Heron Sherwood Collins

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AFFINE IMAGES OF CERTAIN SETS OF MEASURES[1])

HERON S. COLLINS, Baton Rouge

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1. Introduction. In a recent paper [4, Theorems 2.2 and 2.3] the author made use of Choquet's simplex theorem to characterize all $1-1$ affine bicontinuous images of both the set of probability measures and the unit ball of real measures over a compact space (see section 3 for statements of these theorems). It is our purpose here to obtain a corresponding result for the unit ball of *complex* measures (section 4). Some applications of these results are then given in sections 3 and 4, including characterizations of the Banach spaces $C_r(X)$ (= all real continuous functions on X), $C(X)$ (= all complex continuous functions on X), $M_r(X)$ (= all real valued bounded Radon measures on X), and $M(X)$ (= all bounded complex Radon measures on X). Here X is a compact Hausdorff space, and the supremum (resp. total variation) norm is used in the function (resp. measure) spaces. These same two sections of the paper include characterizations of those closed linear subspaces A of $C_r(X)$ (resp. $C(X)$) which are subalgebras containing constants (resp. subalgebras containing constants and closed under conjugation). The conditions given are in terms of the adjoint space A^* and extreme points of its (closed) unit ball; e.g., one requires that $1 \in A$ and if $P = \{L \in A^* : 1 = L(1) = \|L\|\}$ and $T = P_e$ = set of extreme points of P, then $T \subset (\text{ball } A^*)_e$ and Choquet boundary A (see section 2) is all of X. In the final section 5, the preceding work is applied to a study of convolution semigroups and algebras of complex measures over a compact semigroup, as well as to a description of those commutative Banach *-algebras with identity which are B^* algebras.

2. Preliminaries. Throughout this work the symbol F will denote either the field R of real numbers or the field C of complex numbers. If K_1 and K_2 are compact convex subsets of linear topological spaces over F, a function f on K_1 to K_2 is F *affine* provided $x, y \in K_1$, $a \in F$, and $ax + (1-a)\,y \in K_1$ together imply $f(ax + (1-a)\,y) = af(x) + (1-a)f(y)$. K_1 and K_2 are F *affinely equivalent* if there is a $1-1$ bicontinuous F affine map of K_1 onto K_2. If K is a compact convex set and $z \in K$, the symbol $L(K, F)$ (resp. $L_z(K, F)$) denotes the linear space over F of all F

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affine continuous functions on K to F (resp. which also vanish at z). When the adjoint space of either of these is mentioned, it is the adjoint relative to the supremum norm topology. As indicated in section 1, K_e stands for the set of extreme points of K. If $z \in K$, K is *F circled* (resp. *F absorbing*) at z if for every $x \in K$ and $a \in F$ such that $|a| \leqq 1$ we have $a(x - z) \in K - z = \{r - z : r \in K\}$ (resp. if for every vector x

there exists $a \in F$, $a > 0$ such that $ax + z \in K$). If X is compact Hausdorff and $A \subset C_r(X)$ or $C(X)$ is a uniformly closed linear subspace, we follow WRIGHT [10, p. 189] in saying $x \sim y$ $(x, y \in X)$ provided $f(x) = f(y)$ for every $f \in A$. Y denotes the compact Hausdorff space of equivalence classes thus obtained (using the identification topology), and π is the natural map of X onto Y. As in [3, p. 310], the *Choquet boundary of A* consists of those $x \in X$ having the property that every non-negative $\mu \in M(X)$ representing x ($\int f \, d\mu = f(x)$, for all $f \in A$) is carried on $\pi^{-1}(\pi(x))$.

The following (doubtless known) lemma is stated and its proof sketched here since it is needed at several places in the sequel.

