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ON $\beta$-INTEGRATION IN $E_1$

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1. Introduction. By a suitable weakening of the absolute continuity it is possible to extend the domain of the indefinite Lebesgue integral of a given function. An analogous method was studied abstractly by Holec and Mařík in the paper [1]. In this way, Karták and Mařík defined the so called $\beta$-integral in $E_m$ for $m \geq 2$ in [2]. The definition of $\beta$-integral retains its meaning even for $m = 1$. The purpose of this paper is to clear up the relation of this $\beta$-integral in $E_1$ to more usual integrations.

I want to express my gratitude to Professor J. Mařík for his valuable suggestions which led largely to the simplification of the argumentation.

2. Notations and definitions. The terms: outer measure, measure, measurable and so on are related to the Lebesgue measure in $E_1$. The outer measure of the set $M \subset E_1$ is denoted by $|M|$ and the system of all measurable subsets of $E_1$ is denoted by $\mathcal{Z}$. Given $\mathcal{B} \subset \mathcal{Z}$, let $T \mathcal{B}$ denote the system of all sets $T \cap V$ for $V \in \mathcal{B}$.

Further let $\mathcal{A}_0$ denote the system of all subsets of $E_1$ expressible as a finite union of compact nondegenerate intervals. Now, $\mathcal{A}$ stands for the system of all bounded sets $A \subset E_1$ such that there exists a $B \in \mathcal{A}_0$ with $|(A - B) \cup (B - A)| = 0$. Given $A \in \mathcal{A}$, there exists exactly one $B \in \mathcal{A}_0$ possessing the above property; we put $\tilde{A} = B$ and $|A| = 2p$, where $p$ is the number of components of $\tilde{A}$.

Now, let us define the convergence $\to$ on $\mathcal{Z}$ as follows: $Z_n \to Z$ means that $Z_n \subset Z$, $Z - Z_n \in \mathcal{A}$, $\sup_\mathbb{N} |Z - Z_n| < \infty$, $|Z - Z_n| \to 0$. A system $\mathcal{F} \subseteq \mathcal{Z}$ will be called closed, if each limit of a sequence of sets of $\mathcal{F}$ lies in $\mathcal{F}$. Given $\mathcal{B} \subset \mathcal{Z}$, $\mathcal{M}(\mathcal{B})$ denotes the minimal closed system containing $\mathcal{B}$. The set functions under considerations are supposed to be finite and their continuity means the continuity with respect to $\to$.

For a set $M \subset E_1$, let us denote $\overline{M}$ the closure of $M$. Given an open set $G \subset E_1$, let us denote $\mathcal{U}(G)$ the system of all $A \in \mathcal{A}$ such that $\overline{A} \subset G$.

Let $\mathcal{F}$ be the system of all real-valued functions ($\pm \infty$ not excluded) whose domain of definition is a subset of $E_1$. With each $f \in \mathcal{F}$ we associate the system $\mathcal{M}(f)$ of all measurable sets, on which the finite Lebesgue integral of $f$ exists. A point $x \in E_1$ is said to be an $L$-regular point for $f \in \mathcal{F}$ if there exists a neighbourhood $U$ of $x$ such
that $U \in \mathcal{M}(f)$. The set of all $L$-regular points of $f$ will be denoted by $L_f$; this set is evidently open. The Perron (or Lebesgue) integral of $f$ over the set $M$ will be denoted by $\int_M f$.

The function $f \in \mathcal{F}$ is said to be $\beta$-integrable on the set $A \in \mathcal{A}$ if $A \in \mathcal{A}(\mathcal{A} \cap \mathcal{M}(f))$ and if there exists a continuous additive function $\varphi$ defined on $A\mathcal{A}$ such that $\varphi(B) = \int_B f$ for each $B \in \mathcal{M}(f) \cap \mathcal{A}$. The number $\varphi(A)$ will be denoted by $\beta(f, A)$.

3. **Lemma.** Suppose that $A_n \in \mathcal{A}$, $A_n \to A$, $\sup \|A_n\| = 2t$, $\|A\| = 2s$. Then $\tilde{A}_n \to \tilde{A}$ and the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ has at most $t + s$ points.

**Proof.** It is easy to see that $A \subset B$ implies $\tilde{A} \subset \tilde{B}$; whence it follows immediately that $\tilde{A}_n \to \tilde{A}$.

Let us denote $y_1, \ldots, y_s$ the left endpoints of the components of the set $\tilde{A}$ and let $H$ be the set of all these points. Let $x_1 < x_2 < \ldots < x_k$ be arbitrary points of the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n - H$ and let $x_0 = \min H$. Since $|\tilde{A} - \tilde{A}_n| \to \infty$, we can choose such $n$ that $|\tilde{A} - \tilde{A}_n| < |A \cap \langle x_{i-1}, x_i \rangle|$ for $l = 1, \ldots, k$. Hence there exist components $I_1, \ldots, I_k$ of the set $\tilde{A}_n$ lying in the intervals $\langle x_0, x_1 \rangle$, $\langle x_1, x_2 \rangle$, $\ldots$, $\langle x_{k-1}, x_k \rangle$ respectively. It follows that $k \leq t$, and the number of all points of the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ does not exceed $t + s$.

