said to be distinguished iff it is of the form

$$(2.1) \quad P_n(x_1, \dots, x_n) = \alpha \pi_n(x) + \sum_{j=1}^n \beta_j \frac{\pi_n(x)}{x_j} + \sum_{\substack{j,k=1 \\ j < k}}^n \gamma_{jk} \frac{\pi_n(x)}{x_j x_k} + \dots + \sum_{j=1}^n \delta_j x_j + \varepsilon$$

The same definition applies also to other sets of variables.

In what follows we prove that distinguished polynomials in  $x_1, \dots, x_n$  have with respect to  $n$ -dimensional cubes properties analogical to those of linear functions on segments.

We shall use the following convention: the symbol  $\tilde{x}_j$  signifies that  $x_j$  does not enter as a variable in our considerations.

**3. Lemma.** *Let  $P_n(x_1, \dots, x_n)$  be a distinguished polynomial in  $x_1, \dots, x_n$ , and let  $\vartheta_{j_1}, \dots, \vartheta_{j_r}$ ,  $1 \leq r \leq n$ , be real numbers. Then  $P_n(x_1, \dots, \vartheta_{j_1}, \dots, \vartheta_{j_r}, \dots, x_n)$  is a distinguished polynomial in  $x_1, \dots, \tilde{x}_{j_1}, \dots, \tilde{x}_{j_r}, \dots, x_n$ .*

*Proof.* It is sufficient to prove this for  $r = 1$ , but then it is obvious.

4. Put  $K = \langle 0, 1 \rangle \times \dots \times \langle 0, 1 \rangle$  ( $n$  times),  $N = 2^n$ , and given  $j \in \{1, \dots, n\}$ ,  $\vartheta \in \mathcal{R}$ , let  $\{x_j = \vartheta\} = \{[x_1, \dots, x_n] \in \mathcal{R}^n; x_j = \vartheta\}$ . Let  $\{j_1, \dots, j_r\}$  be a non-empty

subset of  $\{1, \dots, n\}$ , and let  $\vartheta_k = 0$  or  $1$  for  $k = 1, \dots, r$ ; then the  $(n - r)$ -dimensional sides of  $K$ , denoted  $K\{x_{j_1} = \vartheta_1, \dots, x_{j_r} = \vartheta_r\}$ , are defined by the formula  $K \cap \{x_{j_1} = \vartheta_1\} \cap \dots \cap \{x_{j_r} = \vartheta_r\}$ . If  $r = n$ , we get the vertices of the cube  $K$ ,  $N$  in number; otherwise we have positive-dimensional sides of  $K$ , considered in the sequel as lower- dimensional cubes.

**5. Lemma.** *Let  $v_j, j = 1, \dots, N$ , denote the vertices of  $K$ , and let  $a_j \in \mathcal{R}$  for  $j = 1, \dots, N$ . There exists one and only one distinguished polynomial  $P_n$  in  $x_1, \dots, x_n$  such that*

$$(5.1) \quad P_n(v_j) = a_j, \quad j = 1, \dots, N$$

*Proof.* If we put  $\varepsilon = P_n(0, 0, \dots, 0)$ ,  $\delta_1 = P_n(1, 0, \dots, 0) - \varepsilon$ , etc., then it is from (2.1) clear that all coefficients may be successively determined for (5.1) to be satisfied.

The numbers  $a_j$  corresponding to the vertices  $v_j$  of  $K, j = 1, \dots, N$ , are considered to be fixed in some further sections; thus, the above polynomial will be denoted simply by  $P_n(x_1, \dots, x_n; K)$ .

**6. Lemma.** *Let  $1 \leq r \leq n - 1$ , and let  $Q = K\{x_{j_1} = \vartheta_1, \dots, x_{j_{n-r}} = \vartheta_{n-r}\}$  be an  $r$ -dimensional side of  $K$ . Then*

$$(6.1) \quad P_n(x_1, \dots, x_n; K) \Big|_Q = P_r(x_1, \dots, \tilde{x}_{j_1}, \dots, \tilde{x}_{j_{n-r}}, \dots, x_n; Q)$$

*Proof.* In virtue of Lemma 3, the left-hand side is a distinguished polynomial in  $x_1, \dots, \tilde{x}_{j_1}, \dots, \tilde{x}_{j_{n-r}}, \dots, x_n$  taking on the prescribed values. Now we apply Lemma 5.

**7. Lemma.** *For each  $j = 1, \dots, n$ , we have*

$$(7.1) \quad P_n(x_1, \dots, x_n; K) = (1 - x_j) P_{n-1}(x_1, \dots, \tilde{x}_j, \dots, x_n; K\{x_j = 0\}) + x_j P_{n-1}(x_1, \dots, \tilde{x}_j, \dots, x_n; K\{x_j = 1\})$$

*Proof.* In virtue of Lemma 5 it is sufficient to note that the right-hand side is a distinguished polynomial in  $x_1, \dots, x_n$  taking on the prescribed values.