Lemma 2.1. *Let E be a linear topological space over F and $K \subset E$ be compact, convex, F circled, and absorbing at 0. Then each $l \in L_0(K, F)$ has a (unique) F linear extension to all of E.*

Proof. If $x \in E$ there is $a > 0$ such that $ax \in K$. Define $\bar{l}(x) = (1/a)\, l(ax)$. If $b > 0$ and $bx \in K$, we can assume $0 < a/b < 1$, so $ax = (a/b)(bx) + (1 - a/b) \,.\, 0 \in K$. But then $l(ax) = (a/b)\, l(bx)$, since l is affine and vanishes at 0, whence $(1/a)\, l(ax) = (1/b)\, l(bx)$ and $\bar{l}$ is well defined. It is obvious $\bar{l}$ extends l. To prove $\bar{l}$ is F homogeneous, let $x \in E$, $a \in F$, and $t > 0$ be such that tax and $t|a|\, x \in K$. Then (can let a be non-zero) $t\ a\ x = (ta/|ta|)\, |ta|\, x + (1 - ta/|ta|) \,.\, 0 \in K$, hence $l(tax) = (ta/|ta|)\, l(|ta|\, x)$, and $\bar{l}(ax) = (1/t)\, l(tax) = (a/|ta|)\, l(|ta|\, x) = a\, \bar{l}(x)$. The argument that $\bar{l}$ is additive is similar: for x, $y \in E$, choose $t > 0$ such that $t(x + y)$, tx, and $ty \in K$. Then $l\frac{1}{2}(tx + ty) = \frac{1}{2}l(tx) + \frac{1}{2}l(ty)$, so $(2/t)\, l[\frac{1}{2}t(x + y)] = (1/t)\, l(tx) + (1/t)\, l(ty)$. This says $\bar{l}(x + y) = \bar{l}(x) + \bar{l}(y)$, concluding the proof.

3. Results on linear spaces over R. Several theorems from [4] are needed in this section and these are stated without proof as Theorems 3.1 and 3.2[2]). The remainder of this part of the paper is devoted to some applications of these theorems, including among them what are hoped to be new descriptions of the Banach spaces $C_r(X)$ and $M_r(X)$. We also characterize those closed linear subspaces A of $C_r(X)$ which are subalgebras containing 1.

It is emphasized that throughout this section and paper X will be a compact Hausdorff space, $P(X)$ will denote the set of probability measures on X (with the weak-$*$ topology), and $B_r(X)$ will be the set of bounded real Radon measures on X

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[2]) The author wishes to thank R. R. PHELPS for pointing out recently that Theorem 3.1 was obtained independently (and somewhat earlier) by H. BAUER [2, Theorem 1].

of variation norm $\leqq 1$ (also with the weak-$*$ topology). The definition of simplex can be found in [8] or in [4].

Theorem 3.1. *For K a compact convex set, the following conditions are mutually equivalent*: (1) *K is R equivalent (see section 2) to some $P(X)$*, (2) (a) *K_e is compact and* (b) *K is a simplex*, (3) (a) *$L(K, R)$ separates points of K*, (b) *K_e is compact*, and (c) *each $f \in C_r(K_e)$ is extendable to $\bar{f} \in L(K, R)$*.

Theorem 3.2. *If K is compact and convex the following are equivalent*: (1) *K is R equivalent to some $B_r(X)$*, (2) (a) *there exists $z \in K$ and compact $T \subset K$ such that $K_e = T \cup (2z - T)$*, (b) *the closed convex hull of T is a simplex*, (c) *$L_z(K, R)$ separates points of K and contains a function which is one on T*, (3) (a) *part* (a) *of* (2) *holds and* (b)*$L_z(K, R)$ separates points of K and each $f \in C_r(T)$ is extendable to $\bar{f} \in L_z(K, R)$*.

In what follows the phrase *linear isometry* will refer to a function between a pair of Banach spaces which is an isometry onto and preserves linear space operations. Recall that $L_0(K, R)$ is the set of real continuous R affine functions on K to R which vanish at 0.

