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DEFORMATION OF SURFACES IN HOMOGENEOUS 3-SPACES

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The local existence questions of manifolds with prescribed properties are treated in many papers. In what follows, I devote myself to the study of deformations of the first order of surfaces in general homogeneous 3-spaces; I restrict my attention to cases in which the fundamental system of equations is immediately involutive.

Be given a homogeneous space G/H and a manifold M, $\dim M < \dim G/H$. Consider an embedding $\pi : M \to G/H$ and its lift $\Pi : M \to G$. To Π, let us associate the 1-form $\omega : T(M) \to \mathfrak{g}$ defined by

$$(1) \qquad \omega(X_m) = (\mathrm{d}L_{\Pi(m)^{-1}})(\mathrm{d}\Pi)_m X \; ; \quad X \in T_m(M) \; ;$$

$L_a : G \to G$ being the left translation $L_a g = ag$; the form ω satisfies the integrability condition

$$(2) \qquad \mathrm{d}\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] \, .$$

Let us write

$$(3) \qquad K(m) = \mathfrak{h} \oplus \omega(T_m(M)) \quad \text{for} \quad m \in M \; ;$$

clearly, $\dim K(m) = \dim \mathfrak{h} + \dim M$. Further, write

$$(4) \qquad \mathfrak{k}^1 = \{v \in \mathfrak{h} \mid [v, K] \subset K\} \, ,$$

$$(5) \qquad \mathfrak{k}^2 = \{v \in \mathfrak{h} \mid [v, K] \subset \mathfrak{h}\} \; ;$$

the spaces $\mathfrak{k}^1$ and $\mathfrak{k}^2$ are Lie algebras. The lift $\Pi : M \to G$ is said to be a *tangent lift* if there is a fixed space K such that

$$(6) \qquad K(m) = K \quad \text{for each} \quad m \in M \, .$$

In [1], I proved the following assertion: *Let $m_0 \in M$ be a fixed point and*

$$(7) \qquad \dim \mathfrak{h}/\mathfrak{k}^1(m_0) = \dim K/\mathfrak{h} \, . \, \dim \mathfrak{g}/K \, ,$$

then there is a neighborhood $O \subset M$ *of* m_0 *and a lift* $\Pi' : M \to G$ *of* $\pi : M \to G/H$ *such that* $K'(m) = K(m_0)$ *for each point* $m \in O$.

Denote by $\mathrm{Gr}^{\dim M}(\mathfrak{h})$ the Grassmann manifold of all spaces K such that $\mathfrak{h} \subset K \subset \mathfrak{g}$, $\dim K = \dim \mathfrak{h} + \dim M$. To the given embedding $\pi : M \to G/H$, let us construct the mapping $p : M \to \mathrm{Gr}^{\dim M}(\mathfrak{h})$ as follows: choose an arbitrary lift $\Pi : M \to G$ and set

$$p(m) = \mathrm{ad}\,(\Pi(m))\,\mathfrak{h} \quad \text{for} \quad m \in M\,; \tag{8}$$

obviously, the mapping p does not depend on Π.

Be given mappings $\pi : M \to G/H$, $\pi' : M' \to G/H$; $\dim M = \dim M'$. Further, let $T: M \to M'$ be a diffeomorphism. T is called a *deformation of order* k if, for each $m_0 \in M$, there is an element $g_0 \in G$ such that

$$j^k_{m_0}(p) = j^k_{m_0}\{\mathrm{ad}\,(g_0)\,(p' \circ T)\}\,, \tag{9}$$

$j^k_m(q)$ being the k-jet of q at m. I have proved in [1]: *Suppose* $N(\mathfrak{h}) = \mathfrak{h}$, $N(\mathfrak{h})$ *being the normalizator of* $\mathfrak{h}$. *Then* T *is the first order deformation if and only if there are lifts* Π, $\Pi' \circ T : M \to G$ *of the embeddings* π, $\pi' \circ T : M \to G/H$ *such that the form*

$$\tau = \omega' - \omega \tag{10}$$

is $\mathfrak{h}$*-valued; the forms* ω, ω' *are associated to* Π *and* $\Pi' \circ T$ *resp. according to* (1).

