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DEFORMATION OF SURFACES IN HOMOGENEOUS 3-SPACES

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The local existence questions of manifolds with prescribed properties are treated in many papers. In what follows, I devote myself to the study of deformations of the first order of surfaces in general homogeneous 3-spaces; I restrict my attention to cases in which the fundamental system of equations is immediately involutive.

Be given a homogeneous space \( G/H \) and a manifold \( M, \dim M < \dim G/H \). Consider an embedding \( \pi : M \to G/H \) and its lift \( \Pi : M \to G \). To \( \Pi \), let us associate the 1-form \( \omega : T(M) \to \mathfrak{g} \) defined by

\[
\omega(X_m) = (dL_{\pi(m)} \cdot \Pi)(d\Pi)_m X; \quad X \in T_m(M);
\]

\( L_a : G \to G \) being the left translation \( L_a g = ag \); the form \( \omega \) satisfies the integrability condition

\[
d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)].
\]

Let us write

\[
K(m) = \mathfrak{h} \oplus \omega(T_m(M)) \quad \text{for} \quad m \in M;
\]

clearly, \( \dim K(m) = \dim \mathfrak{h} + \dim M \). Further, write

\[
t^1 = \{ v \in \mathfrak{h} \mid [v, K] \subset K \},
\]

\[
t^2 = \{ v \in \mathfrak{h} \mid [v, K] \subset \mathfrak{h} \};
\]

the spaces \( t^1 \) and \( t^2 \) are Lie algebras. The lift \( \Pi : M \to G \) is said to be a tangent lift if there is a fixed space \( K \) such that

\[
K(m) = K \quad \text{for each} \quad m \in M.
\]

In [1], I proved the following assertion: Let \( m_0 \in M \) be a fixed point and

\[
\dim \mathfrak{h}/t^1(m_0) = \dim K/\mathfrak{h} \cdot \dim \mathfrak{g}/K,
\]

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then there is a neighborhood $O \subset M$ of $m_0$ and a lift $\Pi' : M \to G$ of $\pi : M \to G/H$ such that $K'(m) = K(m_0)$ for each point $m \in O$.

Denote by $Gr^{\dim M}(h)$ the Grassmann manifold of all spaces $K$ such that $h \subset K \subset g$, $\dim K = \dim h + \dim M$. To the given embedding $\pi : M \to G/H$, let us construct the mapping $p : M \to Gr^{\dim M}(h)$ as follows: choose an arbitrary lift $\Pi : M \to G$ and set

$$p(m) = \text{ad}(\Pi(m))h$$

for $m \in M$; obviously, the mapping $p$ does not depend on $\Pi$.

Be given mappings $\pi : M \to G/H$, $\pi' : M' \to G/H$; $\dim M = \dim M'$. Further, let $T : M \to M'$ be a diffeomorphism. $T$ is called a deformation of order $k$ if, for each $m_0 \in M$, there is an element $g_0 \in G$ such that

$$j_{m_0}^k(\tau) = j_{m_0}^k(\text{ad}(g_0)(\pi' \circ T)),$$

$j_{m_0}^k(\tau)$ being the $k$-jet of $\tau$ at $m$. I have proved in [1]: Suppose $N(h) = h$, $N(h)$ being the normalizer of $h$. Then $T$ is the first order deformation if and only if there are lifts $\Pi, \Pi' \circ T : M \to G$ of the embeddings $\pi, \pi' \circ T : M \to G/H$ such that the form

$$\tau = \omega' - \omega$$

is $h$-valued; the forms $\omega, \omega'$ are associated to $\Pi$ and $\Pi' \circ T$ resp. according to (1).

Let us read “$K$ satisfies the conditions $\mathcal{P}$; $\pi$ is arbitrary and $(\pi', T)$ depends on $x$ functions of $y$ variables” as follows: “Be given manifolds $M$ and $M'$, $\dim M = \dim M'$. Let us write $K_{\mathcal{P}} = \{K \in Gr^{\dim M}(h) | K$ satisfies $\mathcal{P}\}$, and suppose that $\dim K_{\mathcal{P}} = \dim Gr^{\dim M}(h)$. Choose a point $m_0 \in M$ and an embedding $\pi : M \to G/H$ subject to the only condition $K(m_0) \in K_{\mathcal{P}}$. Then there is a neighborhood $O$, $m_0 \in O \subset M$, a diffeomorphism $T : O \to M'$ and an embedding $\pi' : T(O) \to G/H$ such that $T$ is a first order deformation without being an equivalence. $T$ and $\pi'$ depend — in the usual sense — on $x$ functions of $y$ variables.” It is easy to see how to understand to similar statements.

