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ONE GENERALIZATION OF THE FOURTH HARMONIC POINT

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This article contains the discussion concerning the independence of inverse elements on certain choices of coordinatizing ternary rings of a given translation plane. The results obtained are used for the definition of harmonic quadruples on the coordinate axis of the affine plane over a VEBLEN-WEDDERBURN system with both left and right inverse properties. Finally, some generalization of VON STAUDT theorem is given.

I took advice of G. PICKERT who recommended to me the investigation of the independence of harmonic quadruples on changing frames.

By a *frame* $\mathscr{F}$ in an affine plane $\mathscr{P}$ we shall mean any parallelogram OJ_xJJ_y. The lines OJ_x, OJ_y are called *coordinate axes*. $\mathscr{F}$ determines the planar ternary ring $T_{\mathscr{F}}$ ([1], p. 16) for which $\mathscr{P}$ can be identified with $T_{\mathscr{F}} \times T_{\mathscr{F}}$ where $0 = (0, 0)$, $J_x = (1, 0)$, $J = (1, 1)$, $J_y = (0, 1)$. Then to every point $A \in OJ_x \setminus \{0\}$ there exists exactly one point $A'_{\mathscr{F}} \in OJ_x \setminus \{0\}$ such that $A'_{\mathscr{F}} = (a', 0)$ where $a'a = 1$, $A = (a, 0)$.

We shall need for an affine plane $\mathscr{P}$ the condition

(1) *Be given a fixed frame* $\mathscr{F}^* = OJ_xJ^*J_y^*$. *Then for each* $A \in OJ_x \setminus \{0\}$, *the point* $A'_{\mathscr{F}}$ *is independent on the position of the variable frame* $\mathscr{F} = OJ_xJJ_y$ *where* J_y *runs over* OJ_y^*.

Proposition 1. *In an affine plane* $\mathscr{P}$ *let there be given a fixed frame* $\mathscr{F}^* = OJ_xJ^*J_y^*$. *Then the conclusion of* (1) *is equivalent with the "left inverse property"*

$$(2_{\mathscr{F}^*}) \qquad a(a'b) = b \quad \text{for all} \quad a \in T_{\mathscr{F}^*} \setminus \{0\}, \quad b \in T_{\mathscr{F}^*}$$

where the multiplication is taken with respect to $T_{\mathscr{F}^*}$.

Proof. We can construct $A'_{\mathscr{F}}$ using a polygonal line $A_0A_1A_2A_3A_4A_5$ where $A_0 = A$, $A_1 = A_0Y \cap OJ$, $A_2 = A_1X \cap J_xY$, $A_3 = J$, $A_4 = JX \cap OA_2$, $A_5 = A_4Y \cap OX = A'_{\mathscr{F}}$. Here X, Y denote the ideal points of OJ_x and OJ_y^* respectively. Now we construct the analogical polygonal line $A_0^*A_1^*A_2^*A_3^*A_4^*A_5^*$ with respect to $\mathscr{F}^*$ where $A_0^* = A$, $A_5^* = A'_{\mathscr{F}^*}$. Thus with respect to $\mathscr{F}^*$ we obtain $A_0 = (a, 0)$, $A_1 = (a, b)$; $A_2 = (1, b)$; $A_3 = (1, y)$ where y_1 is determined by $b = ay_1$; $A_4 = (x_1, y_1)$

where x_1 is determined by $y_1 = x_1 b$; $A_5 = (x_1, 0)$. The elements a, b belong to $T_{\mathscr{F}^*} \setminus \{0\}$. The equation $A_5 = A_5^* = (a', 0)$ holds now exactly if $b = ay_1 = a(x_1 b) = a(a'b)$ so that the required equivalence is verified.

Corollary. $(2_{\mathscr{F}^*}) \Rightarrow aa' = 1$.

Proof. Putting $b = 1$ in $(2_{\mathscr{F}^*})$ we obtain the required result.

If the element a' determined for each $a \in T_{\mathscr{F}^*} \setminus \{0\}$ by $a'a = 1$, satisfies also $aa' = 1$ then it shall be denoted by a^{-1}.