Theorem 3.3. *If B is a Banach space the following conditions are equivalent*: (1) *B is linearly isometric with some $C_r(X)$*, (2) (a) *if K is the unit ball of B^*, with the weak-$*$ topology, there is a* (*weak-$*$*) *compact set $T \subset K$ such that $K_e = T \cup (-T)$*, (b) *the* (*weak-$*$*) *closed convex hull K_1 of T is a simplex*, (c) *$L_0(K, R)$ contains a function l_0 which is one on T*, (3) (a) *part* (a) *of* (2) *holds and* (b) *each $f \in C_r(T)$ extends to $\bar{f} \in L_0(K, R)$*.

Proof. (1) $\to$ (2). Suppose X is compact Hausdorff and $m : B \to C_r(X)$ is a linear isometry onto. The adjoint mapping $m^* : M_r(X) \to B^*$ is a linear isometry and is weak-$*$ bicontinuous onto. We can now let T be the image under m^* of the set of point measures in $M_r(X)$ and l_0 be the restriction to K of the functional F_0 preceded by the inverse of m^*, where $F_0(\mu) = \mu(X)$, for every $\mu \in M_r(X)$. Condition 2(a) follows from [7, Lemmas 3.1 and 3.2] and 2 (b) follows from Theorem 3.1 above.

(2) $\to$ (3). We use Theorem 3.2 above and let $z = 0$: if $f \in C_r(T)$, Theorem 3.2. yields an extension $\bar{f}$ in $L_0(K, R)$.

(3) $\to$ (1). By (3) each f in $C_r(T)$ has an extension in $L_0(K, R)$ and this in turn has a linear extension $\bar{f}$ to all of B^* (Lemma 2.1). Since $\bar{f}$ is linear on B^* and weak-$*$ continuous on the unit ball of B^*, a well known result [1] implies $\bar{f}$ is weak-$*$ continuous on all of B^*. The extension $\bar{f}$ is unique, so there exists a unique $x \in B$ such that $f(t) = t(x)$, for all $t \in T$. The mapping $x \to x\hat{}$ from B to $C_r(T)$ defined by $x\hat{}(t) = t(x)$ is thus $1-1$ and onto, and it is clearly linear. Further, $\|x\| = \sup\{|g(x)| : g \in B^*, \|g\| \leqq 1\} = \sup\{|g(x)| : g \in K_e\} = \sup\{|t(x)| : t \in T\} = \|x\hat{}\|$, so $x \to x\hat{}$ is an isometry. The proof is complete.

The next theorem is, we feel, of sufficient interest to explicitly state, but the proof is omitted (because of its similarly to that of the preceding). The author knows of no reasonable conditions equivalent to the requirement that a Banach space be the adjoint of some Banach space.

Theorem 3.4. *If B is a (real) Banach space the following conditions are equivalent:* (1) *B is linearly isometric with some $M_r(X)$,* (2) (a) *B is linearly isometric with E^* for some Banach space E,* (b) *if K is the unit ball of B endowed with the weak-$*$ topology, there is a weak-$*$ compact set $T \subset K$ such that $K_e = T \cup (-T)$,* (c) *the weak-$*$ closed convex hull of T is a simplex, and* (d) *there is $l_0 \in L_0(K, R)$ equal one on T.*

There has been considerable interest in uniformly closed subalgebras of $C_r(X)$ which contain constants (cf. [10]), one reason for this being the one to one correspondence between them and closed upper semicontinuous equivalence relations on X. The following result describes such subalgebras in terms of A^* and the Choquet boundary of A.

