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A CHARACTERIZATION OF VERY $k$-SPACES

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We shall be concerned here only with Hausdorff spaces. In this case the definition of a $k$-space runs as follows:

**Definition 1.** (See [1], [2].) A topological space $X$ is said to be a $k$-space if and only if all subsets of $X$ having bicomponent intersection with an arbitrary bicomponent subspace of the space $X$ are closed in $X$.

Thus the topology of a $k$-space is completely determined by the array of all bicomponent subsets of this space. The class of $k$-spaces is very wide. Not only metric spaces and locally bicomponent spaces belong to this class, but also all $G_δ$-spaces (i.e. spaces complete in the sense of E. Čech) do.

Unfortunately, a subspace of a $k$-space need not be a $k$-space: each completely regular $T_1$-space can be embedded into a bicomponent Hausdorff space, and the latter is surely a $k$-space. The purpose of this note is to investigate which spaces are "very $k$-spaces".

**Definition 2.** A topological space $X$ is said to be a very $k$-space if and only if each subspace of the space $X$ is a $k$-space.

**Remark 1.** Obviously, each very $k$-space $X$ must satisfy the following condition:

($k_1$) If $M$ is a subset of $X$ and $x$ is a point such that $x \in [M]$, then there exists a bicomponent subspace $Φ$ of the space $X$ such that $x \in [Φ \cap M].$

It seems quite natural to expect that this condition characterizes the $k$-spaces, but this is not true. There are $k$-spaces which do not satisfy this condition (an example can be found in [3]). For the full treatment of the subject see [4]; a classification of $k$-spaces, based on condition $k_1$, is given there.

**Remark 2.** Here is an obvious reformulation of definition 2.
Proposition 1. A topological space X is a very k-space if and only if for each
subset M ⊆ X and for each point \( x \in [M] \setminus M \) there exists a bicom pact subset
\( \Phi \subset M \cup \{x\} \) such that \( x \in [\Phi \setminus \{x\}] \).

Now we shall state the main theorem.

Theorem 1. A space X is a very k-space if and only if for each subset M ⊆ X
and for each point \( x \in [M] \) there exists a sequence \( \{x_n : n = 1, 2, \ldots\} \) of points in M
such that \( \lim x_n = x \).

Proof. Let \( M \subset X \) and let \( x \) be any point of the set \([M] \setminus M\). Evidently we can
find a set \( L \subseteq M \) such that the two following conditions are fulfilled: 1) \( x \in [L] \);
2) if \( L' \subseteq M \) and \( x \in [L'] \), then the cardinality of \( L \) is less or equal to the cardinality
of \( L' \). Proposition 1 enables us to find a bicom pact \( \Phi \subset L \cup \{x\} \) with the property
\( x \in [\Phi \setminus \{x\}] \). It follows from the choice of the set \( L \) that the cardinality of \( \Phi \) and the
cardinality of \( L \) are equal. We denote it by \( \tau \). Let us show that \( \tau = \aleph_0 \). Then the
theorem will follow. The point \( x \) is not isolated in \( \Phi \); moreover, the character of
the point \( x \) in the space \( \Phi \) is equal to \( \tau \). Consider some base \( \{U_x : x \in A\} \) of \( x \) in \( \Phi \),
such that \( \text{card } A = \tau \). We can suppose that the index set \( A \) is well ordered as the
smallest ordinal corresponding to the cardinal number \( \tau \). Now we are in need of
some transfinite construction.

Let \( O_1 x \) be some neighbourhood of the point \( x \) such that \([O_1 x] \subset U_1 \) and let \( x_1 \)
be some point from \( O_1 x \setminus \{x\} \). Suppose that we have defined, for all \( \alpha < \beta, \beta \in A \),
neighbourhoods \( O_\alpha x \) of the point \( x \) as well as points \( x_{\alpha} \in \Phi \setminus \{x\} \). The cardinality
of the set \( \bigcup_{\alpha < \beta} \{x_{\alpha}\} \) is less than \( \tau \), hence \( \bigcup_{\alpha < \beta} \{x_{\alpha}\} \neq x \). Take for \( O_\beta x \) any neighbourhood
of \( x \) such that \( \bigcup_{\alpha < \beta} \{x_{\alpha}\} \cap [O_\beta x] = \emptyset \) and \([O_\beta x] \subset U_\beta \).