Lemma 1. *Let T be a Veblen Wedderburn system* ([1], *p.* 17) *with the left inverse property. Then for*

$$(3) \qquad a(-1) = -a \quad \text{for all} \quad a \in T,$$

$$(4) \qquad (a(-1)(-1)) = a \quad \text{for all} \quad a \in T$$

it holds (3) $\Leftrightarrow$ (4) *and further, from* (3) *it follows*

$$(5) \qquad a(-b) = -ab \quad \text{for all} \quad a, b \in T.$$

Proof. From $a(-1) = -a$ it follows $(a(-1))(-1) = (-a)(-1) = -(-a) = a$. Secondly, let there hold $(a(-1))(-1) = a$. Determine the solution x of the equation $-x + x(-1) = a$ and multiply on the right by -1. We obtain $(-x)(-1) + (x(-1))(-1) = a(-1)$. The left side can be expressed as $(-x)(-1) + x$ which is the opposite element to $-x + x(-1)$. Thus $-a = a(-1)$. Now let there hold (3). Thus $a^{-1}(-1) = -a^{-1}$ for any $a \in T \setminus \{0\}$. By the left inverse property it follows $a(-a^{-1}) = -1$ and $-(a(-a^{-1})) = 1$. By the identity $(-x)y = -(xy)$ holding in T we obtain $(-a)(-a^{-1}) = 1$ and finally $-a^{-1} = (-a)^{-1}$. Take the equation $-(-b) = b$ and rewrite it as $-(a^{-1}(a(-b))) = b$. From this we deduce $(-a^{-1}) \cdot (a(-b)) = b$ and further by the preceding $(-a)^{-1}(a(-b)) = b$. By the left inverse property it follows $a(-b) = (-a)b$ so that $a(-b) = -(ab)$.

Lemma 2. *Let a translation affine plane $\mathscr{P}$ satisfy* (1). *Then* (3) *holds in $T_{\mathscr{F}^*}$ iff $\mathscr{P}$ satisfies*

$(6_{\mathscr{F}^*})$ *If $A_1B_1C_1$, $A_2B_2C_2$ are triangles such that $A_1, A_2 \in OJ_y^*$; $B_1, B_2 \in OJ_x$; $C_1, C_2 \in OJ^*$; $A_1C_1 \parallel A_2C_2 \parallel OJ_x$; $B_1C_1 \parallel B_2C_2 \parallel OJ_y^*$; $A_1B_1 \parallel J_xJ_y^*$ then $A_2B_2 \parallel J_xJ_y^*$.*

Proof. Without loss of generality choose $A_1 = (0, 1)$, $B_1 = (1, 0)$, $C_1 = (1, 1)$, $A_2 = (a, 0) + (0, 0)$, $B_2 = (0, a)$ with respect to $T_{\mathscr{F}^*}$. Then the line A_2B_2 has the slope ([1], p. 5) $u = a^{-1}(-a)$ and by the left inverse property it follows $au = -a$. Thus $a(-1) = -a$ holds iff $u = -1$.

Lemma 3. *Let a translation affine plane $\mathcal{P}$ satisfy* (1). *Then* (4) *holds in* $T_{\mathcal{F}^*}$ *iff* $\mathcal{P}$ *satisfies*

($7_{\mathcal{F}^*}$) *If* $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ *are parallelograms such that* $A_1, C_1, A_2, C_2 \in OJ^*$; $B_1, C_1, B_2 \in ON$ (N *the ideal point of the line* $J_xJ_y^*$); $C_1D_1 \parallel C_2D_2 \parallel OJ_x$; $A_1D_1 \parallel A_2D_2 \parallel OJ_y^*$ *then* $B_2 \in ON$.

Proof. Without loss of generality take $A_1 = (1, 1)$, $B_1 = (-1, 1)$, $C_1 = (-1, -1)$, $D_1 = (1, -1)$, $A_2 = (a, a) \neq (0, 0)$, $B_2 = (a, a(-1))$, $C_2 = (a(-1), a(-1))$. Then $D_2 = (a(-1), a)$ and consequently $(a(-1))(-1) = a$ iff $D_2 \in ON$ because $y = x(-1)$ is the equation of the line ON.