**8. Lemma.** *For each  $\xi = [\xi_1, \dots, \xi_n] \in K$ , we have*

$$(8.1) \quad \min \{a_1, \dots, a_N\} \leq P_n(\xi_1, \dots, \xi_n; K) \leq \max \{a_1, \dots, a_N\}$$

*Proof.* For  $n = 1$  is it clear. Suppose the assertion is true for  $n - 1$ . As a consequence of (7.1), we have

$$P_n(\xi_1, \dots, \xi_n; K) = (1 - \xi_n) P_{n-1}(\xi_1, \dots, \xi_{n-1}; K\{x_n = 0\}) + \xi_n P_{n-1}(\xi_1, \dots, \xi_{n-1}; K\{x_n = 1\})$$

Using induction assumption, we get from it e.g.

$$\begin{aligned} P_n(\xi_1, \dots, \xi_n; K) &\leq (1 - \xi_n) \max \{a_1, \dots, a_N\} + \xi_n \max \{a_1, \dots, a_N\} = \\ &= \max \{a_1, \dots, a_N\} \end{aligned}$$

**9.** Let  $i \in \mathcal{N}$ . The division  $\mathcal{D}_i^1$  of  $\mathcal{R}$  of the  $i$ -th rank is the set of all points of the form  $k2^{-i}$ , where  $i \in \mathcal{N}$  and  $k$  is an integer. The division  $\mathcal{D}_i^n$  of  $\mathcal{R}^n$  of the  $i$ -th rank is the set  $\mathcal{D}_i^1 \times \dots \times \mathcal{D}_i^1$  ( $n$  times); from now on, we write merely  $\mathcal{D}_i$ , instead of  $\mathcal{D}_i^n$ . It is clear that  $\mathcal{D}_i$  induces a decomposition of  $\mathcal{R}^n$  into non-overlapping cubes, with the edge  $2^{-i}$  each. Let  $\{\mathcal{D}_i\}$  denote the set of all these cubes.

**10.** After the preliminaries, we are going to prove the theorem stated in section 1. Here,  $P(x_1, \dots, x_n; a_1, \dots, a_N)$  is a new notation for the polynomial  $P_n$  of Lemma 5. Also, if  $f$  denotes a point of  $\mathcal{R}^n$  or a vector function, then  $f^{(j)}$ ,  $j = 1, \dots, n$ , denotes the  $j$ -th component of it. The vector-space operations on  $\mathcal{R}^n$  are denoted in the usual manner.

Let  $i \in \mathcal{N}$ . For each  $v \in \mathcal{D}_i \cap G$ , let  $f_v$  be a vector function on  $I$  such that  $[f_v] = v$ . Let us now define a vector function  $f_i$  on  $I \times G$ , generating a classical Carathéodory operator. The construction will be carried out for  $f_i^{(1)}$  only; for other components of  $f_i$  it is analogous.

Let  $K_0 \in \{\mathcal{D}_i\}$  be such that  $K_0 = \{x \in \mathcal{R}^n; 0 \leq x - v \leq 2^{-i}\}$ , for some  $v \in \mathcal{D}_i$ . Suppose that

$$(10.1) \quad K_0 \cap G \neq \emptyset$$

and let  $v_1 = v, v_2, \dots, v_N$  denote the vertices of  $K$ . Let us define the finite functions  $a_j | I, j = 1, \dots, N$ , as follows: if  $v_j \in G$ , put  $a_j(t) = f_{i,v_j}^{(1)}(t)$ ; otherwise put  $a_j(t) = v_j^{(1)}$ . Now, let for  $t \in I, x \in K_0$

$$(10.2) \quad f_i^{(1)}(t, x) = P(2^i(x^{(1)} - v_1^{(1)}), \dots, 2^i(x^{(n)} - v_1^{(n)}); a_1(t), \dots, a_N(t))$$

and similarly for  $f_i^{(j)}, j = 2, \dots, n$ , and all cubes of  $\{\mathcal{D}_i\}$ , satisfying (10.1). It follows from Lemma 6 that using (10.2),  $f_i$  may be well-defined on  $I \times G$ ; we show that it generates a Carathéodory operator, i.e., it satisfies the conditions of Theorem (2,4) in [1].

Let  $x \in G$ . Then  $f_i(\cdot, x)$  is measurable on  $I$ , as a "polynomial" of measurable functions. Let  $t \in I$ . Then  $f_i(t, \cdot)$  is continuous on  $G$ , as a simple consequence of (10.2) and Lemma 6.