Let us read "K satisfies the conditions $\mathscr{P}$; π is arbitrary and (π', T) depends on x functions of y variables" as follows: "Be given manifolds M and M', $\dim M = \dim M'$. Let us write $K_{\mathscr{P}} = \{K \in \mathrm{Gr}^{\dim M}(\mathfrak{h}) \mid K \text{ satisfies } \mathscr{P}\}$, and suppose that $\dim K_{\mathscr{P}} = \dim \mathrm{Gr}^{\dim M}(\mathfrak{h})$. Choose a point $m_0 \in M$ and an embedding $\pi : M \to G/H$ subject to the only condition $K(m_0) \in K_{\mathscr{P}}$. Then there is a neighborhood O, $m_0 \in O \subset M$, a diffeomorphism $T: O \to M'$ and an embedding $\pi' : T(O) \to G/H$ such that T is a first order deformation without being an equivalence. T and π' depend – in the usual sense – on x functions of y variables." It is easy to see how to understand to similar statements.

Theorem. *Be given a homogeneous space* G/H, $\dim G/H = 3$. *By a surface* $\pi : M \to G/H$ *we mean an embedding of a two-dimensional manifold. Let* $N(\mathfrak{h}) = \mathfrak{h}$, $N(\mathfrak{h})$ *being the normalizator of* $\mathfrak{h}$. *Using the just introduced interpretation, we have*:

A$_1$. $\dim \mathfrak{k}^1 = \dim \mathfrak{k}^2 = \dim \mathfrak{h} - 2$, $[\mathfrak{h}, K] = \mathfrak{g}$; (π, π', T) *depends on* 4 *functions of* 1 *variable.*

A$_2$. $\dim \mathfrak{k}^1 = \dim \mathfrak{k}^2 = \dim \mathfrak{h} - 2$, $\dim [\mathfrak{h}, K] = \dim \mathfrak{g} - 1$; π *is arbitrary and* (π', T) *depends on* 2 *functions of* 1 *variable.*

A$_3$. $\dim \mathfrak{k}^1 = \dim \mathfrak{k}^2 = \dim \mathfrak{h} - 2$, $\dim [\mathfrak{h}, K] = \dim \mathfrak{g} - 2$; π *and* π' *are arbitrary and* T *depends on* 2 *constants.*

B$_1$. $\dim \mathfrak{k}^1 = \dim \mathfrak{h} - 2$, $\dim \mathfrak{k}^2 = \dim \mathfrak{h} - 3$ *and there is a* $k \in K$ *such that* $[\mathfrak{h}, k] = \mathfrak{g}$; π *is arbitrary and* (π', T) *depends on* 3 *functions of* 1 *variable.*

B$_2$. $\dim \mathfrak{k}^1 = \dim \mathfrak{h} - 2$, $\dim \mathfrak{k}^2 = \dim \mathfrak{h} - 3$, $\dim [\mathfrak{h}, K] = \dim \mathfrak{g} - 1$; π *and* π' *are arbitrary and* T *depends on* 1 *function of* 1 *variable.*

C. $\dim \mathfrak{k}^1 = \dim \mathfrak{h} - 2$, $\dim \mathfrak{k}^2 = \dim \mathfrak{h} - 4$, *and there is a* $k \in K$ *such that* $[\mathfrak{k}^1, k] \oplus \mathfrak{h} = K$; π *and* π' *are arbitrary and* T *depends on* 2 *functions of* 1 *variable.*

D. $\dim \mathfrak{k}^1 = \dim \mathfrak{h} - 2$, $\dim \mathfrak{k}^2 = \dim \mathfrak{h} - 5$; π *and* π' *are arbitrary and* T *depends on* 1 *function of* 2 *variables.*