4. **Lemma.** Given $Q \subset E_1$, let $\mathcal{A}_Q$ denote the system of all sets $A \in \mathcal{A}$ such that $\tilde{A} - Q$ is countable. Then the system $\mathcal{A}_Q$ is closed.

**Proof.** Suppose that $A_n \in \mathcal{A}_Q$, $A_n \to A$. By the preceding lemma $\tilde{A}_n \to \tilde{A}$ and the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ is finite. Then, by the inclusion $\tilde{A} - Q \subset (\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n) \cup \bigcup_{n=1}^{\infty} (\tilde{A}_n - Q)$, $\tilde{A} - Q$ is countable, i.e. $A \in \mathcal{A}_Q$.

5. **Theorem.** Let $G$ be an open subset of $E_1$. Then $A \in \mathcal{u}(\mathcal{R}(G))$ if and only if $A \in \mathcal{A}$ and $\tilde{A} - G$ is countable.

**Proof.** a) Using the notation of the preceding lemma we have obviously $\mathcal{R}(G) \subset \mathcal{A}_G$ and by that lemma $\mathcal{u}(\mathcal{R}(G)) \subset \mathcal{A}_G$. This means that $A \in \mathcal{A}$ and $\tilde{A} - G$ is countable for $A \in \mathcal{u}(\mathcal{R}(G))$.

b) Suppose now that $A \in \mathcal{A}$ and that $\tilde{A} - G$ is countable. Let us denote $\mathcal{G}$ the system of all open sets $H \subset E_1$ with the following property: If $B \in \mathcal{A}$, $\tilde{B} \subset H$, then $A \cap B \in \mathcal{u}(\mathcal{R}(G))$. We have:

(i) $G \in \mathcal{G}$, $E_1 - \tilde{A} \in \mathcal{G}$. (This is evident.)

(ii) $\bigcup_{H \in \mathcal{G}} H \in \mathcal{G}$ for $\mathcal{G}_1 \subset \mathcal{G}$. (This relation is a consequence of the following
assertion: If \( B \in \mathcal{A} \), \( B = \bigcup_{n \in \mathbb{N}_1} B_n \), then there exists a finite number of sets \( B_i \in \mathcal{A} \), \( i = 1, \ldots, k \), such that \( \bigcup_{i=1}^k B_i = B \) for suitable \( H_i \in \mathcal{G}_1 \).

(iii) If \( \alpha < \beta < \gamma \), \( (\alpha, \beta), (\beta, \gamma) \in \mathcal{G} \), then \( (\alpha, \gamma) \in \mathcal{G} \). (This is obvious.)

Let us put \( H_0 = \bigcup_{n \in \mathbb{N}} H_n \). According to (ii), \( H_0 \in \mathcal{G} \) and according to (i), \( E_1 - H_0 \subset H - G \). Hence the set \( E_1 - H_0 \) is a countable closed set without isolated points (see (iii)). It follows that \( H_0 = E_1 \) whence \( A \in \mathcal{V}(G) \).

6. Lemma. If \( f \in \mathcal{F} \), then \( \mu(A \cap M(f)) = \mu(\mathcal{R}(L_f)) \).

Proof. The obvious inclusion \( \mathcal{R}(L_f) \subset A \cap M(f) \) implies \( \mu(\mathcal{R}(L_f)) \subset \mu(A \cap M(f)) \). Let \( A \in \mathcal{A} \cap M(f) \). Denoting \( <a_i, b_i>, i = 1, 2, \ldots, p \), the components of \( A \), we have \( (a_i, b_i) \in L_f \), whence \( (a_i, b_i) \in \mathcal{U}(\mathcal{R}(L_f)) \). Since \( \mathcal{U}(\mathcal{R}(L_f)) \) is a set ring containing all bounded sets \( M \) with \( \|M\| = 0 \), it follows that \( A \in \mathcal{U}(\mathcal{R}(L_f)) \). Hence \( \mu(A \cap M(f)) = \mu(\mathcal{R}(L_f)) \) also holds.

7. Theorem. Let \( I = (a, b) \) be a compact interval in \( E_1 \).

a) Let \( \phi \) be an additive continuous function on \( \mathbb{R} \). If we put \( f(x) = \phi(<a, x>) \) for \( x \in I \), then the function \( f \) is continuous on \( I \).

b) Conversely, let \( f \) be a continuous function on \( I \). If we put \( \phi(A) = \sum_{j=1}^p \left( f(b_j) - f(a_j) \right) \) for \( A \in \mathbb{R} \) denoting \( <a_j, b_j>, j = 1, 2, \ldots, p \), the components of \( A \), then the function \( \phi \) is additive and continuous on \( \mathbb{R} \).