Theorem 3.5. *Let X be compact Hausdorff, with A a closed linear subspace of $C_r(X)$. The following are then equivalent:* (1) *$1 \in A$ and A is a subalgebra,* (2) (a) *if K is the unit ball of A^* with the weak-$*$ topology and $T = \{L_x : x \in X\}$, where $L_x(f) = f(x)$ for all $f \in A$, then $T \subset K_e$,* (b) *the weak-$*$ closed convex hull K_1 of T is a simplex, and* (c) *there is $l_0 \in L_0(K, R)$ equal one on T,* (3) (a) *$1 \in A$,* (b) *if $P = \{m \in A^* : 1 = m(1) = \|m\|\}$, then $P_e \subset K_e$ and the Choquet boundary of A is all of X,* (c) *the weak-$*$ closed convex hull of P_e is a simplex.*

Proof. (1) $\to$ (2). Let Y be the decomposition space defined by $A : x$ is equivalent to y means $L_x = L_y (x, y \in X)$. Denote by $p : C_r(Y) \to A$ the map defined by the equation $p(h) = h(\pi)$, for all $h \in C_r(Y)$, where $\pi : X \to Y$ is the canonical map. Then [10, p. 189] both p and p^* are linear isometries onto and $p^* : A^* \to M_r(Y)$ is weak-$*$ bicontinuous. Thus K_e is the inverse under p^* of $M \cup (-M)$, where M is the set of point measure on Y, and it is easily verified that $p^*(T) = M$. Since each point of M ia an extreme point of the unit ball of $M_r(Y)$ [7, Lemmas 3.1 and 3.2], it follows that $T \cup (-T) = K_e$. Also, K_1 is a simplex, being the inverse under p^* of $P(Y)$, and the one function on Y yields (in the obvious way) an $l_0 \in L_0(K, R)$ which is one on T.

(2) $\to$ (3). Since K is convex and R-circled it is obvious that 2(a) implies $T \cup (-T) \subset K_e$. On the other hand if $k \in K_e$ and $r : M_r(X) \to A^*$ is the restriction mapping, then $[r^{-1}(k) \cap \text{ball } M_r(X)]_e$ is non-void by the Krein-Milman and Hahn-Banach theorems. Any such extreme point μ is an extreme point of ball $M_r(X)$, so there exists $x \in X$ such that either μ or its minus is the point measure at x. Thus $k = L_x$ or $-L_x$, and we have proved $K_e = T \cup (-T)$. By Theorem 3.2 and Lemma 2.1 each function in $C_r(T)$ has a (unique) extension which is linear on A^* and weak-$*$ continuous on K. Alaoglu's theorem [1] says this extension is weak-$*$ continuous

on A^* and so comes from a point of A. In particular there is $f_0 \in A$ for which $1 = l_0(L_x) = L_x(f_0) = f_0(x)$, for all $x \in X$, so $1 \in A$. Now $L_x \in P$ for each $x \in X$, so 2(a) implies $L_x \in P_e$ and $T \subset P_e$. The latter is $\{L_x : x \in \text{Choquet boundary } A\}$ [3, Lemma 4.3], so equality holds and (3) is proved.

(3) → (1). Using [3, Lemma 4.3] again and 3(b), we see that $T = P_e \subset K_e$. It is then clear (by the same argument employed in (2) → (3)) that A is linearly isometric with $C_r(T)$, and (1) holds.

4. Results on linear spaces over C. In this section we obtain complex analogues of the theorems of section 3. This problem turns out to be reasonably straightforward, although the statements are not always as elegant as could be desired. This is especially true in the basic Theorem 4.1, which is the theorem corresponding to 3.2. Considerable simplification is possible when the set K is absorbing at z, and this is the case in most of the applications. It is emphasized here that from now on we are concerned exclusively with complex linear spaces. X will again be a compact Hausdorff space, while $C(X)$ and $M(X)$ are as defined in section 1.