E. $\dim \mathfrak{k}^1 = \dim \mathfrak{h} - 2$, $\dim \mathfrak{k}^2 = \dim \mathfrak{h} - 6$; π, π' *and* T *are arbitrary.*

Proof. Let us write $\dim \mathfrak{g} = r + 3$, and let us choose a basis $e_1, \ldots, e_{r+3}$ of $\mathfrak{g}$ such that $e_1, \ldots, e_r$ is a basis of $\mathfrak{h}$. Writing

$$(11) \qquad [e_\alpha, e_\beta] = \sum_{\gamma=1}^{r+3} c_{\alpha\beta}^{\gamma} e_\gamma \quad \text{for} \quad \alpha, \beta = 1, \ldots, r + 3\,,$$

we get

$$(12) \qquad c_{ij}^{r+1} = c_{ij}^{r+2} = c_{ij}^{r+3} = 0 \quad \text{for} \quad i, j = 1, \ldots, r\,.$$

Be given a surface $\pi : M \to G/H$, its lift $\Pi : M \to G$ and the associated form

$$(13) \qquad \omega = \sum_{\alpha=1}^{r+3} \omega^\alpha e_\alpha\,.$$

The integrability condition (2) yields

$$(14) \qquad d\omega^\alpha = -\tfrac{1}{2} \sum_{\beta,\gamma=1}^{r+3} c_{\beta\gamma}^{\alpha} \omega^\beta \wedge \omega^\gamma \quad \text{for} \quad \alpha = 1, \ldots, r + 3\,.$$

Let $m_0 \in M$ be a fixed point, and let us investigate π in its neighborhood. Write $K = \omega(T_{m_0}(M))$; obviously, $\dim K = r + 2$. In what follows, we shall be interested only in "general" surfaces satisfying $\dim \mathfrak{k}^1(m) = r - 2$, $K(m) = \omega(T_m(M))$. Each surface of this type has a tangent lift such that $K(m) = K$; let Π be tangent. Let us choose the basis of $\mathfrak{g}$ in such a way that $e_1, \ldots, r_{r+2}$ is the basis of K. The surface π is given by

$$(15) \qquad \omega^{r+3} = 0\,,$$

the exterior differentiation yields

$$(16) \qquad \psi_1 \wedge \omega^{r+1} + \psi_2 \wedge \omega^{r+2} + c_{r+1,r+2}^{r+3} \omega^{r+1} \wedge \omega^{r+2} = 0$$

where

$$(17) \qquad \psi_a = \sum_{i=1}^{r} c_{i,r+a}^{r+3} \omega^i\,; \quad a = 1, 2\,.$$

From the Cartan's lemma, we get

$$\psi_1 = A\omega^{r+1} + \left(B + \tfrac{1}{2}c^{r+3}_{r+1,r+2}\right)\omega^{r+2}, \tag{18}$$
$$\psi_2 = \left(B - \tfrac{1}{2}c^{r+3}_{r+1,r+2}\right)\omega^{r+1} + C\omega^{r+2}.$$

If

$$v = \sum_{i=1}^{r} v^i e_i \in h, \quad k = \sum_{i=1}^{r} k^i e_i + \sum_{a=1}^{2} k^{r+a} e_{r+a} \in K, \tag{19}$$

we get

$$[v, k] = \sum_{i,k=1}^{r} \left(\sum_{j=1}^{r} c^k_{ij} k^j + \sum_{a=1}^{2} c^k_{i,r+a} k^{r+a}\right) v^i e_k + \tag{20}$$
$$+ \sum_{A=1}^{3} \sum_{a=1}^{2} \sum_{i=1}^{r} c^{r+A}_{i,r+a} k^{r+a} v^i e_{r+A}.$$