**Theorem.** Be given a homogeneous space $G/H$, $\dim G/H = 3$. By a surface $\pi : M \to G/H$ we mean an embedding of a two-dimensional manifold. Let $N(h) = h$, $N(h)$ being the normalizer of $h$. Using the just introduced interpretation, we have:

A$_1$. $\dim t^1 = \dim t^2 = \dim h - 2$, $[h, K] = g$; $(\pi, \pi', T)$ depends on 4 functions of 1 variable.

A$_2$. $\dim t^1 = \dim t^2 = \dim h - 2$, $\dim [h, K] = \dim g - 1$; $\pi$ is arbitrary and $(\pi', T)$ depends on 2 functions of 1 variable.

A$_3$. $\dim t^1 = \dim t^2 = \dim h - 2$, $\dim [h, K] = \dim g - 2$; $\pi$ and $\pi'$ are arbitrary and $T$ depends on 2 constants.
B₁. \( \dim t^1 = \dim h - 2, \ \dim t^2 = \dim h - 3 \) and there is a \( k \in K \) such that \( [h, k] = g; \ pi \ is \ arbitrary \ and \ (\pi', T) \ depends \ on \ 3 \ functions \ of \ 1 \ variable. \)

B₂. \( \dim t^1 = \dim h - 2, \ \dim t^2 = \dim h - 3, \ \dim [h, K] = \dim g - 1; \ pi \ and \ \pi' \ are \ arbitrary \ and \ T \ depends \ on \ 1 \ function \ of \ 1 \ variable. \)

C. \( \dim t^1 = \dim h - 2, \ \dim t^2 = \dim h - 4, \ and \ there \ is \ a \ k \in K \) such that \( [t^1, k] \oplus h = K; \ pi \ and \ \pi' \ are \ arbitrary \ and \ T \ depends \ on \ 2 \ functions \ of \ 1 \ variable. \)

D. \( \dim t^1 = \dim h - 2, \ \dim t^2 = \dim h - 5; \ pi \ and \ \pi' \ are \ arbitrary \ and \ T \ depends \ on \ 1 \ function \ of \ 2 \ variables. \)

E. \( \dim t^1 = \dim h - 2, \ \dim t^2 = \dim h - 6; \ pi, \ \pi' \ and \ T \ are \ arbitrary. \)

Proof. Let us write \( \dim g = r + 3 \), and let us choose a basis \( e_1, \ldots, e_{r+3} \) of \( g \) such that \( e_1, \ldots, e_r \) is a basis of \( h \). Writing

\[
[e_x, e_p] = \sum_{\gamma=1}^{r+3} c_{x\gamma}^p e_\gamma \quad \text{for} \quad x, \beta = 1, \ldots, r + 3,
\]

we get

\[
c_{ij}^{i+1} = c_{ij}^{i+2} = c_{ij}^{i+3} = 0 \quad \text{for} \quad i, j = 1, \ldots, r.
\]

Be given a surface \( \pi: M \to G/H \), its lift \( \Pi: M \to G \) and the associated form

\[
\omega = \sum_{x=1}^{r+3} \omega^x e_x.
\]

The integrability condition (2) yields

\[
d\omega^x = -\frac{1}{2} \sum_{\beta, \gamma=1}^{r+3} c_{x\beta}^\gamma \omega^\beta \wedge \omega^\gamma \quad \text{for} \quad x = 1, \ldots, r + 3.
\]

Let \( m_0 \in M \) be a fixed point, and let us investigate \( \pi \) in its neighborhood. Write \( K = \omega(T_{m_0}(M)); \) obviously, \( \dim K = r + 2 \). In what follows, we shall be interested only in “general” surfaces satisfying \( \dim t^1(m) = r - 2, K(m) = \omega(T_{m}(M)). \) Each surface of this type has a tangent lift such that \( K(m) = K; \) let \( \Pi \) be tangent. Let us choose the basis of \( g \) in such a way that \( e_1, \ldots, e_{r+2} \) is the basis of \( h \). The surface \( \pi \) is given by

\[
\omega^{r+3} = 0,
\]

the exterior differentiation yields

\[
\psi_1 \wedge \omega^{r+1} + \psi_2 \wedge \omega^{r+2} + c_{r+1,r+2}^{r+3} \omega^{r+1} \wedge \omega^{r+2} = 0
\]

where

\[
\psi_a = \sum_{i=1}^{r} c_{i,r+3}^{i+3} \omega^i; \quad a = 1, 2.
\]
From the Cartan’s lemma, we get

\begin{align*}
\psi_1 &= A\omega^{r+1} + (B + \frac{1}{2}c_{r+1,r+2}^{+3})\omega^{r+2}, \\
\psi_2 &= (B - \frac{1}{2}c_{r+1,r+2}^{+3})\omega^{r+1} + C\omega^{r+2}.
\end{align*}