Theorem 4.1. *If K is a compact convex subset of a complex linear topological space, the following conditions are equivalent*: (1) *K is C affinely equivalent to the unit ball of some $M(X)$, endowed with the weak-$*$ topology*, (2) (a) *there is compact $T \subset K$ and $z \in K$ such that $K_e = \{at + (1 - a) z : a \in C, |a| = 1, t \in T\}$*, (b) *$L_z(K, C)$ separates points of K and contains l_0 which is one on T*, (c) *the closed convex hull K_1 of T is a simplex*, (d) *if K_2 is the closed convex hull of $T \cup (2z - T)$, $i = \sqrt{(-1)}$, $x, y \in K_2$, and $z = x + iy - iz$, then $x = y = z$ (i.e., z is "uniquely representable" in $K_2 + iK_2 - iz$)*, (e) *if $a_j, b_j \in R$, $x_j, y_j \in K_2$, $\sum a_j = 1 = \sum b_j$, and $\sum a_j x_j + i(\sum b_j y_j) - iz \in K$, then $\sum a_j x_j$ and $\sum b_j y_j \in K_2$ (sums are all finite and over the same set of indices)*, (3) (a) *part* (a) *of* (2) *holds and* (b) *each $f \in C(T)$ has an extension to $\bar{f} \in L_z(K, C)$*.

Proof. (1) → (2). Suppose $B(X)$ is the unit ball of $M(X)$ with the weak-$*$ topology, and $\Phi : B(X) \to K$ is $1-1$ bicontinuous and C affine onto. Let $z = \Phi(0)$. It is known [5, p. 441] that the set of extreme points of $B(X)$ is precisely the set of unimodular complex multiples of point measures on X. Thus if T is the image under Φ of this set of point measures we have $K_e = \{at + (1 - a) z : a \in C, |a| = 1, t \in T\}$, and T is compact. The remainder of (2) is easily verified.

(2) → (3). First note several obvious facts which follow from (2): (A) if $x \in K$ and $|a| \leqq 1$ with $a \in C$, then $ax + (1 - a) z \in K$, (B) $L_z(K_2, R)$ separates points of K_2, (C) l_0 restricted to K_2 is in $L_z(K_2, R)$ and is one on T, and (D) each point of K can be expressed in the form $x + iy - iz$, where $x, y \in K_2$. It then follows from (D) and 2(d) that such representation is unique. Let now $f \in C(T)$ and we wish to extend f to $\bar{f} \in L_z(K, C)$. It may be assumed that $f \in C_r(T)$; Theorem 3.2 together with (B) and (C) imply that f extends to $f_1 \in L_z(K_2, R)$. If $p \in K$, by (E) there exist unique

$x, y \in K_2$ such that $p = x + iy - iz$. Write $\bar{f}(p) = f_1(x) + if_1(y)$. Standard arguments (making strong use of the compactness of K_2) show $\bar{f}$ is a continuous extension of f to all of K. To prove $\bar{f}$ is C affine we need 2(e): suppose $x, y, p, q \in K_2$, $a \in C$, and $w = a[x + iy - iz] + (1 - a)[p + iq - iz] \in K$. If $a = r + is$, with $r, s \in R$, then $w = [rx - sy + p - rp + sq] + i[sx + ry + q - sp - rq] - iz$, and these first two coefficients are in K_2 by 2(e). Since f_1 is R affine it then follows that $\bar{f}(w) = a\bar{f}(x + iy - iz) + (1 - a)\bar{f}(p + iq - iz)$, and $\bar{f} \in L_z(K, C)$.

(3) $\to$ (1). If $x \in K$ and $l \in L_z(K, C)$, let $x'(l) = l(x)$. Then $x \to x'$ is a one to one C affine bicontinuous function between K and $K' \subset L_z(K, C)^*$, where $L_z(K, C)^*$ is the adjoint of $L_z(K, C)$ given the sup norm topology and K' is given the weak-$*$ topology. The resultant map from $B(T)$ onto K' (if $\mu \in B(T)$ and $l \in L_z(K, C)$, the value of the image of μ at l is $\int_T l \, d\mu$) is obviously C affine continuous and onto. The argument that it is $1-1$ follows easily from (3): if $\mu, \nu \in B(T)$ have the same image in K', then $f \in C(T)$ implies $\int_T f \, d\mu = \int_T \bar{f} \, d\mu = \int_T \bar{f} \, d\nu = \int_T f \, d\nu$, where $\bar{f} \in L_z(K, C)$ is the extension of f. The Riesz representation theorem now shows $\mu = \nu$, concluding the proof of the theorem.