Corollary. *Let* $\mathcal{P}$ *satisfy* (1). *Then* ($6_{\mathcal{F}^*}$) *holds iff* ($7_{\mathcal{F}^*}$) *holds.*

Proposition 2. *Let* $\mathcal{P}$ *be a translation affine plane satisfying* (1) *and* ($6_{\mathcal{F}^*}$). *Then* ($6_{\mathcal{F}}$) *is valid for every frame* $\mathcal{F} = OJ_xJJ_y$, $J_y \in OJ_y^*$.

Proof. Without loss of generality take $A_1 = (0, b) \neq (0, 0)$, $B_1 = (1, 0)$, $C_1 = (1, b)$, $B_2 = (a, 0)$, $A_2 = (0, ab)$, $C_2 = (a, ab)$ with respect to $T_{\mathcal{F}^*}$. Then the line A_1B_1 has the slope $u_1 = b$ and the line A_2B_2 has the slope u_2 fulfilling $-ab = au_2$. But $-ab = au_2$ iff $a(-b) = au_2$ by Lemma 1 and $a(-b) = au_2$ iff $u_1 = -b = u_2$ by the left inverse property. Thus $A_1B_1 \parallel A_2B_2$.

Lemma 4. *Let* $\mathcal{P}$ *be an affine plane with a fixed frame* $\mathcal{F}^* = OJ_xJ^*J_y^*$. *Then the "right inverse property"*

$$(8_{\mathcal{F}^*}) \qquad (ab')\,b = a \quad \textit{for all} \quad a \in T_{\mathcal{F}^*}, \quad b \in T_{\mathcal{F}^*} \setminus \{0\}$$

is satisfied in $T_{\mathcal{F}^*}$ *iff*:

(9_{F^*}) *For any parallelograms* $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ *such that* $A_1B_1 \parallel C_1D_1 \parallel A_2B_2 \parallel C_2D_2 \parallel OJ_x$, $A_1D_1 \parallel B_1C_1 \parallel A_2D_2 \parallel B_2C_2 \parallel OJ_y^*$, $B_2 \in OB_1$, $A_1C_1 = A_2C_2 = OJ^*$ *there holds* $D_2 \in OD_1$.

Proof. Without loss of generality choose $C_2 = (a, a) \neq (0, 0)$; $C_1 = (1, 1)$; $B_1 = (1, b')$ where $b \neq 0$; $A_1 = (b', b')$; $D_1 = (1, b')$, $A_2 = (ab', ab')$; $B_2 = (a, ab')$; $D_2 = (ab', a)$ with respect to $T_{\mathcal{F}^*}$. Then $D_2 \in OD_1$ iff $y = xb$ is satisfied for $x = ab'$ and $y = a$.

Proposition 3. *Let* $\mathcal{P}$ *be an affine plane satisfying* (1) *and* ($9_{\mathcal{F}^*}$). *Then* ($9_{\mathcal{F}}$) *holds for all frames* $\mathcal{F} = OJ_xJJ_y$, $J_y \in OJ_y^*$ *iff the following "general right inverse property" is valid in* $T_{\mathcal{F}^*}$

$$(10_{\mathcal{F}^*}) \qquad ((ac)(c^{-1}b))\,c = a(bc) \quad \textit{for all} \quad a, b \in T_{\mathcal{F}^*}, \quad c \in T_{\mathcal{F}^*} \setminus \{0\}\,.$$

Proof. Without loss of generality set $A_1 = (b, bc)$, $B_1 = (1, bc)$, $C_1 = (1, c) \neq (1, 0)$, $D_1 = (b, c)$, $A_2 = (x_0, a(bc)$, $B_2 = (a, a(bc))$, $C_2 = (a, ac)$, $D_2 = (x_0, ac)$

$(x_0$ determined from $a(bc) = x_0c)$ with respect to $T_{\mathscr{F}^*}$. Then $(10_{\mathscr{F}})$ holds for $J = (1, c)$ iff $ac = x_0(b^{-1}c)$ since $b^{-1}c$ is the slope of the line OD_1. Now $a(bc) = x_0c$, $ac = x_0(b^{-1}c)$ are equivalent with $(ac)(b^{-1}c)^{-1} = (a(bc))c^{-1}$ and this last equation is equivalent with $((ac)(c^{-1}b))c = a(bc)$. Here we used $(xy)^{-1} = y^{-1}x^{-1}$ valid by the left and by the right inverse property. For $b = 1$, $ac = d$, $(10_{\mathscr{F}^*})$ yields $(dc^{-1})c = d$ i.e. the right inverse property. For $ac = 1$, $(10_{\mathscr{F}^*})$ yields $(c^{-1}b)c = c^{-1}(bc)$.