Now, \( \bigcap_{\alpha \leq \beta} O_\alpha x \setminus \{x\} = \emptyset \). For the proof we need only to mention that the cardinality
of the family \( \{O_\alpha x : \alpha \leq \beta\} \) is less than \( \tau \) if the character of \( x \) in \( \Phi \) is equal \( \tau \). For \( x_\beta \)
we choose any point from the set \( \bigcap_{\alpha \leq \beta} O_\alpha x \setminus \{x\} \). In such a way we can define \( x_{\alpha} \) and \( O_\alpha x \)
for all \( \alpha \in A \). Consider the subspace \( X^* = \bigcup_{\alpha \in A} \{x_{\alpha}\} \cup \{x\} \) of the space \( X \). Clearly, \( x \)
is not isolated in \( X^* \). On the other hand, the set \( X \setminus \bigcup_{\alpha \leq \beta} \{x_{\alpha}\} \cup [O_{\beta+1} x] \) is a neigh-
bourhood of \( x_\beta \) which does not intersect the set \( X^* \setminus \{x_\beta\} \). Hence all points of the
set \( X^* \setminus \{x\} \) are isolated in \( X^* \). By Proposition 1 we can find a bicom pact \( F \) in \( X^* \)
such that \( x \) is a non-isolated point of this bicom pact. Now, \( F \setminus \{x\} \subset M \). By the
definition of the cardinal number \( \tau \), the cardinality of \( F \) is equal to \( \tau \). Let \( P \) be an
infinite countable subset of the set \( F \setminus \{x\} \). No point of the set \( F \setminus \{x\} \) is an accumula-
tion point of this subset. It follows from the bicom pactness of \( F \) that \([P] \ni x \). Now,
\( P \subseteq M \). Hence, \( \tau = \aleph_0 \). The theorem is proved.

Remark 3. In fact, the following general lemma is established by the argument:
**Lemma.** Let \( X \) be a bicomplete space and let \( x \) be any point of \( X \). Denote the character of \( x \) in \( X \) by \( \tau \). We shall call the point \( x \) "\( \lambda \)-achievable", for some cardinal number \( \lambda \), iff there exists a set \( P \subseteq X \setminus \{x\} \) of the power\(^1\) \( \lambda \) such that \( x \in \text{int}[P] \). If \( x \) is not \( \lambda \)-achievable for any \( \lambda < \tau \), we can find the standard subspace \( X^* \subseteq X \) of the power \( \tau \), only one point of which is not isolated in \( X^* \), such that the neighbourhoods of the point in \( X \) are complements to arbitrary subsets of cardinality less than \( \tau \).

**Remark 4.** The topological spaces in which the sequential closure of a set coincides with the closure of this set are called Frechét-Urysohn spaces (\( FU \)-spaces). So the theorem established may be formulated as follows: The class of all very \( k \)-spaces coincides with the class of all \( FU \)-spaces (among Hausdorff spaces!).

Now we will show how very \( k \)-spaces are related to metric spaces.

**Definition 3.** A map \( f : X \to Y \) is called pseudoopen if for each point \( y \in Y \) and for each open neighbourhood \( U \) of the set \( f^{-1}y \) the interior of the set \( fU \) contains \( y \).

In [4] \( FU \)-spaces we characterized as pseudoopen continuous images of metric spaces. So we have

**Theorem 2.** A topological space \( X \) is a very \( k \)-space if and only if it is a pseudoopen continuous image of some (locally bicomplete) metric space.

**Remark 5.** The \( k_2 \)-spaces [4] have an obvious characterization as pseudoopen continuous images of locally bicomplete spaces (see [4]).

From the main result of this paper, together with the main result of [7, § 7], the following theorem can be deduced.

**Theorem 3.** Let \( X \) be a topological group such that the space of this group is a \( p \)-space\(^2\). Then either of the two following conditions is fulfilled:

1) \( X \) is metrizable;

2) \( X \) contains a subspace, which is not a \( k \)-space.

**Remark 6.** This result is new and non-trivial even in the case when the space of the group under consideration is bicomplete. In fact, a more general result holds: each dyadic bicomplete in which every subspace is a \( k \)-space must be metrizable.

In conclusion we will discuss another phenomena which can occur when dealing with \( k \)-spaces. The fact is that the product of two \( k \)-spaces need not be a \( k \)-space. This may happen even with very \( k \)-spaces. Theorem 2 enables us to give an indirect description of a wide class of \( FU \)-spaces, which is closed with respect to the product.

\(^{1}\) "The power" means the same as "the cardinality".

\(^{2}\) For the definition of a \( p \)-space see [5] or [7]. In particular, any space which is \( G_\delta \) in its bicompleteification, as well as any metric space, is a \( p \)-space.
The elements of the class are pseudoopen bicom pact continuous images of metric spaces. It would be fine to know more about the topological structure of these spaces. I conjecture that all paracompact spaces, belonging to the class, are metrizable. If so, it would be a considerable generalization of the theorem on metrizability of all paracompact spaces which are open continuous bicom pact images of metric spaces (see [6]).

References


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