Proof. a) The continuity from the left of \( f \) is obvious and the continuity from the right follows from the formula \( f(x) = \phi(<a, x>) - \phi(<x, b>) \).

b) The additivity of \( \phi \) is evident. Suppose that \( A_n \to A \), \( A \subset I \) and \( \sup \|A - A_n\| = 2s \). Let \( \varepsilon \) be any positive number. There exists \( \delta > 0 \) such that \( |f(y) - f(x)| < \varepsilon/s \) for \( x, y \in I \), \( |y - x| < \delta \). Further, there exists \( n_0 \) such that \( |A - A_n| < \delta \) for \( n \geq n_0 \). Hence \( |\phi(A) - \phi(A_n)| = |\phi(A - A_n)| < s(\varepsilon/s) = \varepsilon \) for \( n \geq n_0 \). This proves the continuity of \( \phi \).

8. Theorem. Let \( G \) be an open subset of \( E_1 \) and let \( I \) be a compact interval in \( E_1 \). Suppose that the set \( I - G \) is countable. Let \( F \) and \( f \) be two functions on \( I \) such that \( F \) is continuous on \( I \) and is a Perron indefinite integral of \( f \) on each component of \( G \). Then \( F \) is a Perron indefinite integral of \( f \) on \( I \).

Proof. Let \( \varepsilon \) be any positive number. Let \( (a_n, b_n), n \in \mathbb{N} \), be the components of \( G \) and let \( I = (a, b) \). By the well known theorem on Perron integration there exists the Perron integral \( f_{n} = F(b_n) - F(a_n) \) for each \( n \in \mathbb{N} \). Let \( M_n \) be a majorant of \( f \) on \( (a_n, b_n) \) such that \( M_n(b_n) - M_n(a_n) < F(b_n) - F(a_n) + \varepsilon/2^n \). Put \( g_n(x) = 0 \) for
x < a_n, \ g_n(x) = M_n(x) - F(x) - M_n(a_n) + F(a_n) \ \text{for} \ a_n \leq x \leq b_n,\ g_n(x) = g_n(b_n)
\text{for} \ x > b_n \ \text{and} \ g = \sum_{n \in \mathbb{N}} \ g_n. \ \text{Finally put} \ h(x) = \varepsilon \sum_{k} (1/2^k) \ \text{sgn} (x - s_k), \ \text{where} \ \{s_1, s_2, \ldots\} = I - G. \ \text{Then the function} \ M = F + g + h \ \text{is a majorant of} \ f \ \text{on} \ \langle a, b \rangle \ \text{such that} \ M(b) - M(a) < F(b) - F(a) + 3\varepsilon. \ \text{Using a similar construction we find a minorant} \ m \ \text{of} \ f \ \text{on} \ \langle a, b \rangle \ \text{such that} \ m(b) - m(a) > F(b) - F(a) - 3\varepsilon. \ \text{Hence there exists the Perron integral} \int_a^b f = F(b) - F(a).

9. Theorem. Suppose that \( f \in \mathcal{F}, \ I = \langle a, b \rangle. \) Then \( \beta(f, I) \) exists if and only if there exists the Perron integral \( \int_a^b f \) and the set \( I - L_f \) is countable. In this case \( \beta(f, I) = \int_a^b f. \)

Proof. a) Suppose that \( \beta(f, I) \) exists. By the definition of \( \beta \)-integral and by Lemma 6 we have \( I \in u(\mathfrak{N}(L_f)) \), so that by Theorem 5 the set \( I - L_f \) is countable. By Theorem 7 the function \( F, F(x) = \beta(f, \langle a, x \rangle) \) for \( x \in \langle a, b \rangle \), is continuous on \( \langle a, b \rangle \). Now, we can apply Theorem 8 with \( G = L_f \). Hence \( F \) is an indefinite Perron integral of \( f \) on \( \langle a, b \rangle \) and \( \int_a^b f = F(b) - F(a) = \beta(f, I). \)

b) Conversely, suppose that the Perron integral \( \int_a^b f \) exists and the set \( I - L_f \) is countable. By Theorem 5 and Lemma 6 we have \( I \in u(\mathfrak{N}(L_f)) = u(\mathfrak{N} \cap \mathfrak{M}(f)). \)

For \( A \in \mathfrak{M} \) let us put \( \varphi(A) = \sum_{j=1}^p (F(b_j) - F(a_j)), \) where \( \langle a_j, b_j \rangle, \) \( j = 1, 2, \ldots, p, \) are the components of \( A \) and \( F(x) = \int_a^x f. \) By Theorem 7 the function \( \varphi \) is an additive continuous function on \( \mathfrak{M}. \) Since \( \varphi(A) = \int_A f \) for \( A \in \mathfrak{M}(f) \cap \mathfrak{M}, \) there exists \( \beta(f, I) = \varphi(I) = \int_a^b f. \)

References


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