If

\begin{equation}
\tau = \sum_{i=1}^{r} v^i e_i \in h, \quad k = \sum_{i=1}^{r} k^i e_i + \sum_{a=1}^{2} k^{r+a} e_{r+a} \in K,
\end{equation}

we get

\begin{equation}
[v, k] = \sum_{i,k=1}^{r} \left( \sum_{j=1}^{r} c_{ij}^k + \sum_{a=1}^{2} c_{i,r+a}^{k} k^{r+a} \right) v^i e_k + \\
+ \sum_{A=1}^{3} \sum_{a=1}^{2} \sum_{i=1}^{r} c_{i,r+a}^{r+a} k^{r+a} v^i e_{r+a}.
\end{equation}

Thus the Lie algebra \( \mathfrak{t}^1 \) is given by the vectors \( (19) \) satisfying

\begin{equation}
\sum_{i=1}^{r} c_{i,r+a}^{r+3} v^i = 0; \quad a = 1, 2;
\end{equation}

similarly, \( \mathfrak{t}^2 \) is given by the equations \( (21) \) and

\begin{equation}
\sum_{i=1}^{r} c_{i,r+b}^{r+a} v^i = 0; \quad a, b = 1, 2.
\end{equation}

According to the assumption, we have \( \dim \mathfrak{t}^1 = r - 2 \), the equations \( (21) \) are linearly independent, and we have

\begin{equation}
\psi_1 \wedge \psi_2 \neq 0.
\end{equation}

Of course,

\begin{equation}
\omega^{r+1} \wedge \omega^{r+2} \neq 0.
\end{equation}

Now, be given another surface \( \pi': \mathcal{M}' \to G/H \) and a first order deformation \( T: \mathcal{M} \to \mathcal{M}' \). Using a suitable lift of the surface \( \pi' \), the form \( (10) \) is \( \mathfrak{h} \)-valued, and

\begin{align*}
\tau^{r+3} &= 0, \\
\tau^{r+1} &= \tau^{r+2} = 0.
\end{align*}

From \( (14) \), and analogous equations for \( \omega' \), we get

\begin{equation}
d\tau^\alpha = -\sum_{\beta,\gamma=1}^{r+3} c_{\beta,\gamma}^\alpha (\frac{1}{2}\tau^\beta - \omega^\beta) \wedge \tau^\gamma; \quad \alpha = 1, \ldots, r + 3.
\end{equation}

The exterior differentiation of \( (25) \) and \( (26) \) yields

\begin{align*}
\varphi_1 \wedge \omega^{r+1} + \varphi_2 \wedge \omega^{r+2} &= 0, \\
\varphi_a &= \sum_{i=1}^{r} c_{i,r+a}^{r+3} v^i; \quad a = 1, 2;
\end{align*}

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\end{equation}

\begin{equation}
\varphi_1 \wedge \omega^{r+1} + \varphi_2 \wedge \omega^{r+2} = 0.
\end{equation}

\begin{equation}
\varphi_a = \sum_{i=1}^{r} c_{i,r+a}^{r+3} v^i; \quad a = 1, 2;
\end{equation}
and

(30) \[ \varphi_{a1} \wedge \omega^{r+1} + \varphi_{a2} \wedge \omega^{r+2} = 0; \ a = 1, 2; \]

(31) \[ \varphi_{ab} = \sum_{i=1}^{r} e_{i,r+b}^i; \ a, b = 1, 2. \]

The assumption \( \dim \Gamma^1 = r - 2 \) is equivalent to

(32) \[ \varphi_1 \wedge \varphi_2 = 0. \]

From the Cartan's lemma, we get

(33) \[ \varphi_1 = A_1 \omega^{r+1} + A_2 \omega^{r+2}, \quad \varphi_2 = A_2 \omega^{r+1} + A_3 \omega^{r+2}; \]