Remark. If $T_{\mathscr{F}^*}$ has associative multiplication then $(10_{\mathscr{F}^*})$ is fulfilled. If $T_{\mathscr{F}^*}$ is an alternative field ([1], pp. 14–15) then by $((xy)z)y = x(y(zy))$ (cf. [1], p. 15) we obtain at once $((ac)(a^{-1}b))c = a(c(c^{-1}b)c)$. But the expression on the right hand equals to $a(bc)$ because of the relation $(c^{-1}b)c = c^{-1}(bc)$ valid in an alternative field (in an alternative field any two elements generate an associative subfield by the well-known results of Moufang and Zorn). Further,

$$(11_{\mathscr{F}^*}) \qquad (ac)(c^{-1}b) = ab \quad \textit{for all} \quad a, b \in T_{\mathscr{F}^*}, \quad c \in T_{\mathscr{F}^*} \setminus \{0\}$$

is valid iff $T_{\mathscr{F}^*}$ has associative multiplication. In fact, for $d = c^{-1}b$, $(11_{\mathscr{F}^*})$ yields $b = cd$, $ab = a(cd)$ so that $(ac)d = a(cd)$. Conversely, setting $c = b^{-1}d$ in $(ab)c = a(bc)$ we obtain $bc = d$ so that $(ab)(b^{-1}d) = ad$.

Lemma 4'. *Let $\mathscr{P}$ be an affine plane with a fixed frame $\mathscr{F}^* = OJ_xJ^*J_y^*$. Then in $T_{\mathscr{F}^*}$ there holds*

$$(8'_{\mathscr{F}^*}) \qquad a'(ab) = b \quad \textit{for all} \quad a \in T_{\mathscr{F}^*} \setminus \{0\}, \quad b \in T_{\mathscr{F}^*}$$

iff:

$(9'_{\mathscr{F}^*})$ *For parallelograms $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ satisfying $A_1B_1 \parallel C_1D_1 \parallel A_2B_2 \parallel C_2D_2 \parallel OJ_x$, $A_1D_1 \parallel B_1C_1 \parallel A_2D_2 \parallel B_2C_2 \parallel OJ_y^*$, $B_1C_1 = A_2D_2$, $A_1 \in OA_2$, $B_1 \in OB_2$, $C_1 \in OC_2$ it holds $D_1 \in OD_2$.*

Proof. Without losing generality set $B_1 = (1, 1)$, $B_2 = (a, a) \neq (0, 0)$, $C_1 = (1, b)$, $C_2 = (a, ab)$, $A_1 = (a', 1)$, $A_2 = (1, a)$, $D_1 = (a', b)$, $D_2 = (1, ab)$ with respect to $T_{\mathscr{F}^*}$. Then $D_1 \in OD_2$ iff the equation $y = x(ab)$ holds for $x = a'$ and $y = b$.

Proposition 3'. *Let $\mathscr{P}$ be an affine plane with a fixed frame $\mathscr{F}^* = OJ_xJ^*J_y^*$ and let $(8_{\mathscr{F}^*})$, $(8'_{\mathscr{F}^*})$ be satisfied. Then $(8'_{\mathscr{F}})$ holds for every frame $\mathscr{F} = OJ_xJJ_y$, $J_y \in OJ_y^*$.*

Proof. Without losing generality choose the parallelograms $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ in such a way that $B_1 = (1, c) \neq (1, 0)$; $C_1 = (1, b) \neq (1, 0)$ for $b \neq c$; $B_2 = (a, ac)$, $C_2 = (a, ab)$ for $a \neq 0$; $A_1 = (x_0, c)$ for x_0 determined from $c = x_0(ac)$; $A_2 = (1, ac)$; $D_1 = (x_0, b)$; $D_2 = (1, ab)$ with respect to $T_{\mathscr{F}^*}$. Then $D_1 \in OD_2$ iff $b = x_0(ab)$. By the given assumptions $aa' = 1$ and from the left and right inverse properties there follows $(xy)^{-1} = y^{-1}x^{-1}$; we used this fact already in the proof of Proposition 3. So $x_0 = c(c^{-1}a^{-1}) = a^{-1}$ and the equation $b = x_0(ab)$ is satisfied for $x_0 = a^{-1}$ by the left inverse property.