We are going to prove that, for each  $\varphi \in \mathbf{C}(I; G)$ ,  $\lim \varrho([f_i \circ \varphi], T\varphi) = 0$ ; thus, to prove the theorem, it suffices to put  $T_i\varphi = [f_i \circ \varphi]$ .

Let  $\varphi \in \mathbf{C}(I; G)$ . Given  $i \in \mathcal{N}$ , there exists  $\delta_i > 0$  such that

$$(10.3) \quad |t - t'| < \delta_i, \quad t, t' \in I \Rightarrow |\varphi(t) - \varphi(t')| < 2^{-i}$$

Let  $\tau_0 = \tau < \tau_1 < \dots < \tau_k = \tau + \alpha$ ,  $\max \{\tau_j - \tau_{j-1}; j = 1, \dots, k\} < \delta_i$ . In virtue of (10.3), for each  $j = 1, \dots, k$ , there exists  $s_i(j) \in \mathcal{D}_i$  such that

$$(10.4) \quad t \in \langle \tau_{j-1}, \tau_j \rangle \Rightarrow |\varphi(t) - s_i(j)| < 2^{-i}$$

Let us define  $s_i | I = \delta_i(1) | \langle \tau_0, \tau_1 \rangle \oplus \dots \oplus \delta_i(k) | \langle \tau_{k-1}, \tau_k \rangle$ . From (10.4) we infer that  $s_i$  converge to  $\varphi$  uniformly on  $I$ . For each  $i \in \mathcal{N}$ , it holds

$$\varrho(\mathbb{T}\varphi, \mathbb{T}_i\varphi) \leq \varrho(\mathbb{T}\varphi, \mathbb{T}s_i) + \varrho(\mathbb{T}s_i, \mathbb{T}_i s_i) + \varrho(\mathbb{T}_i s_i, \mathbb{T}_i\varphi)$$

As a consequence of the definition of  $\mathbb{T}_i$ ,  $\varrho(\mathbb{T}s_i, \mathbb{T}_i s_i) = 0$ . In virtue of Lemma (2,6) of [1],  $\lim \varrho(\mathbb{T}\varphi, \mathbb{T}s_i) = 0$ . Thus, it is sufficient to prove that

$$(10.5) \quad \lim \varrho(\mathbb{T}_i s_i, \mathbb{T}_i\varphi) = 0$$

Let  $v = 3^n$ , and let  $\varepsilon_1, \dots, \varepsilon_v$  be the points of  $\mathcal{R}^n$  such that  $\varepsilon_k^{(j)}$ ,  $k = 1, \dots, v$ ,  $j = 1, \dots, n$ , equals to  $-1, 0$  or  $1$ , independently.

First we prove an estimation, which could easily be made more precise but which suffices for our purpose. In virtue of Lemma 8, we have for  $i \in \mathcal{N}$ ,  $t \in I$  and each  $j = 1, \dots, n$

$$(10.6) \quad |f_i^{(j)}(t, \varphi(t)) - f_i^{(j)}(t, s_i(t))| \leq \sum_{k, l=1}^v |f_i^{(j)}(t, s_i(t) + 2^{-i}\varepsilon_k) - f_i^{(j)}(t, s_i(t) + 2^{-i}\varepsilon_l)|$$

Now, put  $s_i(t) + 2^{-i}\varepsilon_k = s_i(t; k)$ . The step functions  $s_i(\cdot; k) | I$  thus defined evidently converge to  $\varphi$  uniformly on  $I$ , for each  $k = 1, \dots, v$ .

We have from (10.6)

$$\begin{aligned} |f_i(t, \varphi(t)) - f_i(t, s_i(t))| &= \max \{|f_i^{(1)}(\dots) - f_i^{(1)}(\dots), \dots\} \leq \\ &\leq \max \left\{ \sum_{k, l=1}^v |f_i^{(1)}(t, s_i(t; k)) - f_i^{(1)}(t, s_i(t; l))|, \dots \right\} \leq \\ &\leq \sum_{\substack{k, l=1 \\ k < l}}^v \max \{\dots\} = \sum_{\substack{k, l=1 \\ k < l}}^v |f_i(t, s_i(t; k)) - f_i(t, s_i(t; l))| \end{aligned}$$

