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Higher degrees of distributivity in lattices and lattice-ordered groups

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Let \( \alpha \) and \( \beta \) be cardinal numbers. The \((\alpha, \beta)\)-distributivity in Boolean algebras was studied by several authors (for references, cf. Sikorski [7, p. 61]). For each regular cardinal \( \alpha \) there exists a complete Boolean algebra \( B_\alpha \) that is \((\beta, \gamma)\)-distributive for every \( \beta < \alpha \) and that is not \((\alpha, \alpha)\)-distributive (Scott [6]). Pierce [3] and Weinberg [9, 10] examined the \((\alpha, \beta)\)-distributivity for lattice-ordered groups (\(l\)-groups). Pierce proved the following theorem ([3, Theorem 7.1]): an \( \alpha \)-complete Boolean algebra \( B \) is \( \alpha \)-distributive if and only if the \( l \)-group \( C(X(B)) \) of all continuous real functions defined on the Stone space of \( B \) is \( \alpha \)-distributive. Combining this with the theorem of Scott, we obtain that for each regular cardinal \( \alpha \) there exists an \( l \)-group \( G \) which is \( \beta \)-distributive for each \( \beta < \alpha \) and which is not \( \alpha \)-distributive.

The aim of the present paper is the study of convex sublattices \( S \) of an infinitely distributive lattice satisfying a certain maximality condition with respect to the \((\alpha, \beta)\)-distributivity. The main idea is as follows. Suppose, at first, that \( L \) is a complete modular lattice, \( a \in L \). Let \( \mathcal{L} \) be the system of all distributive intervals \([u, v] (u \leq v)\) of \( L \) containing the element \( a \). The system \( \mathcal{L} \) (partially ordered by the set-theoretical inclusion) need not have a greatest element. To see this it suffices to take for \( L \) the modular lattice with five elements that is not distributive. Let us now replace the condition of modularity of \( L \) and that of distributivity of intervals by the condition of infinite distributivity of \( L \) and the \((\alpha, \beta)\)-distributivity of intervals, respectively. We shall prove that in this case the system \( \mathcal{L} \) always contains a greatest element \( L(a) \).

Let us recall some basic concept and notations. The symbols \( \cap, \cup, \bigcap, \bigcup, \exists, \forall \) and \( \nabla, \cup \),
\( \bigcap, \bigcup \) denote the lattice-theoretical and set-theoretical operations, respectively. Let \( L \) be a lattice, \( a, b \in L, a \leq b \). The interval \([a, b]\) is the set of all \( x \in L \) fulfilling \( a \leq x \leq b \). \([a, b]\) is nontrivial, if \( a < b \). A sublattice \( A \subseteq L \) is convex, if \( a_1, a_2 \in A, a_1 \leq a_2 \) implies \([a_1, a_2] \subseteq A \). A subset \( A \) of \( L \) is a \( c \)-sublattice of \( L \), if any least upper bound and any greatest lower bound of a subset of \( A \) belongs to \( A \). The convex \( c \)-hull of a set \( B \subseteq L \) is the set \( B_0 = \bigcap B_i \) where \( \{B_i\} \) is the system of all convex \( c \)-sublattices \( B_i \) of \( L \) with \( B \subseteq B_i \). A property \((p)\) of convex sublattices of \( L \) is said to be hereditary, if each convex sublattice \( B \) that is a subset of a convex sublattice \( A \) satisfying \((p)\) possesses the property \((p)\) as well. The cardinality of a set \( M \) is denoted by \( \text{card} \ M \).

Let \( \{A_i\}_{i \in I} \) be a system of lattices.\(^1\) The complete direct product \( P = \prod_{i \in I} A_i \) is the system of all functions \( f : I \to \bigcup A_i \) with \( f(i) \in A_i \) for each \( i \in I \). If \( f(i) = a_i \), then we write also \( f = (\ldots, a_i, \ldots) \); \( a_i \) is the component of \( a \) in \( A_i \). The lattice-operations in \( P \) are performed component-wise. A complete direct product is nontrivial if at least two of the factors \( A_i \) have the cardinalities greater than one. If \( I = \{1, 2, \ldots, n\} \), then we write also \( P = A_1 \times \ldots \times A_n \); \( P \) is the direct product of the lattices \( A_1, \ldots, A_n \).

Returning to the general case, let \( f_0 \in P \) and let us denote (for a fixed \( i \in I \) and for any \( f \in P \))
\[
A_i(f_0) = \{ f : f \in P, f(j) = f_0(j) \text{ for each } j \in I, j \neq i \}.
\]

Let \( Q \) be a sublattice of \( P \) such that \( \bigcup_{i \in I} A_i(f_0) \subseteq Q \). Then we shall say that \( Q \) is a complete subdirect product of lattice \( A_i \) with respect to the element \( f_0 \). (For the concept of the complete subdirect product cf. \textsc{Raney} [5], \textsc{Sik} [8] and \textsc{Weinberg} [10].) If \( \varphi \) is an isomorphism of a lattice \( L \) into the lattice \( P \), \( a \in L \), \( \varphi(a) = f, f(i) = a_i \), then \( a_i \) is said to be the projection of \( a \) into \( A_i \).

\[1. \text{ FUNDAMENTAL NOTIONS}\]

In this section some definitions concerning \((\alpha, \beta)\)-distributivity are given and simple consequences of these definitions are deduced.

1.1. A lattice \( L \) is said to be infinitely distributive if the following condition \((d_1)\) and the condition dual to \((d_1)\) are satisfied:

\[
(d_1) \text{ If } x \in L, \{x_i\} \subseteq L \text{ and if the element } \bigcup x_i \text{ exists in } L, \text{ then } x \cap (\bigcup x_i) = \bigcup (x \cap x_i).
\]

\(^1\) We suppose that \( I \neq \emptyset \); in the sequel, this assumption remains valid for any set of indices.
Let \( \alpha \) and \( \beta \) be cardinal numbers and let \( T, S \) be non-empty sets with \( \text{card } T \leq \alpha \), \( \text{card } S \leq \beta \). The lattice \( L \) is \((\alpha, \beta)\)-distributive if the following identities hold in \( L \):

\[
\bigcap_{t \in T} \bigcup_{s \in S} x_{t,s} = \bigcup_{\varphi \in S^T} \bigcap_{t \in T} x_{t,\varphi(t)},
\]

\[
\bigcup_{t \in T} \bigcap_{s \in S} x_{t,s} = \bigcap_{\varphi \in S^T} \bigcup_{t \in T} x_{t,\varphi(t)}.
\]

under the assumption that all meets and unions standing in (1.1) and (1.1') exist in \( L \). (The symbol \( S^T \) denotes the system of all mappings of the set \( T \) into the set \( S \).) \( L \) is \( \alpha \)-distributive if it is \((\alpha, \alpha)\)-distributive. \( L \) is completely distributive, if it is \( \alpha \)-distributive for each cardinal \( \alpha \).

1.1.1. Thorough the present paper it is supposed that \( L \) is infinitely distributive and \( \text{card } L = \alpha_0 \). By examining whether a convex sublattice \( A \) of \( L \) is \( \alpha \)-distributive or not we shall consider only the condition (1.1); by a dual argument one can verify whether the dual condition (1.1') is true or not. Analogously, by deducing consequences from the supposition that \( A \) is not \((\alpha, \beta)\)-distributive we shall suppose that (1.1) is not satisfied; the case when (1.1') does not hold can be treated in a similar manner.

1.2. Let us remark the following simple fact. Suppose that \( A \) is a convex sublattice of \( L \), \( \{a_i\} \subset A \). If the least upper bound of \( \{a_i\} \) in the lattice \( L \) or in \( A \) exists, then we denote this element by \( \bigcup a_i \) and \( \text{sup}_A a_i \), respectively. If \( \text{sup}_A a_i = b \) exists, then \( \bigcup a_i \) also exists and \( b = \bigcup a_i \). An analogical statement holds for the operation \( \cap \). From this it follows: if a convex sublattice \( A \) of \( L \) is not \((\alpha, \beta)\)-distributive, then \( L \) is not \( (\alpha, \beta)\)-distributive, (or, conversely, if \( L \) is \((\alpha, \beta)\)-distributive, then each convex sublattice of \( L \) is \((\alpha, \beta)\)-distributive).

1.3. Suppose that \( L \) is not \((\alpha, \beta)\)-distributive. Then (cf. 1.1.1) there exists a system \( \{x_{t,s}\}_{t \in T, s \in S} \) (card \( T \leq \alpha \), card \( S \leq \beta \)) such that

\[
x = \bigcap_{t \in T} \bigcup_{s \in S} x_{t,s},
\]

\[
y = \bigcup_{\varphi \in S^T} \bigcap_{t \in T} x_{t,\varphi(t)},
\]

and \( y < x \). Let us denote \( (x_{t,s} \cap x) \cup y = z_{t,s} \). We have \( y \leq z_{t,s} \leq x \) and from (1.2), (1.2') using the infinite distributivity we get

\[
x = \bigcap_{t \in T} \bigcup_{s \in S} z_{t,s},
\]

\[
y = \bigcup_{\varphi \in S^T} \bigcap_{t \in T} z_{t,\varphi(t)}.
\]

Hence the interval \([y, x]\) is not \((\alpha, \beta)\)-distributive. From (1.3) and from \( z_{t,s} \in [y, x] \) it follows

\[
\bigcup_{s \in S} z_{t,s} = x \quad \text{for each } t \in T;
\]

in the same way from (1.3') we get

\[
\bigcap_{\varphi \in S^T} z_{t,\varphi(t)} = y \quad \text{for each } \varphi \in S^T.
\]
1.3.1. **Corollary.** If each interval of $L$ is $(\alpha, \beta)$-distributive, then $L$ is $(\alpha, \beta)$-distributive as well.

1.3.2. It is easy to see that if (1.4) and (1.4') hold and if $y < x$, then the interval $[y, x]$ is not $(\alpha, \beta)$-distributive.