Definition 1. Let $\mathscr{P}$ be a translation affine plane satisfying (1). Let $T_{\mathscr{F}^*}$ satisfy the condition $1 + 1 \neq 0$. Any ordered triple of pairwise distinct points A, B, C on the coordinate axis OJ_x where $C \neq M_{AB}$ [1]) will be called *admissible*. To any admissible triple (A, B, C) on OJ_x we associate the point $H^{\mathscr{F}^*}_{ABC}$ in the following manner: If $A = (a, 0)$, $B = (b, 0)$, $C = (c, 0)$ with respect to $T_{\mathscr{F}^*}$ (where, according to the preceding assumptions $a \neq b \neq c \neq a$ and $c + c \neq a + b$) then construct the points[2]) $SB \cap J^*Y = B_1$, $SC \cap J^*Y = C_1$ with $S = (a, 1)$, further the point D_1 such that $B_1 = M_{C_1D_1}$ and finally the point $H^{\mathscr{F}^*}_{ABC} = SD_1 \cap OJ_x$.

Proposition 4. *By the assumption of Definition* 1 *there holds* $H^{\mathscr{F}}_{ABC} = H^{\mathscr{F}^*}_{ABC}$ *for every* $\mathscr{F} = OJ_xJJ_y$, $J_y \in OJ_y$ *and for every admissible triple* (A, B, C) *on* OJ_x.

Proof. It can be easily verified that $B_1 = (1, (a - b)^{-1})$, $C_1 = (1, (a - c)^{-1})$, $D_1 = (1, (a - d)^{-1})$ for $H^{\mathscr{F}^*}_{ABC} = (d, 0)$ with respect to $T_{\mathscr{F}^*}$. By the construction of D_1 there is then

$$(a - b)^{-1} + (a - b)^{-1} = (a - c)^{-1} + (a - d)^{-1} . \tag{12}$$

Regarding (1) and the assumption that $\mathscr{P}$ is a translation plane, we conclude that the equation (12) retains its form also when transited to each frame $\mathscr{F} = OJ_xJJ_y$, $J_y \in OJ^*$ so that $H^{\mathscr{F}}_{ABC} = H^{\mathscr{F}^*}_{ABC}$.

Remark. If, in particular, $T_{\mathscr{F}^*}$ is an alternative field (of characteristic $\neq 2$), then the equation (12) is geometrically interpreted in [3], p. 98, or in [5], p. 79.

Lemma 6. *Let* $\mathscr{P}$ *be a translation affine plane satisfying* (1), $(6_{\mathscr{F}^*})$, $(9_{\mathscr{F}^*})$ *and* $1 + 1 \neq 0$ *in* $T_{\mathscr{F}^*}$. *Then for* $A = (1, 0)$, $B = (-1, 0)$, $C = (c, 0) \neq (0, 0)$ *it follows* $H^{\mathscr{F}^*}_{ABC} = (c^{-1}, 0)$.

Proof. For the investigated point $H^{\mathscr{F}^*}_{ABC} = (d, 0)$ we obtain $2^{-1} + 2^{-1} = (1 - c)^{-1} + (1 - d)^{-1}$. The left side is equal to 1 since $2^{-1} + 2^{-1} = (1 + 1)\, 2^{-1} = 2 \,.\, 2^{-1}$. Further $1 = (1 - c)^{-1} + (1 - d)^{-1} \Leftrightarrow 1 - d = (1 - c)^{-1}(1 - d) + 1 \Leftrightarrow -d = (1 - c)^{-1}(1 - d) \Leftrightarrow (1 - c)(-d) = 1 - d \Leftrightarrow -d + (-c)(-d) = 1 - d \Leftrightarrow (-c)(-d) = 1 \Leftrightarrow -d = (-c)^{-1} = -c^{-1} \Leftrightarrow d = c^{-1}$. To these arrangements there was used the distributive law $(x + y)\, z = xz + yz$, the left and the right inverse properties and at the last step the relation $(-c)^{-1} = -c^{-1}$ which is equivalent to $c^{-1}(-1) = -c^{-1}$.