Remark. *If in the preceding theorem K is absorbing at z, then condition* 2(e) *may be deleted.* To see this, let $f \in C_r(T)$ and f_1 be its extension in $L_z(K_2, R)$. As in the proof of (2) $\to$ (3) above, the formula $\bar{f}(p) = f_1(x) + if_1(y)$ gives a continuous extension of f on all of K to C, and it is easily proved that $\bar{f}$ is R affine. If $h(x) = x + z$ and $g = \bar{f}(h)$, then g on $K - z$ to C is continuous, R affine, and vanishes at 0. Since $K - z$ is absorbing it follows as in Lemma 2.1 that g has an extension $\bar{g}$ to the whole space which is complex valued, R linear, and continuous on $K - z$. Now let $\alpha(x) = \bar{g}(x) - i\bar{g}(ix)$. Then $\frac{1}{2}\alpha$ is a complex linear functional which is continuous on $K - z$ and $x \to \frac{1}{2}\alpha(x - z)$ is complex affine. But this last function restricted to K is $\bar{f}$, concluding the proof.

Theorem 4.2. *For B a complex Banach space the following conditions are equivalent*: (1) *B is linearly isometric with some $C(X)$*, (2) (a) *if K is the unit ball of B^* with the weak-$*$ topology, there is* (*weak-$*$*) *closed $T \subset K$ such that $K_e = \{at : a \in C, |a| = 1, t \in T\}$*, (b) *the weak-$*$ closed convex hull K_1 of T is a simplex*, (c) 0 *is uniquely representable in $K_2 + iK_2$, where K_2 is the weak-$*$ closed convex hull of $T \cup (-T)$*, (d) *there is $l_0 \in L_0(K, C)$ which is one on T*, (3) (a) *part* (a) *of* (2) *holds and* (b) *each $f \in C(T)$ has an extension $\bar{f} \in L_0(K, C)$*.

Proof. The implication (1) $\to$ (2) is simply the complex analogue of (1) $\to$ (2) of Theorem 3.3. In proving (2) $\to$ (3), we use Theorem 4.1 and the remark following it, while (3) $\to$ (1) is proved as in the preceding theorem. This completes the proof.

As in the previous section, where Theorem 3.4 was not proved, we state now its complex analogue without any verification.

Theorem 4.3. *For a complex Banach space B the following are equivalent*: (1) *B is linearly isometric with some $M(X)$*, (2) (a) *B is linearly isometric with E^* for some complex Banach space E*, (b) *if K is the unit ball of B with the weak-$*$ topology, there is weak-$*$ compact $T \subset K$ such that $K_e = \{at : a \in C, |a| = l, t \in T\}$*, (c) *the weak-$*$ closed convex hull of T is a simplex*, (d) *there is $l_0 \in L_0(K, C)$ which is one on T, and* (e) *0 is uniquely representable in $K_2 + iK_2$, where K_2 is the weak-$*$ closed convex hull of $T \cup (-T)$*.

The final result of this section corresponds in this (the complex) case to Theorem 3.5. Its proof is so similar that we content ourselves with merely stating the theorem.

Theorem 4.4. *Suppose A is a closed linear subspace of $C(X)$. The following conditions are then equivalent*: (1) *A is a subalgebra containing constants and closed under conjugation*, (2) (a) *if K is the unit ball of A^* with the weak-$*$ topology and $T = \{L_x : x \in X\}$, where $L_x(f) = f(x)$, for all $f \in A$, then $T \subset K_e$*, (b) *the weak-$*$ closed convex hull of T is a simplex*, (c) *there is $l_0 \in L_0(K, C)$ which is one on T, and* (d) *if K_2 is the weak-$*$ closed convex hull of $T \cup (-T)$, then 0 is uniquely expressible in $K_2 + iK_2$*, (3) (a) *$1 \in A$*, (b) *if $P = \{m \in A^* : 1 = m(1) = \|m\|\}$, then $P_e \subset K_e$ and the Choquet boundary of A is all of X*, (c) *the weak-$*$ closed convex hull of P_e is a simplex, and* (d) *0 is uniquely representable in $S + iS$, where S is the weak-$*$ closed convex hull of $P_e \cup (-P_e)$*.