If the lattice $L$ is $(\alpha, \beta)$-distributive and if $(\alpha_1, \beta_1) < (\alpha, \beta)$, then clearly $L$ is $(\alpha_1, \beta_1)$-distributive, too. Denote $2^{\omega_0} = \omega^*$. From the axiom of choice it follows $\omega^* = \omega^{\omega_0}$.

1.4. If $L$ is $(\omega^*, \omega_0)$-distributive, then $L$ is completely distributive.

**Proof.** Let us assume that $L$ is not completely distributive and that it is $(\omega^*, \omega_0)$-distributive. Then there exists $(\alpha, \beta) > (\omega^*, \omega_0)$ such that $L$ is not $(\alpha, \beta)$-distributive. We shall be using the same notations as in 1.3. Let $t \in T$ be fixed. Since $\text{card } L = \omega_0$, there exists a set $S_1 = S_1(t)$ and a system $\{v_{t,s}\}_{s \in S_1}$ such that $\text{card } S_1 = \omega_0$, $\{v_{t,s}\}_{s \in S_1} = \{z_{t,s}\}_{s \in S}$. We may suppose that $S_1(t) = S_1(t')$ holds for each $t, t' \in T$ (since only the power of the set $S_1(t)$ is essential for our consideration). Hence we have by (1.4)

$$
U_{s \in S_1} v_{t,s} = x \quad \text{for each } t \in T.
$$

For every $\varphi \in S^T$ there exists $\varphi_1 \in S_1^T$ such that

$$
(z_{t,\varphi(t)} = v_{t,\varphi_1(t)} \quad \text{for each } t \in T
$$

(since each $z_{t,s}$ equals to some suitably chosen element $v_{t,s}$). Conversely, for each $\varphi_1 \in S_1^T$ there exists $\varphi \in S^T$ such that (1.6) holds. From (1.6) and (1.4') we get

$$
\cap_{t \in T} v_{t,\varphi(t)} = y \quad \text{for each } \varphi_1 \in S_1^T.
$$

From (1.5) and (1.5') it follows that $L$ is not $(\omega, \omega_0)$-distributive. For any $t, t' \in T$ let us put $t \sim t'$ if $v_{t',s} = v_{t,s}$ for every $s \in S_1$. Then $\sim$ is an equivalence relation on the set $T$; we pick out an element from each class of the corresponding partition of $T$ and we denote the set of all these elements by $T_1$. If $t \in T_1$, let $\psi_t : S_1 \to L$ be a function satisfying $\psi_t(s) = v_{t,s}$ for each $s \in S_1$. Then the correspondence $t \to \psi_t$ is a one-to-one mapping of the set $T_1$ into the set $L^{S_1}$; hence card $T_1 \leq \omega^*$.

Let us now consider the system $\{v_{t,s}\}_{t \in T_1, s \in S_1}$. For each $\varphi_2 \in S_1^{T_1}$ there exists $\varphi_1 \in S_1^T$ such that

$$
\{v_{t,\varphi_2(t)}\}_{t \in T_1} = \{v_{t,\varphi_1(t)}\}_{t \in T}
$$

(for $t' \in T$ it suffices to choose $\varphi_1(t') = \varphi_2(t)$, where $t \sim t'$, $t \in T_1$). Therefore by (1.5') we have

$$
\cap_{t \in T_1} v_{t,\varphi_2(t)} = y \quad \text{for each } \varphi_2 \in S_1^{T_1}.
$$

Since $T_1 \subset T$, card $S_1 \leq \omega_0$ and card $T_1 \leq \omega^*$ it follows from (1.5) and (1.5") that the lattice $L$ is not $(\omega^*, \omega_0)$-distributive; this is a contradiction.
1.4.1. Corollary. If $L$ is $\alpha^*$-distributive, then it is completely distributive.

1.5. Assume that $L$ is not $(\alpha, \beta)$-distributive. Let $x, y$ have the same meaning as in 1.3; $y < x$. Then no non-trivial interval of the lattice $[y, x]$ is $(\alpha, \beta)$-distributive.

Proof. Let $[u, v]$ be a non-trivial interval of the lattice $[y, x]$; put $y_{t,s} = (z_{t,s} \cap v) \cup u$. From the infinite distributivity and from (1.3), (1.3') it follows

$$v = \cap_{t \in T} \cup_{s \in S} Y_{t,s} > \cup_{t \in T} \cap_{s \in S} Y_{t,s} = u.$$

1.6. Let $A$ be a convex sublattice of $L$. If $A$ is not completely distributive, then we shall denote by $dA$ the least cardinal number $\gamma$ for which $A$ is not $\gamma$-distributive. For any cardinals $\alpha, \beta$ we shall examine the following conditions:

- $(p_1)$ $A$ is $(\alpha, \beta)$-distributive.
- $(p_2)$ If $\langle \alpha_1, \beta_1 \rangle < (\alpha, \beta)$, then $A$ is $(\alpha_1, \beta_1)$-distributive.
- $(p_3)$ $A$ is $\alpha$-distributive.
- $(p_4)$ $A$ is $\alpha_1$-distributive for each $\alpha_1 < \alpha$.
- $(p_5)$ $A$ is $\alpha$-distributive for every non-trivial interval $[a, b] \subset A$.
- $(p_6)$ $\langle a, b \rangle = \alpha$ for every non-trivial interval $[a, b] \subset A$.

If necessary, we shall be using a more detailed notation, i.e. $(p_j(\alpha, \beta))$ instead of $(p_j)$ for $j = 1, 2$, and $(p_j(\alpha))$ instead of $(p_j)$ for $j = 3, 4, 5, 6$. Let us put $J = \{1, \ldots, 6\}$. The following statements are immediate consequences of the definition 1.6:

1.7. Each condition $(p_j)$ $(j \in J)$ is hereditary. If $a \in L$, then the one-element interval $\{a\}$ satisfies $(p_j)$ for each $j \in J$. The implications

$$(p_1(\alpha, \beta)) \Rightarrow (p_2(\alpha, \beta)), \quad (p_3(\alpha)) \Rightarrow (p_4(\alpha)), \quad (p_5(\alpha)) \Rightarrow (p_6(\alpha + ))$$

are fulfilled for any cardinals $\alpha, \beta$.

1.8. For each $j \in J$ we shall consider also the following condition which is in a certain sense complementary to the condition $(p_j)$:

- $(p'_j)$ No non-trivial interval of the lattice $A$ satisfies the condition $(p_j)$.

It is obvious that the condition $(p'_j)$ is hereditary.

2. THE CONDITIONS $(p_j)$ FOR INTERVALS

In this section it is assumed that $\alpha$ and $\beta$ are fixed cardinals. We shall prove at first some simple lemmas on transposed intervals (in 2.1, 2.2 and 2.3 it would be sufficient to suppose that $L$ is distributive rather than infinitely distributive). Let us remark that for $a, b \in L$, $a \cap b = u, a \cup b = v$ the intervals $[u, a], [b, v]$ are called transposed.
2.1. (Cf. Birkhoff [1].) Any transposed intervals are isomorphic.

2.2. Let \( a, b, c \in L, a \leq b \leq c \) and let \([x_1, x_2]\) be a non-trivial interval of \( L \), \([x_1, x_2] \subseteq [a, c] \). Then there exists a non-trivial interval \([y_1, y_2] \subseteq [x_1, x_2]\) which is transposed to an interval contained in \([a, b]\) or in \([b, c]\).

**Proof.** For each \( z \in [a, c] \) let us denote \( z' = b \cap z, z'' = b \cup z \). The following statement follows from the distributivity of \( L \): if \( z_1, z_2 \in [a, c] \) and \( z_1' = z_2', z_1'' = z_2'' \), then \( z_1 = z_2 \). Therefore we have \( x_1' < x_2' \) or \( x_1'' < x_2'' \). Let the first case be considered. Then it suffices to put \( y_1 = x_1, y_2 = x_1 \cup x_2 \). The second case is analogical to the first one.

2.3. Let \( a, b, c \in L, a \leq b \). The interval \([a \cup c, b \cup c]\) is transposed to an interval contained in \([a, b]\).

**Proof.** For this purpose it is sufficient to take the interval \([b \cap (a \cup c), b]\).

**Remark.** Obviously, the statement dual to 2.3 also holds.

2.4. Let \( a \in L, \{b_i\} \subseteq L, a \leq b_i \) for each \( b_i \), \( \cup b_i = b \). Let \([x_1, x_2]\) be a non-trivial interval of \( L, [x_1, x_2] \subseteq [a, b] \). Then there exists a non-trivial interval \([y_1, y_2] \subseteq [x_1, x_2]\) and an element \( b_i^0 \in \{b_i\} \) such that \([y_1, y_2]\) is transposed to an interval contained in \([a, b_i^0]\).

**Proof.** For each \( z \in [a, b] \) we put \( z^i = z \cap b_i \). Then we have

\[(2.1) \quad z = z \cap b = z \cap (\cup b_i) = \cup(z \cap b_i) = \cup z^i.\]

From this and from \( x_1 < x_2 \) it follows that there exists at least one \( i \) satisfying \( x_1^i < x_2^i \). It suffices to apply now the statement dual to 2.3.

2.5. Let \( a, b, c \in L, a \leq b \leq c \). If the intervals \([a, b], [b, c]\) satisfy \((p_1)\), then \([a, c]\) also fulfills \((p_1)\).

**Proof.** For \( z \in [a, c] \) let \( z' \) and \( z'' \) have the same meaning as in 2.2. Let \( \{x_t\}_{t \in T}, \cup S \subseteq [a, c], \text{card } T \leq \alpha, \text{card } S \leq \beta \). Let us assume that all meets and unions standing in \((1.1)\) exist in \( L \). The element on the left or on the right side of \((1.1)\) will be denoted by \( x \) or \( y \) respectively. From the infinite distributivity of \( L \) it follows

\[x' = \cap_{t \in T} \cup_{s \in S}(x_t,s)' , \quad y' = \cup_{q \in S^T} \cap_{t \in T}(x_{t,q(t)}').\]

Since we assume that \([a, b]\) satisfies \((p_1)\), we have \( x' = y' \). Similarly \( x'' = y'' \) holds, and therefore \( x = y \).