Thus the Lie algebra $\mathfrak{k}^1$ is given by the vectors (19_1) satisfying

$$\sum_{i=1}^{r} c^{r+3}_{i,r+a} v^i = 0; \quad a = 1, 2; \tag{21}$$

similarly, $\mathfrak{k}^2$ is given by the equations (21) and

$$\sum_{i=1}^{r} c^{r+a}_{i,r+b} v^i = 0; \quad a, b = 1, 2. \tag{22}$$

According to the assumption, we have $\dim \mathfrak{k}^1 = r - 2$, the equations (21) are linearly independent, and we have

$$\psi_1 \wedge \psi_2 \neq 0. \tag{23}$$

Of course,

$$\omega^{r+1} \wedge \omega^{r+2} \neq 0. \tag{24}$$

Now, be given another surface $\pi' : M' \to G/H$ and a first order deformation $T : M \to M'$. Using a suitable lift of the surface π', the form (10) is $\mathfrak{h}$-valued, and

$$\tau^{r+3} = 0 \tag{25}$$

$$\tau^{r+1} = \tau^{r+2} = 0. \tag{26}$$

From (14), and analogous equations for ω', we get

$$d\tau^\alpha = -\sum_{\beta,\gamma=1}^{r+3} c^\alpha_{\beta\gamma}\left(\tfrac{1}{2}\tau^\beta - \omega^\beta\right) \wedge \tau^\gamma; \quad \alpha = 1, \ldots, r+3. \tag{27}$$

The exterior differentiation of (25) and (26) yields

$$\varphi_1 \wedge \omega^{r+1} + \varphi_2 \wedge \omega^{r+2} = 0, \tag{28}$$

$$\varphi_a = \sum_{i=1}^{r} c^{r+3}_{i,r+a} \tau^i; \quad a = 1, 2; \tag{29}$$

and

(30) $$\varphi_{a1} \wedge \omega^{r+1} + \varphi_{a2} \wedge \omega^{r+2} = 0 \; ; \quad a = 1, 2 \; ;$$

(31) $$\varphi_{ab} = \sum_{i=1}^{r} c_{i,r+b}^{r+a} \tau^i \; ; \quad a, b = 1, 2 \, .$$

The assumption $\dim \mathfrak{k}^1 = r - 2$ is equivalent to

(32) $$\varphi_1 \wedge \varphi_2 \neq 0 \, .$$

From the Cartan's lemma, we get

(33) $$\varphi_1 = A_1 \omega^{r+1} + A_2 \omega^{r+2} \, , \qquad \varphi_2 = A_2 \omega^{r+1} + A_3 \omega^{r+2} \; ;$$

(34) $$\varphi_{a1} = A_{a1} \omega^{r+1} + A_{a2} \omega^{r+2} \, , \quad \varphi_{a2} = A_{a2} \omega^{r+1} + A_{a3} \omega^{r+2} \; ; \quad a = 1, 2 \, .$$

A. Let $\dim \mathfrak{k}^2 = r - 2$. The equations (22) are linear combinations of the equations (21), and there are numbers α_b^{ac} such that

(35) $$c_{i,r+b}^{r+a} = \sum_{c=1}^{2} \alpha_b^{ac} c_{i,r+c}^{r+3} \; ; \quad a, b = 1, 2 \; ; \quad i = 1, \ldots, r \, .$$

The expression (20) reduces to

(36) $$[v, k] = \sum_{i=1}^{r} (\cdot) \, e_i + \sum_{a,b=1}^{2} k^{r+a} w_{r+b} f_a^b \, ,$$

where

(37) $$w_{r+a} = \sum_{i=1}^{r} c_{i,r+a}^{r+3} v^i \; ; \quad a = 1, 2 \; ;$$

(38) $$f_a^b = \sum_{c=1}^{2} \alpha_a^{cb} e_{r+c} + \delta_a^b e_{r+3} \; ; \quad a, b = 1, 2 \, .$$