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[1]) If P, Q are points of $\mathscr{P}$ then by the given assumptions there exists precisely one point M_{PQ} such that the translation sending P into M_{PQ} takes M_{PQ} into Q (cf. [2], p. 6).

[2]) Y denotes the ideal point of the line OJ^*_y.

Definition 2. Let $\mathscr{P}$ be a translation affine plane satisfying the assumptions of Lemma 6. Then by a *von Staudt projectivity* on OJ_x we shall mean a $1-1$ mapping σ of the line OJ_x onto OJ_x which reproduce at both sides each admissible triple on OJ_x and satisfies $(H^{\mathscr{T}^*}_{ABC})^\sigma = H^{\mathscr{T}^*}_{A^\sigma B^\sigma C^\sigma}$ for each admissible triple (A, B, C) on OJ_x.

Remark. It may be easily shown that the mapping σ in Definition 2 satisfies the condition $(H^{\mathscr{T}^*}_{ABC})^{\sigma^{-1}} = H^{\mathscr{T}^*}_{A^{\sigma^{-1}}B^{\sigma^{-1}}C^{\sigma^{-1}}}$ for each admissible triple (A, B, C) on OJ_x.

Proposition 5. *Let $\mathscr{P}$ be a translation affine plane satisfying the assumptions of Lemma 6. If σ is a von Staudt projectivity of OJ_x with fixed points O, J_x then the mapping $\sigma_0 : T_{\mathscr{T}^*} \to T_{\mathscr{T}^*}$ defined by the prescription $A^\sigma = (a^{\sigma_0}, 0)$ for each $A = (a, 0) \in OJ_x$ satisfies the conditions*

$$(i_{\sigma_0}) \qquad (a + b)^{\sigma_0} = a^{\sigma_0} + b^{\sigma_0} \quad \text{for each} \quad a, b \in T_{\mathscr{T}^*},$$

$$(ii_{\sigma_0}) \qquad (a^{-1})^{\sigma_0} = (a^{\sigma_0})^{-1} \quad \text{for each} \quad a \in T_{\mathscr{T}^*} \setminus \{0\}.$$

Conversely, if ϱ is a $1-1$ mapping of $T_{\mathscr{T}^*}$ onto $T_{\mathscr{T}^*}$ with fixed elements 0, 1 satisfying (i_ϱ) and (ii_ϱ) [3]) *then the mapping $\varrho^0 : OJ_x \to OJ_x$ defined by $\varrho^0 A = (a^\varrho, 0)$ for each $A = (a, 0) \in T_{\mathscr{T}^*} \times \{0\}$ is von Staudt projectivity of OJ_x.*