5. Further applications in the complex case. If K is a compact convex subset of a complex linear topological space which is also a topological semigroup (relative to some multiplication on K), then K is a C *affine semigroup* provided multiplication is also separately C affine. The most interesting example of such a semigroup is the convolution semigroup $S^{\approx}$ of bounded complex Radon measures (over the compact semigroup S) of variation norm $\leqq 1$. The first theorem of this section presents a characterization of such semigroups (see [6] for information about $S^{\approx}$). For information about Banach algebras with involution, positive functionals, indecomposable positive functionals; etc., the reader is referred to [9]. The B^* algebras mentioned in Theorem 5.2. (we prefer this to Naimark's completely regular) are, of course, those algebras satisfying $\|xx^*\| = \|x\|^2$. As is customary the symbol $\mu * \nu$ is used for the *convolution* product of the measures μ and ν.

Theorem 5.1. *If K is a compact C affine semigroup, the following conditions are equivalent*: (1) *K is the $1-1$ C affine bicontinuous and isomorphic image of $S^{\approx}$ for some compact semigroup S*, (2) *the conditions* (2) *of Theorem* 4.1 *hold, T is a semigroup, and z is a zero of K*.

Proof. (1) $\rightarrow$ (2) is obvious, while (2) $\rightarrow$ (1) follows as in the proof of the preceding Theorem 4.1 and Lemma 3.1 of [4]. However, the proof given there of Lemma 3.1 was unnecessarily complicated, so we include the following simple argument (assuming (2)) that the resultant map $\mu \rightarrow x_\mu$ of the complex ball semigroup $T^{\approx}$ onto K is a homomorphism. By definition, if $\mu \in T^{\approx}$, x_μ is the unique point of K satisfying $l(x_\mu) = \int_T l\, d\mu$, for all $l \in L_z(K, C)$. If then μ and ν are in $T^{\approx}$, $l(x_{\mu*\nu}) = \int\int l(xy)\,.$ $.\, d\mu(x)\, d\nu(y)$. Since multiplication in K is separately C affine and z is a zero for K, it is easy to verify that $y \to \int l(xy)\, d\mu(x)$ is in $L_z(K, C)$, whence $l(x_{\mu*\nu}) = \int l(xx_\nu)\, d\mu(x) =$ $= \int l^{x_\nu}\, d\mu$, where $l^t(s) = l(st)$. Since $l^{x_\nu} \in L_z(K, C)$, it follows that $l(x_{\mu*\nu}) = l^{x_\nu}(x_\mu) =$ $= l(x_\mu \,.\, x_\nu)$. The fact that $L_z(K, C)$ separates points now implies $x_{x*\nu} = x_\mu \,.\, x_\nu$, and the proof is completed.

Theorem 5.2. *Let E be a commutative (complex) Banach algebra with identity and involution. These are equivalent:* (1) *E is a B^* algebra,* (2) (a) *if K is the unit ball of E^* with the weak-$*$ topology and T is the set of indecomposable normalized positive functionals on E, then T is weak-$*$ closed,* (b) *the set of normalized positive functionals on E is a simplex, and* (c) *K_e is $\{at : a \in C, |a| = 1, t \in T\}$.*

Proof. (1) $\to$ (2). By the well known theorem on B^* algebras (cf. [9, p. 230]), E is isometric and $*$ isomorphic with some $C(X)$. Then T is the image in E^* (under the adjoint of this mapping) of the set of point measures in $M(X)$, and the image of $P(X)$ is the set of normalized positive functionals on E. The conditions of (2) are then all clear.