(34) \[ \varphi_{a1} = A_{a1} \omega^{r+1} + A_{a2} \omega^{r+2}, \quad \varphi_{a2} = A_{a2} \omega^{r+1} + A_{a3} \omega^{r+2}; \ a = 1, 2. \]

A. Let \( \dim \Gamma^2 = r - 2 \). The equations (22) are linear combinations of the equations (21), and there are numbers \( a_{bc}^a \) such that

(35) \[ c_{i,r+b}^r = \sum_{c=1}^{2} a_{bc}^a e_{i,r+c}; \ a, b = 1, 2; \ i = 1, \ldots, r. \]

The expression (20) reduces to

(36) \[ [v, k] = \sum_{i=1}^{r} e_i + \sum_{a,b=1}^{2} k^{r+a} w_{r+b} f_a, \]

where

(37) \[ w_{r+a} = \sum_{i=1}^{r} e_{i,r+a}^i; \ a = 1, 2; \]

(38) \[ f_a^b = \sum_{c=1}^{2} a_{ac}^b e_{r+c} + \delta_{ac}^b e_{r+3}; \ a, b = 1, 2. \]

Let us write

(39) \[ R_1 = \text{rang} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{11} & \alpha_{21} \\ \alpha_{21} & \alpha_{22} & \alpha_{21} & \alpha_{22} \\ 1 & 0 & 0 & 1 \end{pmatrix}; \]

obviously,

(40) \[ \dim [\mathfrak{g}, \mathfrak{k}] = r + R_1. \]

The equations (30) reduce to

(41) \[ \sum_{a=1}^{2} \varphi_a \wedge \left( \sum_{b=1}^{2} a_{bc}^d \omega^{r+b} \right) = 0; \ c = 1, 2. \]

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and we get
\[
(42) \quad \alpha_2^{n_1} A_1 + (\alpha_2^{n_2} - \alpha_1^{n_1}) A_2 - \alpha_1^{n_2} A_3 = 0; \quad a = 1, 2;
\]
from (33).

Let \( R_1 = 3 \). The equations (28), (41) are linearly independent as well as the equations (42). The system (16), (28), (41) being in involution, we have proved \( A_1 \).

Let \( R_1 = 2 \). Then one of the equations (41) is the linear combination of the second one and the equation (28). Suppose, e.g., that (28) and (41) are linearly independent; substituting from (33) into (41), we get (42). For a given surface \( \pi \), the couple \( (\pi', T) \) is given by the involutive system (28) + (41), and \( A_2 \) has been proved.

Let \( R_1 = 1 \). The equations (41) are the multiples of (28). \( \pi \) and \( \pi' \) being given, \( T \) is given by the completely integrable equations (26), and we have proved \( A_3 \).

B. Let \( \dim t^2 = r - 3 \). Three of the equations (22) are linear combinations of the remaining one and of (21); we may suppose the existence of numbers \( \alpha_1, \ldots, \gamma_3 \) such that
\[
(43) \quad \xi_{i,r+2}^{r+1} = \alpha_1 \xi_{i,r+1}^{r+3} + \alpha_2 \xi_{i,r+2}^{r+3} + \alpha_3 \xi_{i,r+3}^{r+3}, \quad \xi_{i,r+1}^{r+2} = \beta_1 \xi_{i,r+1}^{r+3} + \beta_2 \xi_{i,r+2}^{r+3} + \beta_3 \xi_{i,r+3}^{r+3},
\]
\[
\xi_{i,r+2}^{r+2} = \gamma_1 \xi_{i,r+1}^{r+3} + \gamma_2 \xi_{i,r+2}^{r+3} + \gamma_3 \xi_{i,r+3}^{r+3} \quad \text{for} \quad i = 1, \ldots, r.
\]

The expression (20) reduces to
\[
(44) \quad [v, k] = \sum_{i=1}^{r} \xi_{i,r+1}^{r+1} w_{r+1} (\beta_1 \xi_{i,r+2}^{r+3} + \xi_{i,r+3}) + k^{r+1} w_{r+2} \beta_2 \xi_{r+2} + \xi_{i,r+3}^{r+3} + k^{r+1} w_{r+3} (\beta_3 \xi_{i,r+2}^{r+3} + \xi_{i,r+3}) + k^{r+2} \xi_{i,r+3}^{r+3} + \xi_{i,r+3}^{r+3}.
\]