2.6. Let \( a, b, c \in L, a \leq b \leq c, j \in J \). If the intervals \([a, b], [b, c]\) satisfy the condition \((p_j)\), then \([a, c]\) also fulfills \((p_j)\).

**Proof.** For \( j = 1 \) the statement is proved in 2.5 and for \( j = 2 \) it is an easy consequence of 2.5. For \( j = 3 \) it suffices to put \( x = \beta \). The statement for \( j = 4 \) follows easily from the case \( j = 3 \).
For $j = 6$ we proceed as follows. Let $\alpha_j < \alpha$. By our assumption the intervals $[a, b], [b, c]$ are $\alpha_j$-distributive, hence by 2.5 the interval $[a, c]$ is $\alpha_j$-distributive, too. Let us suppose now that there exists a non-trivial $\alpha$-distributive interval $[x_1, x_2] \subset [a, c]$. Then by 2.2 and 2.1 there exists a non-trivial $\alpha$-distributive interval contained in $[a, b]$ or in $[b, c]$. This is a contradiction.

From the statement for $j = 6$ the statement for $j = 5$ follows.

2.7. Let $a \in L$, $\{b_i\} \subset L$, $a \leq b_i$ for each $b_i$, $\bigcup b_i = b$, $j \in J$. If each interval $[a, b_i]$ satisfies $(p_j)$, then $[a, b]$ also fulfills $(p_j)$.

Proof. Let us consider at first the case $j = 1$. For $z \in [a, b]$ let $z'$ have the same meaning as in 2.4. Let $\{x_{t, s}\}_{t \in T, s \in S} \subset [a, b]$, $\operatorname{card} T \leq \alpha$, $\operatorname{card} S \leq \beta$. Let us assume that (1.2) and (1.2') holds. By the infinite distributivity

$$x^n = \bigcap_{\sigma \in T} \bigcup_{\sigma \in S} (x_{t, s})^n, \quad y^n = \bigcup_{\sigma \in S} \bigcap_{t \in T} (x_{t, \sigma(t)})^n.$$

The elements standing in these equations belong to $[a, b_1]$; since $[a, b_i]$ is $(\alpha, \beta)$-distributive, we have $x^n = y^n$. Therefore by (2.1) $x = y$. The statements for $j = 2, 3$, are easy consequences of the case $j = 1$. Let $j = 6$ and $\alpha_j < \alpha$. As we have already proved the interval $[a, b]$ is $\alpha_j$-distributive. Assuming the existence of a non-trivial $\alpha$-distributive interval $[x_1, x_2] \subset [a, b]$ we get that by 2.4 and 2.1 there exists a non-trivial $\alpha$-distributive interval which is contained in some interval $[a, b_i]$; this is a contradiction. The proof for $j = 6$ is complete. Hence by 1.7 the statement holds also for the case $j = 5$.

2.8. Let $a, b, c \in L$, $a \leq b$, $j \in J$. If $[a, b]$ satisfies $(p_j)$, then $[a \cup c, b \cup c]$ satisfies $(p_j)$, as well. The proof follows from 2.3 and 2.1.

Remark. By an analogical argument one can prove the statements dual to 2.7 and 2.8.

2.9. The statements 2.6, 2.7 and 2.8 remain valid if the condition $(p_j)$ is replaced by $(p_j')$ ($j \in J$).

This follows from 2.1, 2.2, 2.4 and 2.3 and from the fact that $(p_j')$ is hereditary.

3. THE SUBLATTICES $L_j(a)$

Let $\alpha$ and $\beta$ be fixed cardinals and let $a \in L$, $j \in J$. With the aid of 2.6, 2.7 and 2.8 some further results on the condition $(p_j)$ will be deduced. Let us denote by $L_j^j(a)$ the set of all elements $z \in L$, $z \geq a$ such that the interval $[a, z]$ satisfies $(p_j)$. Analogously let us put $L_j(a) = \{z : z \in L, z \leq a, [z, a]$ satisfies $(p_j)\}$.

3.1. $L_j^j(a)$ is a convex $c$-sublattice of $L$ with the least element $a$.

Proof. Obviously, $a$ is the least element of $L_j^j(a)$. Since $(p_j)$ is hereditary, it follows from $a \leq z_1 \leq z_2, z_2 \in L_j^j(a)$ that $z_1 \in L_j^j(a)$ and hence $L_j^j(a)$ is a convex subset of $L$. 362
Let \( z_i \in L_j(a) \), \( \bigcup z_i = z \). By 2.7 we have \( z \in L_j(a) \). If \( \cap z_i = v \), then \( a \leq v \leq z_i \), and therefore by the convexity of \( L_j(a) \) the element \( v \) belongs to \( L_j(a) \).

By the dual argument one can show:

3.1'. \( L_j(a) \) is a convex c-sublattice of \( L \) with the greatest element \( a \).

We shall denote by \( L_j(a)^2 \) the set-theoretical union of all intervals \([b, c]\) with \( b \in L_j(a) \), \( c \in L_j(a) \).

3.2. The set \( L_j(a) \) is a convex c-sublattice of \( L \) satisfying the condition \((p_j)\).

Proof. Let \( \{a_i\} \subset L_j(a) \), \( \bigcup a_i = a_0 \). To each \( a_i \) there exist elements \( b_i, c_i \) such that \( a_i \in [b_i, c_i] \), \( b_i, c_i \in L_j(a) \). Hence by 3.1 \( a \cup a_i \in L_j(a) \), \( a \cup a_0 = a \cup \cup (\cup a_i) = \cup (a \cup a_i) \in L_j(a) \). For each \( b_i \) we have \( b_i \leq a_0 \leq a \cup a_o \), hence \( a_o \in L_j(a) \). Analogously one can prove: if \( \cap a_i \) exists in \( L \), then \( \cap a_i \in L_j(a) \). This proves that \( L_j(a) \) is a c-sublattice of \( L \). Let \( a_1, a_2 \in L_j(a) \), \( a_3 \in L \), \( a_1 \leq a_3 \leq a_2 \). If we use the notations analogous to those used above, we have \( b_1 \leq a_3 \leq c_2 \), hence \( L_j(a) \) is a convex sublattice of \( L \). Since \([a_1, a_2] \subset [b_1, c_2] \), the interval \([a_1, a_2] \) satisfies \((p_j)\) by 2.6. The proof is complete.

3.3. Let \( A \) be a convex sublattice of \( L \) satisfying \((p_j)\) and let \( a \in A \). Then \( A \subset L_j(a) \).

Proof. If \( z \in A \), then \( z \cap a, z \cup a \in A \), therefore the intervals \([z \cap a, a], [a, z \cup a] \) satisfy \((p_j)\) and \( z \cap a \in L_j(a) \), \( z \cup a \in L_j(a) \). This implies \( z \in L_j(a) \).

3.4. If \( a_i \in L_j(a) \), then \( L_j(a_{i}) = L_j(a) \).

Proof. Let \( a_i \in L_j(a) \). By 3.2 and 3.3 we have \( L_j(a) \subset L_j(a_{i}) \). But then \( a \in L_j(a_{i}) \), and hence \( L_j(a_{i}) \subset L_j(a) \).

Since \( a \in L_j(a) \), it follows from 3.4 that the system \( \{L_j(a)\}_{a \in L} \) is a partition of the set \( L \); this partition (and also the corresponding equivalence relation) will be denoted by \( R_j \).

3.5. \( R_j \) is a congruence relation on \( L \).

Proof. For \( z \in L \) we shall denote \( \bar{z} = L_j(z) \). Let \( a, b, c \in L \), \( \bar{a} = \bar{b} \). Put \( a \cap b = u \), \( a \cup b = v \). Since \( \bar{a} \) is a sublattice of \( L \), we have \( u, v \in \bar{a} \), therefore the interval \([u, v] \) satisfies \((p_j)\). By 2.8 the interval \([u \cup c, v \cup c] \) also satisfies \((p_j)\) and hence by 3.3 \( u \cup c = v \cup c \). Since \( a \cup c, \bar{b} \cup c \in [u \cup c, v \cup c] \), we have also \( \bar{a} \cup c = \bar{b} \cup c \). Analogically (by using the statement dual to 2.8) we can prove \( a \cap c = \bar{b} \cap c \). This completes the proof.

In the sections 3.6—3.9 the symbol \( \bar{z} \) has the same meaning as in 3.5.

3.6. The congruence \( R_j \) has the following property:

1) \( \bigcap \{a_i\} \subset L \), \( \bigcup a_i = a \), then in the factor-lattice \( L/R_j \cup \bar{a}_i = \bar{a} \) holds.

2) If necessary, we write \( L_j(a, a) \) instead of \( L_j(a) \) for \( j = 3, 4, 5, 6 \).

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Proof. From $\bigcup a_i = a$ it follows $\bar{a}_i \leq \bar{a}$ for each $\bar{a}_i$. Let $\bar{c} \in L/R_j$ and let $\bar{a}_i \leq \bar{c}$ for each $\bar{a}_i$. Then we obtain $a_i \cup c = c_i \in \bar{c}$ for each $a_i$. Since $\bar{c}$ is a c-sublattice of $L$, the equation $c \cup a = c \cup (\bigcup a_i) = \bigcup (c \cup a_i) = \bigcup c_i$ implies $c \cup a \in \bar{c}$, hence $\bar{c} \cup \bar{a} = \bar{c}$, $\bar{a} \leq \bar{c}$. This proves that $\bigcup \bar{a}_i = \bar{a}$.