(2) $\to$ (1). We are assuming here that $\|x^*\| = \|x\|$, for all $x \in E$. Denote by K_1 the set of normalized positive functionals on E and by K_2 the closed (weak-$*$) convex hull of $T \cup (-T)$. Then [9, p. 266] K_1 is the weak-$*$ closed convex hull of T. Since this is so, it is easy to prove that K_2 is the union of all line segments $[f, g]$, with $f \in K_1$ and $g \in -K_1$. But then each functional in K_2 is *real valued* (i.e., real valued on Hermitian elements of E). Since each $x \in E$ can be written $x = x_1 + ix_2$, with x_1, x_2 Hermitian, it follows that if the zero functional on E is written as $f + ig$, with $f, g \in K_2$, then $f = 0 = g$. But now all the conditions of (2) of Theorem 4.2 hold, so the mapping $x \to x\hat{}$ of E to $C(T)$ defined by $x\hat{}(t) = t(x)$ is a linear isomorphism and isometry onto. However, [9, *p.* 266] and the remark on page 272 of [9] together imply that each $t \in T$ is a *symmetric* homomorphism of $E(t(x^*) = \overline{t(x)}$, for each $x \in E)$. But then $(xy)\hat{}\,(t) = t(xy) = t(x)\,t(y) = x\hat{}(t)\, y\hat{}(t)$ and $(x^*)\hat{}\,(t) = t(x^*) =$ $= \overline{t(x)} = \overline{x\hat{}(t)}$, for all $t \in T$, whence E is isometric and $*$ algebra isomorphic with $C(T)$. Invoking [9, p. 230] again we obtain (1), and this concludes the proof.

Our final theorem gives a description of those (complex) Banach spaces admitting a multiplication making them into a Banach algebra which is isometric and algebra isomorphic with the convolution algebra $M(S)$ of measures over a compact Hausdorff topological semigroup S.

Theorem 5.3. *The following conditions for a (complex) Banach space Bare equivalent:* (1) *there is a multiplication on B making B into a Banach algebra isometric and isomorphic with some convolution algebra $M(S)$,* (2) (a) *B is linearly isometric with E^* for some Banach space E,* (b) *the unit ball K of B (with the weak-$*$ topology borrowed from E^*) contains a compact set T such that $K_e =$*

$= \{at : a \in C, |a| = 1, t \in T\}$, (c) *the (weak-*) closed convex hull of* T *is a simplex,* (d) *there is* $l_0 \in L_0(K, C)$ *which is one on* T, (e) 0 *is uniquely expressible in* $K_2 + iK_2$ *where* K_2 *is the weak-* closed convex hull of* $T \cup (-T)$, *and* (f) T *with the weak-* topology is a topological semigroup (relative to some multiplication on* T).

Proof. (1) $\to$ (2). If $\alpha : B \to M(S)$ is an isometry and algebra isomorphism (S a compact semigroup), let $E = C(S)$. Then previous work implies all of (2) save 2(f). Here T is the inverse under α of the set of point measures in $M(S)$. Since this set of point measures is a topological semigroup with respect to the weak-$*$ topology and α is a weak-$*$ bicontinuous algebra isomorphism, it is obvious then that 2(f) holds.

(2) $\to$ (1). As in Theorems 4.2 and 4.3, $x \to x\hat{}$ from E to $C(T)$ is a linear isometry onto. If Φ is the adjoint of this map, Φ is a linear isometry and a weak-$*$ bicontinuous function from $M(T)$ onto E^*. Since by 2(f) $M(T)$ is a Banach algebra (relative to convolution, of course) and there is a linear isometry β of E^* onto B, it is clear that $\Phi(\beta)$ can be used to impose a multiplication on B so that (1) holds.

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Author's address: Louisiana State University, Baton Rouge, Louisiana, U.S.A.