Let us write
\[
(45) \quad R_2 = \begin{vmatrix} 0 & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 & \gamma_1 & \gamma_2 & \gamma_3 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}
\]
obviously, \( \dim [h, K] = r + R_2 \). The equations (30) reduce to
\[
(46) \quad \alpha_1 \varphi_1 + \omega^{r+2} + \alpha_2 \varphi_2 + \omega^{r+2} + \varphi_3 (\omega^{r+1} + \alpha_3 \omega^{r+2}) = 0,
\]
\[
\varphi_1 (\beta_1 \omega^{r+1} + \gamma_1 \omega^{r+2}) + \varphi_2 (\beta_2 \omega^{r+1} + \gamma_2 \omega^{r+2}) + \varphi_3 (\beta_3 \omega^{r+1} + \gamma_3 \omega^{r+2}) = 0.
\]
The polar matrix of the system (28) + (46) is
\[
(47) \quad \begin{vmatrix} \omega^{r+1} & \omega^{r+2} & 0 \\ \alpha_1 \omega^{r+2} & \alpha_2 \omega^{r+2} & \omega^{r+1} + \alpha_3 \omega^{r+2} \\ \beta_1 \omega^{r+1} + \gamma_1 \omega^{r+2} & \beta_2 \omega^{r+1} + \gamma_2 \omega^{r+2} & \beta_3 \omega^{r+1} + \gamma_3 \omega^{r+2} \end{vmatrix}.
\]
Let us choose a vector \( k \in K \) \((19\text{)}_2\). \((44)\) yields that the space \([h,k]\) is spanned by the vectors \( e_1, \ldots, e_r \) and

\[
\begin{align*}
g_1 &= \alpha_1 k^{r+2} e_{r+1} + (\beta_1 k^{r+1} + \gamma_1 k^{r+2}) e_{r+2} + k^{r+1} e_{r+3}, \\
g_2 &= \alpha_2 k^{r+2} e_{r+1} + (\beta_2 k^{r+1} + \gamma_2 k^{r+2}) e_{r+2} + k^{r+2} e_{r+3}, \\
g_3 &= (k^{r+1} + \alpha_3 k^{r+2}) e_{r+1} + (\beta_3 k^{r+1} + \gamma_3 k^{r+2}) e_{r+2}.
\end{align*}
\]

If \( \dim [h,k] = r + 3 \) for some vector \( k \in K \), the determinant of \((47)\) is not equal to zero. Of course, \( \dim [h,K] = r + 3 \), and the equations \((28) + (46)\) are linearly independent. This proves \( B_1 \).

Let \( R_2 = 2 \). The equations \((28) \) and \((46_1)\) are linearly independent, and \((46_2)\) is the linear combination of them. The surfaces \( \pi \) and \( \pi' \) being given, the deformation \( T \) is given by the system \((26)\) and the quadratic equation \((46_1)\). \( B_2 \) has been proved.

C. Let \( \dim t^2 = r - 4 \); the Lie algebra \( t^2 \) be given by the equations \((21)\) and

\[
\varrho_a \equiv \sum_{i=1}^{r} \varrho_{ai} v^i = 0; \quad a = 1, 2.
\]

Hence, there are numbers \( \alpha^{ae}_{bc}, \beta^{ae}_{bc} \) such that

\[
\varepsilon^{r+a}_{i,r+b} = \sum_{c=1}^{2} (\alpha^{ac}_{bc} \epsilon^{r+3}_{i,r+c} + \beta^{ac}_{bc} \varrho_{ci}).
\]

Writing

\[
\chi_a = \sum_{i=1}^{r} \varrho_{ai} \tau^i; \quad a = 1, 2;
\]

the forms \( \varphi_1, \varphi_2, \chi_1, \chi_2 \) are linearly independent, and the equations \((30)\) reduce to

\[
\sum_{a,b=1}^{2} (\beta^{ab}_{ac} \kappa_b \wedge \omega^{r+a} + \alpha^{ab}_{ac} \varphi_a \wedge \omega^{r+b}) = 0; \quad c = 1, 2.
\]

Consider the vectors \((19)\) such that \( v \in t^1 \). We have

\[
[v,k] = \sum_{i=1}^{r} (\cdot) e_i + \sum_{a,b,c=1}^{2} \beta^{ca}_{bc} \varrho_{a,b} k^{r+b} e_{r+c}.
\]

If \([t^1,k] \oplus h = K\) for some vector \( k \), the polar matrix of the system \((52)\) is regular. \( D. \) and \( E. \) are evident.

References


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