Remark. Analogically one can prove the statement dual to (v). It is well-known that a congruence on a general lattice need not satisfy the condition (v).

3.7. If $L$ is a complete lattice, then each class of the congruence $R_j$ contains a least and a greatest element.

Proof. If $L$ is complete and $z \in L$, then there exists the least upper bound $z_1$ and the greatest lower bound $z_2$ of the set $\bar{z} = L_j(z)$. By 3.2 $\bar{z}$ is a c-sublattice of $L$, therefore $z_1, z_2 \in \bar{z}$.

The previous results 3.1 – 3.7 will be summarized in the following theorem:

3.8. Theorem. Let $L$ be an infinitely distributive lattice. There exists a partition $R_j$ of the set $L$ with the following properties:

(a) If $a \in L$, then the class $\bar{a}$ of the partition $R_j$ containing the element $a$ is the greatest element in the system of all convex sublattices of $L$ satisfying $(p_j)$ and containing $a$.

(b) $R_j$ is a congruence relation on $L$ fulfilling the condition (v) and the dual one.

(c) Each congruence class of $R_j$ is a c-sublattice of $L$. If $L$ is complete, then each such class has the least and the greatest element.

In proving the theorem 3.8 we have been using the propositions 2.6, 2.7 and 2.8 only (without the explicit use of the definition of $(p_j)$); hence we obtain by 2.9 the following result:

3.8'. The theorem 3.8 remains true if the condition $(p_j)$ is replaced by $(p'_j)$.

3.9. Let $j \in \{1, 2, 3, 4\}$. Let us put $L = L/R_j$ and let $\bar{a}, \bar{b} \in L$, $\bar{a} < \bar{b}$. The interval $[\bar{a}, \bar{b}]$ does not fulfill the condition $(p_j)$.

It is easy to see that it suffices to prove this for $j = 1$. Let $\bar{a}, \bar{b} \in L/R_j$, $\bar{a} < \bar{b}$. Then there exists $a_1 \in \bar{a}$ and $b_1 \in \bar{b}$ such that $a_1 < b_1$. Since $\bar{a}_1 \neq \bar{b}_1$, the interval $[a_1, b_1]$ is not $(\alpha, \beta)$-distributive. Hence there exists a system $\{z_{t,s}\}_{t \in \mathbb{T}, s \in S}$ with the properties as in 1.3. By 3.6, (1.3) and (1.3')

\begin{align}
(3.1) & \quad \bar{x} = \bigcap_{t \in \mathbb{T}} \bigcup_{s \in S} \bar{z}_{t,s}, \\
(3.1') & \quad \bar{y} = \bigcup_{s \in S} \bigcap_{t \in \mathbb{T}} \bar{z}_{t,s}. 
\end{align}

Since $[\bar{y}, \bar{x}] \subset [\bar{a}_1, \bar{b}_1] = [\bar{a}, \bar{b}]$ and the interval $[\bar{y}, \bar{x}]$ is not $(\alpha, \beta)$-distributive, we have $\bar{y} < \bar{x}$. By (3.1) and (3.1') the interval $[\bar{a}, \bar{b}]$ is not $(\alpha, \beta)$-distributive.

Let the symbol $L_j(a)$ have the analogical meaning as $L_j(a)$ with the distinction that
instead of (p_\j) we are dealing with the condition (p'\j). In the section 5 the following simple proposition will be used:

3.10. L_j(a) \cap L'_j(a) = \{a\} holds for each a \in L.

Proof. If z \in L_j(a) \cap L'_j(a), let us denote u = z \cap a, v = z \cup a. Since L_j(a) and L'_j(a) are convex sublattices of L, we have [u, v] \subseteq L_j(a) \cap L'_j(a). Therefore [u, v] satisfies (p_j) and no non-trivial interval contained in [u, v] satisfies (p_j); hence u = v, z = a.

4. THE RELATIONS AMONG THE SETS L_j(a)

In this section the relations among the sets L_j(a, \alpha_1), L_j(a, \alpha_2) (j_1, j_2 \in J) will be studied where \alpha_1 and \alpha_2 are any cardinals. An immediate consequence of the definition 1.6 (cf. also 1.7) is the following proposition:

4.1. (a) \alpha_1 < \alpha_2 \Rightarrow L_3(a, \alpha_2) \subset L_3(a, \alpha_1), L_4(a, \alpha_2) \subset L_4(a, \alpha_1).

(b) L_3(a, \alpha_1) \subset L_4(a, \alpha_1), L_5(a, \alpha_1) \subset L_5(a, \alpha_1), L_6(a, \alpha_1) \subset L_6(a, \alpha_1).

(c) L_5(a, \alpha_1) = L_6(a, \alpha_1 +).

4.2. L_3(a, \alpha_1) \cap L_0(a, \alpha_1) = \{a\} = L_2(a, \alpha_1) \cap L_0(a, \alpha_1).

The first statement follows from the fact that each interval of the lattice L_3(a, \alpha_1) is \alpha_1-distributive and no non-trivial interval of L_0(a, \alpha_1) is \alpha_1-distributive. The second statement follows from the first one and from 4.1 (b).

4.3. \alpha_1 \neq \alpha_2 \Rightarrow L_6(a, \alpha_1) \cap L_0(a, \alpha_2) = \{a\}.

Proof. If a non-trivial interval [z_1, z_2] is a subset of L_6(a, \alpha_1) \cap L_0(a, \alpha_2), then \alpha_1 = d[z_1, z_2] = \alpha_2 would be true and this is a contradiction.

4.4. \alpha_1 \leq \alpha_2 \Rightarrow L_6(a, \alpha_1) \cap L_3(a, \alpha_2) = \{a\}.

Proof. If \alpha_1 \leq \alpha_2, then no non-trivial interval of L_6(a, \alpha_1) is \alpha_1-distributive and each interval of L_3(a, \alpha_2) is \alpha_1-distributive.

4.5. Let [z_1, z_2] be a non-trivial interval of L. There exists a non-trivial interval [y_1, y_2] \subseteq [z_1, z_2] satisfying (p_3(x*)) or (p_0(x)) for some infinite \alpha \leq x*.

Proof. If [z_1, z_2] is completely distributive, then the interval [y_1, y_2] = [z_1, z_2] fulfils the condition p_3(x*). Let us assume that [z_1, z_2] is not completely distributive and d[z_1, z_2] = \alpha_1. Since L is distributive, \aleph_0 \leq \alpha_1 and by 1.4.1 \alpha_1 \leq \alpha^* in the interval [z_1, z_2] there exist elements x, y and a system \{x_{t,s}\}_{t \in T, s \in S} satisfying the same conditions as in 1.3 (where we set \alpha = \beta = \alpha_1). Let us put [y_1, y_2] = [y, x]. Let [v_1, v_2] be a non-trivial interval contained in [y_1, y_2]. By 1.5 [v_1, v_2] is not \alpha_1-distributive. If \alpha_2 < \alpha_1, then [z_1, z_2] is \alpha_2-distributive, hence [v_1, v_2] is \alpha_2-distributive, too. This implies d[v_1, v_2] = \alpha_1, hence [y_1, y_2] fulfils the condition (p_0(x_1)).
Remark. Let \([a, z]\) be a non-trivial interval of \(L\). There need not exist, in general, a non-trivial interval \([a, y]\) \(\subset [a, z]\) satisfying \((p_3(\alpha^*))\) or \((p_0(\alpha))\) for some \(\alpha\), \(\mathbb{N}_0 \leq \alpha \leq \alpha^*\).

Example. Let \(\{x_i\}_{i=1,2,\ldots}\) be an ascending sequence of regular cardinals. Let \(B_i\) \((i = 1, 2, \ldots)\) be a Boolean algebra which is \(\beta\)-distributive for every \(\beta < \alpha_i\) and which is not \(\alpha_i\)-distributive. The least and the greatest element of \(B_i\) will be denoted by \(u_i\) and \(v_i\), respectively. Let \(B_0\) be an one-element Boolean algebra \(\{\emptyset\}\). We define a partial order on the set \(L = \bigsqcup B_i\) \((i = 0, 1, 2, \ldots)\) as follows: on each set \(B_i\) the partial order has its original meaning; the element 0 will be the least element of \(L\); if \(b_1 \in B_i\), \(b_2 \in B_{i'}, i_1, i_2 \geq 1, i_1 \neq i_2\), we put \(b_1 < b_2\) if \(i_2 < i_1\). It is easy to see that \(L\) is an infinitely distributive lattice. Let \([0, z], [0, y]\) be non-trivial intervals of \(L\), \([0, y]\) \(\subset [0, z]\). Then \(y \in B_i\) for some \(i \geq 1\). Since \([u_{i+1}, v_{i+1}] \subset [0, y]\) and by our assumption \(d[u_{i+1}, v_{i+1}] = \alpha_{i+1}\), the interval \([0, y]\) is not completely distributive and \(d[0, y] \leq \alpha_{i+1}\). Further we have \([u_{i+2}, v_{i+2}] \subset [0, y]\), \(d[u_{i+2}, v_{i+2}] = \alpha_{i+2} > d[0, y]\). From this we obtain that \([0, y]\) does not fulfil any condition \((p_3(\alpha^*)), (p_0(\alpha))\) \((\mathbb{N}_0 \leq \alpha \leq \alpha^*)\).

We shall be using the following notations:

\[ I = \{0\} \cup \{x : \mathbb{N}_0 \leq x \leq \alpha^*\}, \]
\[ H_0(\alpha) = L_0(\alpha, x) \text{ (for } \mathbb{N}_0 \leq x \leq \alpha^*\), \]
\[ H_0(\alpha) = L_0(\alpha, x^*), \]
\[ H_1(i) = \{z : z \in H_0(i), z \geq a\} \text{ (} i \in I\), \]
\[ H_2(i) = \{z : z \in H_0(i), z \leq a\} \text{ (} i \in I\). \]

4.6. If \(i_1, i_2 \in I, i_1 \neq i_2, z_1 \in H_1(i), z_2 \in H_1(i), \) then \(z_1 \cap z_2 = a\).

This follows immediately from 4.3 and 4.4.