Let us write

(39) $$R_1 = \operatorname{rang} \begin{Vmatrix} \alpha_1^{11} & \alpha_1^{12} & \alpha_2^{11} & \alpha_2^{12} \\ \alpha_1^{21} & \alpha_1^{22} & \alpha_2^{21} & \alpha_2^{22} \\ 1 & 0 & 0 & 1 \end{Vmatrix} \; ;$$

obviously,

(40) $$\dim [\mathfrak{h}, K] = r + R_1 \, .$$

The equations (30) reduce to

(41) $$\sum_{a=1}^{2} \varphi_a \wedge \Big(\sum_{b=1}^{2} \alpha_b^{ca} \omega^{r+b} \Big) = 0 \; ; \quad c = 1, 2 \; ;$$

and we get

$$(42) \qquad \alpha_2^{a1} A_1 + (\alpha_2^{a2} - \alpha_1^{a1}) A_2 - \alpha_1^{a2} A_3 = 0 ; \quad a = 1, 2 ;$$

from (33).

Let $R_1 = 3$. The equations (28), (41) are linearly independent as well as the equations (42). The system (16), (28), (41) being in involution, we have proved A_1.

Let $R_1 = 2$. Then one of the equations (41) is the linear combination of the second one and the equation (28). Suppose, e.g., that (28) and (41_1) are linearly independent; substituting from (33) into (41_1), we get (42_1). For a given surface π, the couple (π', T) is given by the involutive system (28) + (41_1), and A_2 has been proved.

Let $R_1 = 1$. The equations (41) are the multiples of (28). π and π' being given, T is given by the completely integrable equations (26), and we have proved A_3.

B. Let $\dim \mathfrak{k}^2 = r - 3$. Three of the equations (22) are linear combinations of the remaining one and of (21); we may suppose the existence of numbers $\alpha_1, \ldots, \gamma_3$ such that

(43)

$$c_{i,r+2}^{r+1} = \alpha_1 c_{i,r+1}^{r+3} + \alpha_2 c_{i,r+2}^{r+3} + \alpha_3 c_{i,r+1}^{r+1} , \quad c_{i,r+1}^{r+2} = \beta_1 c_{i,r+1}^{r+3} + \beta_2 c_{i,r+2}^{r+3} + \beta_3 c_{i,r+1}^{r+1} ,$$
$$c_{i,r+2}^{r+2} = \gamma_1 c_{i,r+1}^{r+3} + \gamma_2 c_{i,r+2}^{r+3} + \gamma_3 c_{i,r+1}^{r+1} \quad \text{for} \quad i = 1, \ldots, r .$$

The expression (20) reduces to

$$(44) \qquad [v, k] = \sum_{i=1}^{r} (.)\, e_i + k^{r+1} w_{r+1}(\beta_1 e_{r+2} + e_{r+3}) + k^{r+1} w_{r+2} \beta_2 e_{r+2} +$$
$$+ k^{r+1} w_{r+3}(e_{r+1} + \beta_3 e_{r+2}) + k^{r+2} w_{r+1}(\alpha_1 e_{r+1} + \gamma_1 e_{r+2}) +$$
$$+ k^{r+2} w_{r+2}(\alpha_2 e_{r+1} + \gamma_2 e_{r+2} + e_{r+3}) + k^{r+2} w_{r+3}(\alpha_3 r_{r+1} + \gamma_3 e_{r+2}) .$$

Let us write

$$(45) \qquad R_2 = \operatorname{rang} \begin{Vmatrix} 0 & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{Vmatrix} ;$$

obviously, $\dim [\mathfrak{h}, K] = r + R_2$. The equations (30) reduce to

$$(46) \qquad \alpha_1 \varphi_1 \wedge \omega^{r+2} + \alpha_2 \varphi_2 \wedge \omega^{r+2} + \varphi_3 \wedge (\omega^{r+1} + \alpha_3 \omega^{r+2}) = 0 ,$$
$$\varphi_1 \wedge (\beta_1 \omega^{r+1} + \gamma_1 \omega^{r+2}) + \varphi_2 \wedge (\beta_2 \omega^{r+1} + \gamma_2 \omega^{r+2}) +$$
$$+ \varphi_3 \wedge (\beta_3 \omega^{r+1} + \gamma_3 \omega^{r+2}) = 0 .$$