Hence we get

$$\min(1, |\mathbb{T}_i\varphi - \mathbb{T}_i s_i|) \leq \sum_{\substack{k, l=1 \\ k < l}}^v \min(1, |\mathbb{T}_i s_i(\cdot; k) - \mathbb{T}_i s_i(\cdot; l)|)$$

so that

$$\begin{aligned} \varrho(\mathbb{T}_i\varphi, \mathbb{T}_i s_i) &\leq \sum_{\substack{k, l=1 \\ k < l}}^v \varrho(\mathbb{T}_i s_i(\cdot; k), \mathbb{T}_i s_i(\cdot; l)) = \\ &= \sum_{\substack{k, l=1 \\ k < l}}^v \varrho(\mathbb{T}s_i(\cdot; k), \mathbb{T}s_i(\cdot; l)) \leq \sum_{\substack{k, l=1 \\ k < l}}^v \varrho(\mathbb{T}s_i(\cdot; k), \mathbb{T}\varphi) + \\ &\quad + \sum_{\substack{k, l=1 \\ k < l}}^v \varrho(\mathbb{T}\varphi, \mathbb{T}s_i(\cdot; l)) \end{aligned}$$

Now, (10.5) follows in virtue of Lemma (2,6) of [1]. This proves Theorem A.

**11. Theorem B.** Let  $\mathbb{T}$  be a Carathéodory operator on  $\mathbf{C}(I; G)$  and let  $\{f_i\}$  be the sequence of vector functions on  $I \times G$  defined in the proof of Theorem A. Suppose that there exists  $N \subset I$  such that

$$(11.1) \quad p(N) = 0$$

$$(11.2) \quad t \in I - N \Rightarrow f_i(t, \cdot) \text{ converge uniformly on compact subsets of } G$$

Then  $f|I \times G$ , defined for  $t \in I - N$ ,  $x \in G$  as  $f(t, x) = \lim f_i(t, x)$ , satisfies the conditions (2.4.1), (2.4.2) of Theorem (2.4) of [1], generating thus a classical Carathéodory operator. Further,  $\mathbb{T}\varphi = [f \circ \varphi]$  for each  $\varphi \in \mathbf{C}(I; G)$ .

Proof. From (11.2) we see that  $f(t, \cdot)$  is continuous on  $G$  for each  $t \in I - N$ . Further,  $f(\cdot, x)$  is measurable on  $I$  for each  $x \in G$ , as a limit a.e. of measurable functions. To prove the second assertion of this theorem, it is sufficient to note that  $\mathbb{T}\xi = [f \circ \xi]$  for each  $\xi \in \bigcup_{i=1}^{\infty} D_i \cap G$ ; see Corollary (2.7) in [1].

**12.** The author believes that the assumptions of Theorem B are fulfilled for each Carathéodory operator; to solve Problem C of [1], it would be enough to prove this hypothesis. In section 14, we prove that this is true for classical Carathéodory operators.

**13. Lemma.** Let  $f$  be a finite continuous function on  $\langle 0, 1 \rangle$ . For each  $i \in \mathcal{N}$ , let  $t_0, t_1, \dots, t_{2^i}$ , where  $0 = t_0 < t_1 < \dots < t_{2^i} = 1$ , denote the points of  $\mathcal{D}_i^1 \cap \langle 0, 1 \rangle$ . Let  $l_i(t_k) = f(t_k)$ ,  $k = 0, 1, \dots, 2^i$ , and let  $l_i$  be linear on each  $\langle t_{k-1}, t_k \rangle$ . Then  $l_i$  converge to  $f$  uniformly on  $I$ .

Proof. Given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $t'_1, t'_2 \in \langle 0, 1 \rangle$ ,  $|t'_1 - t'_2| \leq \delta \Rightarrow |f(t'_1) - f(t'_2)| \leq \varepsilon$ . For each  $i \in \mathcal{N}$ , let  $t(i) \in \mathcal{D}_i^1 \cap \langle 0, 1 \rangle$  be such that  $|t - t(i)| \leq 2^{-i}$ . Let  $i_0 \in \mathcal{N}$  be such that  $2^{-i_0} \leq \delta$ . Now using linearity of  $l_i$ , we get for each  $i \geq i_0$  and  $t \in I$  that  $|f(t) - l_i(t)| \leq |f(t) - f(t(i))| + |f(t(i)) - l_i(t(i))| + |l_i(t(i)) - l_i(t)| \leq \varepsilon + 0 + \varepsilon = 2\varepsilon$  which proves the lemma.

**14. Theorem C.** Let  $\mathbb{T}$  be a classical Carathéodory operator on  $\mathbf{C}(I; G)$ , represented by a vector function  $f|I \times G$ . Then, the assumptions of Theorem B are fulfilled.

Proof. This is an easy consequence of the preceding lemma.

**15. Corollary.** Let  $f$  be the vector function of Theorem C. Let  $\lambda^n$  denote the Lebesgue measure on  $R^n$ . Then  $f$  is measurable on  $(I \times G, p \times \lambda^n)$ .

#### Reference

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