The convex c-envelope of the set-theoretical union of the system \(\{H_1(i)\}_{i \in I}\) will be denoted by \(H^1(\alpha)\).

4.7. Let \(L\) be a conditionally complete lattice. Then each element \(z \in H_1(\alpha)\) can be written uniquely in the form \(z = \bigcup_{i \in I} z_i\), where \(z_i \in H_1(i)\) for each \(i \in I\). Let \(z = \bigcup_{i \in I} z_i, v = \bigcup_{i \in I} v_i(z, v \in H^1(\alpha), z_i, v_i \in H_1(i), i \in I)\); then the inequality \(z \leq v\) holds if and only if \(z_i \leq v_i\) for each \(i \in I\).

Proof. Let \(W\) be the set-theoretical union of all intervals \([a, w]\) where \(w\) can be expressed in the form \(w = \bigcup_{k \in K} \{w_k\}_{k \in K} \subset \bigcup_{i \in I} H_1(i)\). Since \(H^1(\alpha)\) is a convex \(c\)-sublattice of \(L\) and \(\bigcup_{i \in I} H_1(i) \subset H^1(\alpha)\), we obtain \(W \subset H^1(\alpha)\). On the other hand the least upper bound of any subset of \(W\) (if it exists) obviously belongs to \(W\); since \(W\) has the least element and \(W\) is convex in \(L\), an analagical assertion holds for the meets of any subset of \(W\). Therefore \(W\) is a convex \(c\)-sublattice of \(L, W \supset \bigcup_{i \in I} H_1(i)\), which implies \(W = H^1(\alpha)\). Under the same notations for \(w \in W\), let us denote \(K_i = \{k : k \in E, w_k \in H_1(i)\}\). If \(K_i = \emptyset\), we put \(w_i = a\). If \(K_i \neq \emptyset\), then the set \(\{w_k\}_{k \in K_i}\) is bounded;
since $L$ is conditionally complete and since $H^1_1(a)$ is a c-sublattice of $L$, there exists $w_i = \bigcup_{k \in K} w_k$ and $w_i \in H^1_1(a)$ holds. We obtain $w = \bigcup_{i \in I} w_i$.

Let us now suppose that $z \in H^1_1(a)$. Since $H^1_1(a) = W$, we can put $w = z$, $w_i = z_i$ and we have $z = \bigcup_{i \in I} z_i$, $z_i \in H^1_1(a)$. If at the same time $z = \bigcup_{i \in I} t_i$, $t_i \in H^1_1(a)$, then by 4.6 $z_i \cap t_i = 0$ holds for $i_1, i_2 \in I$, $i_1 \neq i_2$ and therefore $z_i = z_i \cap z = z_i \cap (\bigcup_{i \in I} t_i) = z_i \cap t_i$, $z_i \leq t_i$. Consequently $z_i = t_i$ for each $i \in I$. The proof of the first assertion is complete.

Let $z = \bigcup_{i \in I} z_i$, $v = \bigcup_{i \in I} v_i$, $z_i, v_i \in H^1_1(a)$. If $z_i \leq v_i$ for each $i \in I$, then, obviously, $z \leq v$. If $z \leq v$, then by 4.6 $z_i = z_i \cap v = z_i \cap v_i$, hence $z_i \leq v_i$.

Under the same notation as in 4.7 let us consider the mapping $\varphi : H^1_1(a) \rightarrow \bigcap_{i \in I} H^1_1(a)$ defined as follows: $\varphi(z) = (..., z_i, ...)$ for each $z \in H^1_1(a)$. Put $\varphi(H^1_1(a)) = B$. By 4.7 the partially ordered sets $H^1_1(a)$ and $B$ are isomorphic. Moreover, we have:

4.8. $B$ is a complete subdirect product of the lattices $H^1_1(a)$ $(i \in I)$ with respect to the element $a$.

Proof. Let $b_1 = (..., z_i, ...)_{i \in I} \in B$, $b_2 = (... v_i, ...)_{i \in I} \in B$. Then $b_1 = \varphi(z)$ and $b_2 = \varphi(v)$, where $z = \bigcup_{i \in I} z_i$, $v = \bigcup_{i \in I} v_i$. From this we obtain $z \cup v \in H^1_1(a)$, $z \cup v = \bigcup_{i \in I} (z_i \cup v_i)$, hence $\varphi(z \cup v) = (..., z_i \cup v_i, ...)$ and therefore $b_1 \cup b_2 \in B$. By the infinite distributivity and by 4.6 $z \cap v = \bigcup_{i \in I} (z_i \cap v_i)$, hence $b_1 \cap b_2 \in B$ too. This proves that $B$ is a sublattice of $\bigcap_{i \in I} H^1_1(a)$. If $z = a$, then we have obviously $z_i = a$, for each $i \in I$. To complete the proof we have to show the following (cf. the introduction): if $i_0 \in I$ and $z_i = a$ for each $i \in I$, $i \neq i_0$, $z_{i_0} \in H^1_1(a)$, then $b = (..., z_i, ...) \in B$. But under our assumptions $z_{i_0} \in H^1_1(a) \subset H^1_1(a)$ and hence $\varphi(z_{i_0}) = b, b \in B$.

4.9. Theorem. Let $L$ be a conditionally complete infinitely distributive lattice. Then the lattice $H^1_1(a)$ is isomorphic to a complete subdirect product of lattices $H^1_1(a)_{i \in I}$ with respect to the element $a$. If for any system $\{z_i\}_{i \in I}, z_i \in H^1_1(a)$ there exists in $L$ the element $\bigcup_{i \in I} z_i$, then $H^1_1(a)$ is isomorphic to the lattice $\bigcap_{i \in I} H^1_1(a)$.

The first assertion follows from 4.7 and 4.8. The second one follows from the fact that under our assumption $\varphi(H^1_1(a)) = \bigcap_{i \in I} H^1_1(a)$ holds.

By a dual argument we can prove the analogical proposition for the convex c-envelope of the set-theoretical union of the system $\{H^2_1(a)\}_{i \in I}$.

5. RELATIVELY COMPLEMENTED DISTRIBUTIVE LATTICES

In this section we shall assume that $L$ is a relatively complemented distributive lattice (because of the following lemma 5.1 we need not explicitly suppose that $L$ is infinitely distributive). If $z \in [u, v] \subset L$, we shall denote by $z_{[u,v]}$ the relative comple-
ment of \( z \) with respect to the interval \([u, v]\). It is well-known that any Boolean algebra is infinitely distributive; hence each interval of \( L \) is infinitely distributive.

5.1. The lattice \( L \) is infinitely distributive.

**Proof.** It suffices to prove that the condition (d) from 1.1 is satisfied; the proof of the dual condition is a similar one. Let \( \{x_k\}_{k \in K} \subseteq L \), \( \bigcup x_k = x, y \in L \). Let us choose a fixed index \( k_0 \in K \) and denote \( x_{k_0} \cup x = x_0 \). Put \( x_{k_0} \cap y = u, x \cup y = v \). Then the elements \( x, y, x_0 \) belong to the Boolean algebra \([u, v]\), hence \( y \cap x = y \cap (\bigcup x_k) = y \cap (\bigcup x_k^0) = z \). Since \( x_k \leq x_0 \), it follows \( y \cap x_k \leq z \) for each \( k \in K \). If \( z \neq \bigcup (y \cap x_k) \), then there exists an element \( z_1 < z \) satisfying \( y \cap x_k \leq z_1 \) for each \( k \in K \). Thus \( y \cap x_k \cup (y \cap x_{k_0}) \leq z_1 \) whence \( y \cap x_k \leq z_1 \) and therefore \( z \leq z_1 \), which is a contradiction.

Let \( a \in L, j \in J \); analogously as in the section 3 we set \( L^j_j(a) = \{z : z \in L^j(a), z \leq a\} \), \( L^j_1(a) = \{z : z \in L^j(a), z \leq a\} \). Further we put \( L^1_j(a) = \{z : z \in L, z \leq a\} \), \( L^2_j(a) = \{z : z \in L, z \leq a\} \). We shall prove some propositions on \( L^1_j(a), L^2_j(a) \); analogical propositions hold for \( L^j_1(a), L^j_2(a) \).

5.2. If the interval \([a, c] \subseteq L \) contains a non-trivial interval satisfying (p), then there exists a non-trivial interval \([a, c]\) satisfying (p).

**Proof.** If a non-trivial interval \([u, v] \subseteq [a, c]\) fulfills (p), then the non-trivial interval \([a, v \cap u]\) also satisfies (p).

**Remark.** An analogical assertion holds for (p).

Let us denote by (1), (2) and (3) the condition that \( L, L^1_j(a) \) and \( L^2_j(a) \), respectively, contains a greatest element.

5.3. If any two conditions from (1), (2), (3) hold true, then the remaining one also holds. In such a case the greatest element of the set \( L^j_j(a) \) is the relative complement of the greatest element of the set \( L^1_j(a) \) with respect to the interval \([a, u]\), where \( u \) is the greatest element of \( L \).

This proposition is a consequence of the following lemmas 5.3.1—5.3.4.

5.3.1. Let \( u \) be an upper bound of the set \( L_j^1(a) \cup L_j^2(a) \). Then \( u \) is the greatest element of \( L \).

**Proof.** Let us suppose that there exists an element \( v \in L, v > u \). Assume at first that there does not exist a non-trivial interval contained in \([u, v]\) and satisfying (p).