The polar matrix of the system (28) + (46) is

$$(47) \qquad \begin{Vmatrix} \omega^{r+1} & \omega^{r+2} & 0 \\ \alpha_1 \omega^{r+2} & \alpha_2 \omega^{r+2} & \omega^{r+1} + \alpha_3 \omega^{r+2} \\ \beta_1 \omega^{r+1} + \gamma_1 \omega^{r+2} & \beta_2 \omega^{r+1} + \gamma_2 \omega^{r+2} & \beta_3 \omega^{r+1} + \gamma_3 \omega^{r+2} \end{Vmatrix} .$$

Let us choose a vector $k \in K$ (19_2). (44) yields that the space $[\mathfrak{h}, k]$ is spanned by the vectors $e_1, \ldots, e_r$ and

$$(48) \qquad \begin{aligned} g_1 &= \alpha_1 k^{r+2} e_{r+1} + (\beta_1 k^{r+1} + \gamma_1 k^{r+2})\, e_{r+2} + k^{r+1} e_{r+3}\,, \\ g_2 &= \alpha_2 k^{r+2} e_{r+1} + (\beta_2 k^{r+1} + \gamma_2 k^{r+2})\, e_{r+2} + k^{r+2} e_{r+3}\,, \\ g_3 &= (k^{r+1} + \alpha_3 k^{r+2})\, e_{r+1} + (\beta_3 k^{r+1} + \gamma_3 k^{r+2})\, e_{r+2}\,. \end{aligned}$$

If $\dim [\mathfrak{h}, k] = r + 3$ for some vector $k \in K$, the determinant of (47) is not equal to zero. Of course, $\dim [\mathfrak{h}, K] = r + 3$, and the equations (28) + (46) are linearly independent. This proves B_1.

Let $R_2 = 2$. The equations (28) and (46_1) are linearly independent, and (46_2) is the linear combination of them. The surfaces π and π' being given, the deformation T is given by the system (26) and the quadratic equation (46_1). B_2 has been proved.

C. Let $\dim \mathfrak{k}^2 = r - 4$; the Lie algebra $\mathfrak{k}^2$ be given by the equations (21) and

$$(49) \qquad \varrho_a \equiv \sum_{i=1}^{r} \varrho_{ai} v^i = 0\,; \quad a = 1, 2\,.$$

Hence, there are numbers α_b^{ac}, β_b^{ac} such that

$$(50) \qquad c_{i,r+b}^{r+a} = \sum_{c=1}^{2} (\alpha_b^{ac} c_{i,r+c}^{r+3} + \beta_b^{ac} \varrho_{ci})\,.$$

Writing

$$(51) \qquad \chi_a = \sum_{i=1}^{r} \varrho_{ai} \tau^i\,; \quad a = 1, 2\,;$$

the forms $\varphi_1, \varphi_2, \chi_1, \chi_2$ are linearly independent, and the equations (30) reduce to

$$(52) \qquad \sum_{a,b=1}^{2} (\beta_a^{cb} \chi_b \wedge \omega^{r+a} + \alpha_b^{ca} \varphi_a \wedge \omega^{r+b}) = 0\,; \quad c = 1, 2\,.$$

Consider the vectors (19) such that $v \in \mathfrak{k}^1$. We have

$$(53) \qquad [v, k] = \sum_{i=1}^{r} (.)\, e_i + \sum_{a,b,c=1}^{2} \beta_b^{ca} \varrho_a k^{r+b} e_{r+c}\,.$$

If $[\mathfrak{k}^1, k] \oplus \mathfrak{h} = K$ for some vector k, the polar matrix of the system (52) is regular.

D. and E. are evident.

References

[1] *A. Švec:* Cartan's method of specialization of frames. Czech. Math. J., *16* (*91*), 1966, 552—599.

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