Proof. 1) Evidently, (i_{σ_0}) is valid for $a = 0$ or for $b = 0$. If $a \neq 0$ then a triple of mutually distinct points $(0, 0)$, $(a + a, 0)$, $(a, 0)$ is not admissible so that $((0, 0), ((a + a)^{\sigma_0}, 0), (a^{\sigma_0}, 0))$ is not admissible, i.e. $(a + a)^{\sigma_0} = a^{\sigma_0} + a^{\sigma_0}$. If we define $x/2$ for each $x \in T_{\mathscr{T}^*}$ by $x/2 + x/2 = x$ then for $b = a + a$ we have by the preceding $b^{\sigma_0} = (b/2)^{\sigma_0} + (b/2)^{\sigma_0}$ and this means $b^{\sigma_0}/2 = (b/2)^{\sigma_0}$. Let $a \neq b$. Then the triple of mutually distinct points $((a, 0), (b, 0), (\frac{1}{2}(a + b), 0)$ is not admissible so that $((a^{\sigma_0}, 0), (b^{\sigma_0}, 0), (\frac{1}{2}(a + b))^{\sigma_0}, 0))$ is not admissible, i.e. $(\frac{1}{2}(a + b))^{\sigma_0} = \frac{1}{2}(a^{\sigma_0} + b^{\sigma_0})$. By the preceding we have then $(a + b)^{\sigma_0} = a^{\sigma_0} + b^{\sigma_0}$. Further $(-1)^{\sigma_0} = -1$ since the triples of mutually distinct points $((1, 0), (-1, 0), (0, 0))$, $((1, 0), ((-1)^{\sigma_0}, 0), (0,0))$ are not admissible at the same time. The equation (ii_{σ_0}) is of course satisfied for $a = \pm 1$. Further, take $a \neq 0, 1, -1$. By Lemma 6 it follows $(H^{\mathscr{T}^*}_{(1,0)(-1,0),(a,0)})^\sigma = ((a^{-1})^{\sigma_0}, 0) = H^{\mathscr{T}^*}_{(1,0)(-1,0)(a^{\sigma_0},0)} = ((a^{\sigma_0})^{-1}, 0)$ so that $(a^{-1})^{\sigma_0} = (a^{\sigma_0})^{-1}$. The first part of Proposition 5 is proved.

2) From (i_ϱ) it follows $(a/2)^\varrho = a^\varrho/2$ so that to a not admissible triple $((0, 0), (a, 0), (a/2, 0))$ there corresponds the not admissible triple $((0, 0), (a^\varrho, 0), (a^\varrho/2, 0))$. Similarly for ϱ^{-1}.

If $a \neq b$ then the triple of mutually distinct points $((a, 0), (b, 0), (\frac{1}{2}(a + b), 0))$ is not admissible. From (i_ϱ) and from the above identity $(x/2)^\varrho = x^\varrho/2$ it follows $(\frac{1}{2}(a + b))^\varrho = \frac{1}{2}(a^\varrho + b^\varrho)$ so that the corresponding triple is $((a^\varrho, 0), (b^\varrho, 0), (\frac{1}{2}(a^\varrho + b^\varrho), 0))$. This triple consists of mutually distinct points and it is also not admissible. Similarly for ϱ^{-1}. If $((a, 0), (b, 0), (c, 0))$ is an admissible triple then by

[3]) Here $0^\varrho = 0$ follows already from (i_ϱ) whereas from (ii_ϱ) it follows only $(1^\varrho)^2 = 1$.

the preceding it follows that also $((a^\varrho, 0), (b^\varrho, 0), (c^\varrho, 0))$ is an admissible triple. If $H^{\mathscr{F}^*}_{(a,0)(b,0)(c,0)} = (d, 0)$ then d is well-determined by $(a - b)^{-1} + (a - b)^{-1} = (a - c)^{-1} + (a - d)^{-1}$. By (i_ϱ), (ii_ϱ) and $(-x)^{-1} = -x^{-1}$ we obtain $(a^\varrho - b^\varrho)^{-1} + (a^\varrho - b^\varrho)^{-1} = (a^\varrho - c^\varrho)^{-1} + (a^\varrho - d^\varrho)^{-1}$ i.e. $H^{\mathscr{F}^*}_{(a^\varrho,0)(b^\varrho,0)(c^\varrho,0)} = (d^\varrho, 0) = (H^{\mathscr{F}^*}_{(a,0)(b,0)(c,0)})^{\varrho^0}$. So we have proved also the second part of Proposition 5.

Remark. The assumptions in Proposition 5 are fulfilled especially if $\boldsymbol{T}_{\mathscr{F}^*}$ is a Veblen-Wedderburn system with associative multiplication (i.e. a nearfield) or if $\boldsymbol{T}_{\mathscr{F}^*}$ is an alternative field. It is an open question whether, except these two cases, any further case is possible for $\boldsymbol{T}_{\mathscr{F}^*}$ in Proposition 5. Notice that Proposition 5 in the case that $\boldsymbol{T}_{\mathscr{F}^*}$ is an alternative field gives the von Staudt theorem studied in [4], p. 165 (cf. also [4], p. 165).

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