Then \([u, v]\) fulfills (p), hence \([a, u'_v]\) satisfies (p), too, and therefore \( u'_v \in L_j^1(a) \), \( u'_v \leq u \). This implies \( u = v \), which is a contradiction. Let us now suppose, that in the interval \([u, v]\) there exists a non-trivial interval satisfying (p). Since the intervals \([u, v]\) and \([a, u'_v]\) are transposed, it follows by 5.2 that there exists a non-trivial interval \([a, c] \subseteq [a, u'_v]\) satisfying (p). Therefore we have \( c \in L_j^1(a), c \leq u \) and at the same time \( c \leq u'_v \), thus \( c = a \), which is impossible.

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5.3.2. Let \( b \) and \( c \) be the greatest element of \( L_j^1(a) \) and of \( L_j^3(a) \), respectively. Then \( u = b \cup c \) is the greatest element of \( L \) and \( b'_{[a,u]} = c \).

Proof. The first assertion follows from 5.3.1. The convexity of the sets \( L_j^1(a) \), \( L_j^3(a) \) and 3.10 imply the second statement.

5.3.3. Let \( u \) and \( b \) be the greatest element of \( L \) and \( L_j^1(a) \), respectively. Then \( L_j^1(a) \) has a greatest elements as well.

Proof. It is easy to see that \([b, u] \) satisfies \((p_j')\) (in the opposite case there exists an element \( b_1 > b \) such that \([b, b_1] \) satisfies \((p_j)\); then \([a, b_1] \) also satisfies \((p_j)\) and therefore \( b \) is not the greatest element of \( L_j^1(a) \), which is a contradiction). Let us put \( c = b'_{[a,u]} \). The interval \([a, c] \) is isomorphic to \([b, u] \), hence it satisfies \((p_j)\). If \( z \in L_j^1(a) \), then \([a, z] \) satisfies \((p_j)\), whence \( b \cap z = a, z \leq c \). This proves that \( c \) is the greatest element of the set \( L_j^1(a) \).

5.3.4. Let \( u \) and \( c \) be the greatest element of \( L \) and \( L_j^1(a) \), respectively. Then the set \( L_j^1(a) \) also has a greatest element.

Proof. Let us put \( c'_{[a,u]} = b \). The interval \([b, u] \) fulfills \((p_j)\) since \([a, c] \) and \([b, u] \) are transposed. From 3.10 it follows that \( b \) is an upper bound of the set \( L_j^1(a) \). Let us suppose that there exists an element \( b_1 < b \) which is an upper bound of the set \( L_j^1(a) \). If there exists in \([b_1, b] \) a non-trivial interval satisfying \((p_j)\), then according to 5.2 there exists a non-trivial interval \([b_1, b_2] \subset [b_1, b] \) fulfilling \((p_j)\). In such a case \([a, b'_{[a,b_2]}] \) satisfies \((p_j)\), thus \( b'_{[a,b_2]} \in L_j^1(a) \) and at the same time \( b'_{[a,b_2]} \leq b_1 \); this is a contradiction. Hence \([b_1, b] \) does not contain any non-trivial interval satisfying \((p_j)\) and therefore \([b_1, b] \) fulfills \((p_j')\). It follows that \([a, b'_{[a,b]}] \) also satisfies \((p_j)\), whence \( b'_{[a,b]} \in L_j^1(a) \), \( b'_{[a,b]} \leq c \). This implies \( a \leq b'_{[a,b]} \leq b \cap c = a \) and hence \( b'_{[a,b]} = a, b_1 = b \), a contradiction.

From the distributivity of \( L \) and from 5.3 it follows:

5.4. If any two conditions from (1), (2), (3) hold, then the lattice \( L_j^1(a) \) is isomorphic to the direct product of lattices \( L_j^1(a) \) and \( L_j^3(a) \).

5.5. Let the lattice \( L_j^1(a) \) be conditionally complete. Then \( L_j^1(a) \) is isomorphic to the direct product of lattices \( L_j^1(a) \) and \( L_j^3(a) \).

Proof. For \( z \in L_j^1(a) \) we shall denote \( z_1 = \sup \{t : t \in L_j^1(a), t \leq z\} \), \( z_2 = z'_{[a,z]} \).
By 5.2 and 2.6 the interval \([z_1, z_2] \) satisfies \((p_j')\), hence \([a, z_2] \) also satisfies \((p_j')\) and \( z_2 \in L_j^1(a) \). With the aid of 3.10 one can easily prove that the correspondence \( \phi(z) = (z_1, z_2) \) is a one-to-one mapping of the set \( L_j^1(a) \) onto the direct product of lattices \( L_j^1(a), L_j^3(a) \) and that for \( z, v \in L_j^1(a) \) we have \( z \leq v \) if and only if \( z_1 \leq v_1, z_2 \leq v_2 \).

5.6. The mapping \( \psi : z \to (z \cup a, z \cap a) \) is an isomorphism of the lattice \( L \) onto the direct product of lattices \( L_j^1(a), L_j^3(a) \).
Proof. If \( z \in L \), let us put \( z_1 = z \lor a \), \( z_2 = z \land a \). From the distributivity of \( L \) it follows that \( \psi \) is one-to-one. If \( u \leq a \leq v \), let us denote \( z = a^{[u,v]} \); then \( z_1 = v \), \( z_2 = u \), hence \( \psi(L) = L_1^1(a) \times L_1^2(a) \). Let \( z, v \in L \). If \( z \leq v \), then, obviously, \( z_1 \leq v_1 \), \( z_2 \leq v_2 \). Conversely, let \( z_1 \leq v_1 \), \( z_2 \leq v_2 \). Then we have

\[
z = z \land z_1 \leq z \land v_1 = (z \land v) \lor (z \land a) \leq (z \lor v) \lor (v \land a) = (z \lor a) \land v \leq v.
\]

5.7. Theorem. If any two conditions from (1), (2), (3) and any two analogical dual conditions (concerning the lattices \( L_i, L_i^1(a), L_i^2(a) \) hold, then the lattice \( L \) is isomorphic to the direct product of lattices \( L_i(a) \) and \( L_i^1(a) \).

Proof. Let us denote by \( \simeq \) the lattice-theoretical isomorphism. By 5.6, 5.4 and by the statement dual to 5.4 we obtain

\[
L \simeq L_1^1(a) \times L_1^2(a) \simeq (L_i^1(a) \times L_i^1(a)) \times (L_i^2(a) \times L_i^2(a)).
\]

Since the operation of forming direct product is commutative and associative, applying 5.6 once more we get

\[
L \simeq (L_i^1(a) \times L_i^2(a)) \times (L_i^1(a) \times L_i^2(a)) \simeq L_i^1(a) \times L_i^2(a).
\]

5.8. Theorem. If \( L \) is conditionally complete, then \( L \) is isomorphic to the direct product of lattices \( L_i(a) \) and \( L_i^1(a) \).

The proof is similar to that of 5.7 with the distinction that we use 5.5 instead of 5.4. An immediate consequence of the definition 1.6 is that if \( L = L_4(a) \), then \( L_3^1(a) = L_0(a) \). From this it follows:

5.9. The propositions 5.3, 5.4, 5.5, 5.7 and 5.8 remain true if \( L, L_i(a), L_i^1(a) \) are replaced by \( L_4(a), L_3(a), L_0(a) \).

6. A SUBDIRECT PRODUCT DECOMPOSITION OF A RELATIVELY COMPLEMENTED LATTICE

In this section it is assumed that the lattice \( L \) is relatively complemented and conditionally complete. We shall use the same notations as in the sections 4 and 5.

6.1. Let \( [z_1, z_2] \) be a non-trivial interval of \( L \). There exists a non-trivial interval \( [z_1, z] \subset [z_1, z_2] \) satisfying \( (p_3(\alpha^*)) \) or one of the conditions \( (p_6(\alpha)) \) \((\Sigma_0 \leq \alpha \leq \Sigma_*))\).

This follows from 4.5 and 5.2.

6.2. \( H^1(a) = L_1^1(a) \).

Proof. We obviously have \( H^1(a) \subset L_1^1(a) \). Let \( c \in L_1^1(a) \) and let us denote \( M = H^1(a) \cap [a, c] \). Suppose that the element \( c \) is not the least upper bound of the
set $M$. Then there exists an element $b_1 < c$ which is an upper bound of $M$. Let us put $b = b_1^{1[a,c]}$. Then $a < b$, hence there exists an element $z \in L$ with the properties as in 6.1 (where $[z_1, z_2] = [a, b]$). Obviously, $z \in H^1(a)$, therefore $z \leq b_1$, $a \leq z \leq b \cap b_1 = a$, which is a contradiction. From this it follows $c = \sup M$; since $H^1(a)$ is a $c$-sublattice of $L$, we obtain $c \in H^1(a)$.

6.3. Theorem. The lattice $L^1(a)$ is isomorphic to a complete subdirect product of lattices $H^1_i(a)$ $(i \in I)$ (with respect to the element $a$). If for each system $\{z^i\}_{i \in I}$ where $z^i \in H^1_i(a)$ there exists in $L$ the join $\bigvee z^i$, then the lattice $L^1(a)$ is isomorphic to the complete direct product of lattices $H^1_i(a)$ $(i \in I)$.

This is a consequence of 4.9 and 6.2.

Remark. The dual statement concerning the lattice $L^2(a)$ can be proved similarly.

6.4. Theorem. The lattice $L$ is isomorphic to a complete subdirect product of lattices $H^1_i(a)$ $(i \in I)$ (with respect to the element $a$).

Proof. The situation can be represented by the following scheme:

$$
L \overset{(1)}{=} L^1(a) \times L^2(a) \overset{(2)}{=} \prod_{i \in I} H^1_i(a) \times \prod_{i \in I} H^2_i(a) \cong \prod_{i \in I} (H^1_i(a) \times H^2_i(a)) \overset{(3)}{=} \prod_{i \in I} H^i_i(a).
$$

Here, the isomorphism (1) is constructed by 5.6; (2) is an isomorphic map of $L^1(a) \times L^2(a)$ into $\prod_{i \in I} H^1_i(a) \times \prod_{i \in I} H^2_i(a)$ which is constructed with the aid of the theorem 6.3 and the theorem dual to 6.3; the isomorphism (3) follows from 5.6. Hence the lattice $L$ is isomorphic to a sublattice of the complete direct product of lattices $H^i_i(a)$ $(i \in I)$. It remains to prove that the condition contained in the definition of the complete subdirect product is satisfied. For $z \in L$ let us denote by $f_z$ the image of $z$ with respect to the mapping $L \rightarrow \prod_{i \in I} H^i_i(a)$ which is defined in (6.1). It is easy to see that $f_z(i) = a$ for each $i \in I$. Choose a fixed $i_0 \in I$ and $c \in H^{i_0}_i(a)$. Let $f(i) = a$ for $i \neq i_0$ and $f(i_0) = c$. Let us put $z = c$. In the isomorphism (1) the image of $z$ is the pair $(z_1, z_2)$, $z_1 = c \cup a$, $z_2 = c \cap a$. Obviously, $z_1, z_2 \in H^{i_0}_i(a)$. In the mapping (2) we have $(z_1, z_2) \rightarrow (f_{z_1}, f_{z_2})$, where $f_{z_1}(i) = f_{z_2}(i) = a$ for $i \neq i_0$ and $f_{z_1}(i_0) = z_1$, $f_{z_2}(i_0) = z_2$. The image of $(f_{z_1}, f_{z_2})$ is an element $g \in \prod_{i \in I} H^i_i(a)$ such that for each $i \in I$ $g(i)$ is the relative complement of $a$ with respect to the interval $[f_{z_2}(i), f_{z_1}(i)]$. From this we obtain $g(i) = a$ for $i \neq i_0$, and $g(i_0) = c$, whence $g = f$. We have proved that $f = f_z$; the proof is complete.

6.5. Theorem. Suppose that $L$ satisfies the following conditions:

(a) For each system $\{a^i\}_{i \in I}$ where $a^i \in H^1_i(a)$ the element $\cup a^i$ exists in $L$.

(b) For each system $\{b^i\}_{i \in I}$ where $b^i \in H^2_i(a)$ the element $\cap b^i$ exists in $L$.

Then $L \cong \prod_{i \in I} H^1_i(a)$. 

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Proof. By 4.9, and 6.2 and by the dual statements we have

\[ L^1(a) \times L^2(a) \cong \prod_{i \in I} H^1_i(a) \times \prod_{i \in I} H^2_i(a) \]

from this by (6.1) the assertion of the theorem follows.

From 6.5 we get as a corollary:

**6.6. Theorem.** Let \( L \) be a complete Boolean algebra. Then \( L \cong \prod_{i \in I} H_i(0) \).

6.7. For any convex sublattice \( A \) of \( L \) we put \( \pi A = \sup \{ \alpha : \alpha \leq \alpha^* \}, \) \( A \) is \( \alpha \)-distributive. The lattice \( A \) is said to be \( \pi \)-homogeneous, if \( \pi[z_1, z_2] = \pi[z_3, z_4] \) holds for any two non-trivial intervals \([z_1, z_2] \subset A, [z_3, z_4] \subset A\). The following theorem is known (Pierce [4]):

**6.7.1. If \( L \) is a complete Boolean algebra, then \( L \) is isomorphic to a complete direct product of \( \pi \)-homogeneous Boolean algebras.**

Remark. Our definition of the cardinal \( \pi A \) depends (in the case when \( A \) is completely distributive) not only on \( A \) but on the cardinality of \( L \), too; this could be removed e.g., by dealing with the “infinite” cardinal \( \infty \); in such a case we should have \( \pi A = \infty \) for any completely distributive lattice \( A \).

6.8. Now we can compare two direct decompositions of a complete Boolean algebra that are constructed by means of the function \( d \) (cf. 1.6 and 6.6) and by means of \( \pi \) (cf. 6.7.1). Let us consider at first the lattices \( H_i(a) \); let \( i \in I \) be fixed. From the definition of the set \( H_i(a) \) it follows that for each non-trivial interval \([z_1, z_2] \subset H_i(a)\) the following statements hold (cf. 6.7):

- \( \pi[z_1, z_2] = \alpha^* \), if \( i = 0 \).
- \( \pi[z_1, z_2] = \alpha \), if \( 0 \neq i = \alpha \) and if \( \alpha \) is a limit cardinal.
- \( \pi[z_1, z_2] = \beta \), where \( \beta^+ = \alpha \), if \( 0 \neq i = \alpha \) and \( \alpha \) is a non-limit cardinal.

Hence each lattice \( H_i(a) \) is \( \pi \)-homogeneous. From this it follows that the theorem 6.7.1 is a corollary of 6.6. From 6.4 and 6.5 we get the following result:

**6.8.1. Theorem.** If \( L \) is conditionally complete and relatively complemented, then \( L \) is isomorphic to a complete subdirect product of \( \pi \)-homogeneous lattices. If, moreover, the conditions (a) and (b) from 6.5 hold, then \( L \) is isomorphic to a complete direct product of \( \pi \)-homogeneous lattices.

6.9. Let us suppose that \([z_1, z_2]\) and \([z_3, z_4]\) are intervals of \( L \) which are not completely distributive. Suppose that \( \alpha \) is a limit cardinal, \( d[z_1, z_2] = \alpha, d[z_3, z_4] = \alpha^+. \) Then we have \( \pi[z_1, z_2] = \alpha = \pi[z_3, z_4] \). From this we can see that if \([z_1, z_2] \subset A, [z_3, z_4] \subset A \) and if \( A \) is a \( \pi \)-homogeneous lattice, then the behaviour of intervals of \( A \) with respect to higher orders of distributivity need not be equal. The following example shows that such a situation can actually happen. Let \( \alpha \) be a regular
cardinal. Let us put \( \beta = \alpha + \); obviously, \( \beta \) is a regular cardinal, too. Let \( B_\alpha \) be a complete Boolean algebra which is \( \alpha \)-distributive for every \( \alpha_1 < \alpha \) and which is not \( \alpha \)-distributive; let \( B_\beta \) have analogical properties (with \( \beta \) instead of \( \alpha \)). Let us denote \( L = B_\alpha \times B_\beta \). Then \( L \) is a complete Boolean algebra. Each non-trivial interval \([u, v]\) of \( L \) is \( \alpha_1 \)-distributive for every \( \alpha_1 < \alpha \) and it is not \( \alpha_2 \)-distributive for \( \alpha_2 > \alpha \). Since \( \alpha \) is a limit cardinal, \( \pi(u, v) = \alpha \) holds, hence \( L \) is \( \pi \)-homogeneous. Let \( u_1 \) and \( v_1 \) be the least and the greatest element of \( B_\alpha \), respectively, and let \( u_2 \) and \( v_2 \) have the analogical meaning with respect to \( B_\beta \). Denote \( z_0 = (u_1, u_2) \), \( z_1 = (v_1, u_2) \), \( z_2 = (u_1, v_2) \). Then \( d[z_0, z_1] = dB_\alpha = \alpha \), \( d[z_0, z_2] = dB_\beta = \beta > \alpha \). This example shows also that for a \( \pi \)-homogeneous Boolean algebra \( L \) the direct decomposition treated in 6.6 can be non-trivial; in our case we obviously have \( H_\alpha(z_0) = [z_0, z_1] \), \( H_\beta(z_0) = = [z_0, z_2] \), \( H_i(z_0) = \{z_0\} \) for \( i \in I \), \( \alpha \neq i \neq \beta \).

7. \((\alpha, \beta)\)-DISTRIBUTIVITY IN \( l \)-GROUPS

In this section we shall denote by \( L \) a lattice-ordered group (\( l \)-group). The terminology of [1, chap. XIV] will be used. If \( L \) is regarded merely as a lattice, then it will be denoted by \( L(\leq) \). The \( l \)-group \( L \) is said to be complete if the lattice \( L(\leq) \) is conditionally complete. By the same notations as in the previous sections we have \( L^+ = L^+(0), L^- = L^-(0) \). For \( z \in L \) let us put \( z^+ = z \cup 0, z^- = z \cap 0 \); then \( z = z^+ + + z^- \). If \( z_1, z_2 \in L \), \( z_1 \cap z_2 = 0 \), then \( z_1 \cup z_2 = z_1 + z_2 \). The direct product, the complete direct product and the complete subdirect product of \( l \)-groups are defined analogously as in the case of lattices; the complete subdirect product is always taken with respect to the element 0.

It is well-known that every \( l \)-group is infinitely distributive (cf. [1]). Let \( \alpha, \beta \) be infinite cardinals, \( j \in J \). In considering \( l \)-groups one often uses the fact that for any elements \( a, b \in L \) the mappings \( \varphi_1(z) = a + z + b \) and \( \varphi_2(z) = a - z + b \) is an automorphism of \( L(\leq) \) and a dual automorphism of \( L(\leq) \), respectively.

7.1. Let \( z_i \in L_j(0) \), \( z_i \geq 0 \), \( i = 1, 2 \). Then \( z_1 + z_2 \in L_j(0) \).

Proof. By our assumption the intervals \([0, z_i] (i = 1, 2) \) satisfy the condition \((p_j)\). The interval \([z_1, z_1 + z_2] \) is isomorphic to \([0, z_2] \), hence \([z_1, z_1 + z_2] \) also satisfies \((p_j)\). Therefore by 2.6 the interval \([0, z_1 + z_2] \) fulfils \((p_j)\), hence \( z_1 + z_2 \in L_j(0) \).

By a similar argument we can prove the analogical assertion for \( z_1 \leq 0, z_2 \leq 0 \).

7.2. \( L_j(0) \) is a convex \( l \)-subgroup of \( L \).

Proof. From the previous considerations we know that \( L_j(0) \) is a convex sublattice of \( L \). Let \( z_1, z_2 \in L_j(0) \). Then \( z_i^+, z_i^- (i = 1, 2) \) also belong to \( L_j(0) \). Since \( z_i^- \leq z_i \leq \leq z_i^+, \quad -z_2^+ \leq -z_2 \leq -z_2^- \), we obtain \( z_1^- - z_2^+ \leq z_1 - z_2 \leq z_1^+ - z_2^- \) and therefore applying 7.1 and the convexity of the set \( L_j(0) \) we get \( z_1 - z_2 \in L_j(0) \).
7.3. $L_j(0)$ is an l-ideal of $L$.

Proof. Let $a \in L$. Let us consider the mapping $\phi(z) = a + z - a$. Since $\phi$ is an automorphism of $L(\leq)$, the set $\phi(L_j(0))$ has analogical properties as $L_j(0)$, i.e., for each $z_1 \in \phi(L_j(0))$ the set $\phi(L_j(0))$ is the greatest convex sublattice of $L$ containing $z_1$ and satisfying $(p_j)$. Since $0 \in \phi(L_j(0))$, we obtain $\phi(L_j(0)) = L_j(0)$.

7.4. $L_j(a) = a + L_j(0)$ for each $a \in L$ (i.e., the classes of the partition $R_j$ are congruence classes with respect to the l-ideal $L_j(0)$).

This can be proved in a similar way as in 7.3 by considering the mapping $\phi(z) = a + z$.

7.5. An element $z \in L$ belongs to $L_j'(0)$ if and only if $|z| \cap |x| = 0$ for each $x \in L_j(0)$.

Proof. The “only if” part holds by 3.10. Let us suppose that $|z| \cap |x| = 0$ for each $x \in L_j(0)$. If there exists a non-trivial interval $[u, v] \subset [0, |z|]$ such that $[u, v]$ satisfies $(p_j)$ then $0 < u_1 \leq |z|, u_1 \in L_j(0)$, where $u_1 = v - u$. If we set $x = u_1$, we have a contradiction. From this it follows that $[0, |z|]$ satisfies $(p_j)$ and therefore $|z| \in L_j'(0)$. The interval $[-|z|, 0]$ is dually isomorphic to $[0, |z|]$, hence $-|z|$ belongs to $L_j'(0)$, too. From $-|z| \leq z \leq |z|$ and from the convexity of $L_j'(0)$ it follows $z \in L_j'(0)$.

7.6. Let $L$ be a complete l-group. For $z \in L, z \geq 0, j \in J$ let us denote

$$z_j = \sup \{t : t \in L_j(0), t \leq z\},$$

$$z'_j = \sup \{t : t \in L_j'(0), t \leq z\}.$$

Then $z = z_j \cup z'_j = z_j + z'_j$.

Proof. Let us suppose that $z_j \cup z'_j = u < z$ holds. If the interval $[u, z]$ satisfies $(p'_j)$, then $-u + z \in L_j'(0)$, hence by 2.6 $z'_j < z'_j + (-u + z) \in L_j'(0), z'_j + (-u + z) \leq z$, which is a contradiction to the definition of $z'_j$. Therefore there exists a nontrivial interval $[b, v] \subset [u, z]$ satisfying $(p_j)$. But then we have $-t + v \in L_j(0)$, thence $z_j < z_j + (-t + v) \in L_j(0), z_j + (-t + v) \leq z$; according to the definition of $z_j$ his is a contradiction. Thus $z = z_j \cup z'_j$ holds and $z = z_j + z'_j$ by 7.5.

7.7. Theorem. Any complete l-group $L$ is isomorphic to the direct product of l-groups $L_j(0), L_j'(0)$.

Proof. Let $z \in L$. Since $z^+, -z^- \geq 0$, we can construct the elements $(z^+)_j, (-z^-)_j, (z^+)'_j, (-z^-)'_j$ by 7.6; let us put $z_j = (z^+)_j - (-z^-)_j, z'_j = (z^+)'_j - (-z^-)'_j$; if $z \geq 0$, then this definition of elements $z_j, z'_j$ obviously coincides with the definition 7.6. By 7.5 any two elements $a \in L_j(0), b \in L_j'(0)$ are permutable, hence $z = z_j + z'_j$. This proves that the group $L(\oplus)$ is isomorphic to the direct product of groups $L_j(0), L_j'(0)$, where the isomorphism is given by the mapping $\phi(z) = (z_j, z'_j)$. It remains to
prove that \( \varphi \) is an isomorphism also with respect to the partial order. To do this, it suffices to verify that \( z \geq 0 \) holds if and only if \( z_j \geq 0, z_j' \geq 0 \). From \( z = z_j + z_j' \) it follows that our condition is sufficient; moreover it is also necessary by 7.6.

Let the symbols \( I, H_i(0) (i \in I), H^1, H(0) \) have the same meaning as in the section 4. For any \( i \in I \) let \( z_i \) be the projection of the element \( z \) into the direct factor \( H_i(0) \) (cf. 7.6 and 7.7).

7.8. Let \( L \) be a complete \( l \)-group, \( z \in L, z \geq 0 \). Then \( z = \bigcup_{i \in I} z_i \).

Proof. Obviously, \( z_i \leq \alpha \) for each \( i \in I \). Assume that \( \bigcup_{i \in I} z_i = u < z \). By 4.5 there exists a non-trivial interval \([t, v] \subset [u, z] \) satisfying \( \beta_3(z^*) \) or \( \beta_4(\alpha) \) for some \( \alpha \in \mathbb{N}_0 \leq x \leq x^* \). Let us consider the case when \([t, v] \) satisfies \( \beta_3(\alpha) \) (in other cases the proof would be analogical). Then we have \(-t + v \in H_0(0) \) and \( z_0 < z_0 + (-t + v) \in H_0(0) \). By the definition of \( z_0 \) (cf. 7.6) \( z_0 = \sup \{ t : t \in H_0(0), t \leq z \} \), whence \( z_0 + (-t + v) \leq z_0 \), which is a contradiction.

Corollary. \( H^1 = L^* \) (and dually, \( H^2(0) = L^- \)). From this we obtain \( H(0) = L \).

7.9. Theorem. Any complete \( l \)-group \( L \) is isomorphic to the complete subdirect product of \( l \)-groups \( H_i(0) (i \in I) \).

Proof. Let us consider the mapping \( \varphi(z) = (\ldots, z_i, \ldots) \) of \( L \) into the complete direct product of \( l \)-groups \( H_i(0) (i \in I) \). Let us put \( \varphi(L) = L_1 \). By 7.7 \( \varphi \) is a homomorphism with respect to the operations \( +, \cap, \cup \) (since each \( l \)-group \( H_i(0) \) is equal to some \( l \)-group \( L_i(0) \); cf. the section 4). Let us suppose that there exists an element \( z \neq 0 \) such that \( \varphi(z) = 0 \), hence \( z_i = 0 \) for each \( i \in I \). Without the loss of generality we can suppose \( z > 0 \) (in the opposite case we take the element \(|z| \) rather than \( z \)).

By 7.8 \( z = 0 \); a contradiction. Hence the mapping \( \varphi \) is an isomorphism of the \( l \)-group \( L \) onto the \( l \)-group \( L_1 \).

Let \( i_0 \in I, \ z \in H_{i_0}(0) \). Then \( z^+ \in H_{i_0}(0) \), hence \( z^+ \cap \{ t \} = 0 \) for each \( t \in H_{i_0}(0) \), where \( i \in I, i \neq i_0 \). From this it follows \( (z^+)^{i} = 0 \) by 7.6; analogously we get \( (-z^-)^{i} = 0 \), hence \( z_i = 0 \). Obviously, \( z_{i_0} = z \). This proves that \( L_1 \) is a complete subdirect product of \( l \)-groups \( H_{i_0}(0) \).

7.10. Let \( L \) be a complete \( l \)-group, \( 0 \leq x^i \in H_i(0) \) for each \( i \in I \), \( \bigcup_{i \in I} x^i = x \). Then \( x_i = x^i \).

Proof. Obviously, \( x^i \leq x \) holds and hence \( x^i \leq x_i \) by 7.6. Further \( x_i = x_i \cap x = x_i \cap \bigcup_{i \in I} x^i = x_i \cap \bigcup_{i \in I} x^i = x^i, x_i \leq x^i \) holds.

7.11. Theorem. If for each system \( \{x^i\}_{i \in I} \) such that \( x^i \in H_i(0), x^i \geq 0 \) there exists the element \( \bigcup x^i \) in \( L \), then the \( l \)-group \( L \) is isomorphic to the complete direct product of \( l \)-groups \( H_i(0) (i \in I) \).
Proof. Put \( v = (..., v^i, ...) \in \prod_{i \in I} H_i(0) \). According to the assumption the elements
\[ a = \bigcup (v^i)^+, \quad b = \bigcup -(v^i)^- \]
exist in \( L \). From 7.10 it follows that \( a_i = (v^i)^+ \), \( b_i = -(v^i)^- \) for each \( i \in I \). Denote \( z = a - b \); then \( z_i = (v^i)^+ + (v^i)^- = v^i \) for each \( i \in I \), hence \( \varphi(z) = v \), \( v \in L_1 \). Since by 7.9 \( \varphi : L \to L_1 \) is an isomorphism, the proof is complete.

An \( l \)-group \( L \) is said to be orthogonally complete if for each system \( \{a_k\}_{k \in K} \), where
\[ a_{k_1} \cap a_{k_2} = 0 \text{ for any } k_1, k_2 \in K, \ k_1 \neq k_2 \text{ there exists the element } \bigcup_{k \in K} a_k \text{ in } L \text{ (cf. e.g., [2])}. \]
As \( H_1(0) \bigcap H_{i_2}(0) = \{0\} \) holds for \( i_1, i_2 \in I, \ i_1 \neq i_2 \), the following proposition follows from 7.11:

7.12. If \( L \) is a complete \( l \)-group which is also orthogonally complete, then \( L \) is isomorphic to the complete direct product of \( l \)-groups \( H_i(0) (i \in I) \).

References


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