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Czechoslovak Mathematical Journal, Vol. 18 (1968), No. 3, 523–526

Persistent URL: `http://dml.cz/dmlcz/100849`

Czechoslovak Mathematical Journal, 18 (93) 1968, Praha

ON β-INTEGRATION IN E_1

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(Received March 24, 1967)

1. Introduction. By a suitable weakening of the absolute continuity it is possible to extend the domain of the indefinite Lebesgue integral of a given function. An analogous method was studied abstractly by HOLEC and MAŘÍK in the paper [1]. In this way, KARTÁK and MAŘÍK defined the so called β-integral in E_m for $m \geqq 2$ in [2]. The definition of β-integral retains its meaning even for $m = 1$. The purpose of this paper is to clear up the relation of this β-integral in E_1 to more usual integrations.

I want to express my gratitude to Professor J. MAŘÍK for his valuable suggestions which led largely to the simplification of the argumentation.

2. Notations and definitions. The terms: outer measure, measure, measurable and so on are related to the Lebesgue measure in E_1. The outer measure of the set $M \subset E_1$ is denoted by $|M|$ and the system of all measurable subsets of E_1 is denoted by $\mathfrak{Z}$. Given $\mathfrak{V} \subset \mathfrak{Z}$, $T \in \mathfrak{Z}$, let $T\mathfrak{V}$ denote the system of all sets $T \cap V$ for $V \in \mathfrak{V}$.

Further let $\mathfrak{A}_0$ denote the system of all subsets of E_1 expressible as a finite union of compact nondegenerate intervals. Now, $\mathfrak{A}$ stands for the system of all bounded sets $A \subset E_1$ such that there exists a $B \in \mathfrak{A}_0$ with $|(A - B) \cup (B - A)| = 0$. Given $A \in \mathfrak{A}$, there exists exactly one $B \in \mathfrak{A}_0$ possessing the above property; we put $\tilde{A} = B$ and $\|A\| = 2p$, where p is the number of components of $\tilde{A}$.

Now, let us define the convergence $\rightarrow$ on $\mathfrak{Z}$ as follows: $Z_n \rightarrow Z$ means that $Z_n \subset Z$, $Z - Z_n \in \mathfrak{A}$, $\sup_n \|Z - Z_n\| < \infty$, $|Z - Z_n| \rightarrow 0$. A system $\mathfrak{F} \subset \mathfrak{Z}$ will be called closed, if each limit of a sequence of sets of $\mathfrak{F}$ lies in $\mathfrak{F}$. Given $\mathfrak{V} \subset \mathfrak{Z}$, $\boldsymbol{u}(\mathfrak{V})$ denotes the minimal closed system containing $\mathfrak{V}$. The set functions under considerations are supposed to be finite and their continuity means the continuity with respect to $\rightarrow$.

For a set $M \subset E_1$ let us denote $\overline{M}$ the closure of M. Given an open set $G \subset E_1$, let us denote $\mathfrak{K}(G)$ the system of all $A \in \mathfrak{A}$ such that $\bar{A} \subset G$.

Let $\mathscr{F}$ be the system of all real- valued functions ($\pm\infty$ not excluded) whose domain of definition is a subset of E_1. With each $f \in \mathscr{F}$ we associate the system $\mathfrak{M}(f)$ of all measurable sets, on which the finite Lebesgue integral of f exists. A point $x \in E_1$ is said to be an L-regular point for $f \in \mathscr{F}$ if there exists a neighbourhood U of x such that $U \in \mathfrak{M}(f)$. The set of all L-regular points of f will be denoted by L_f; this set is evidently open. The Perron (or Lebesgue) integral of f over the set M will be denoted by $\int_M f$.

The function $f \in \mathscr{F}$ is said to be β-integrable on the set $A \in \mathfrak{A}$, if $A \in \boldsymbol{u}(\mathfrak{A} \cap \mathfrak{M}(f))$ and if there exists a continuous additive function φ defined on $A\mathfrak{A}$ such that $\varphi(B) = \int_B f$ for each $B \in \mathfrak{M}(f) \cap \mathfrak{A}$. The number $\varphi(A)$ will be denoted by $\beta(f, A)$.

3. Lemma. *Suppose that $A_n \in \mathfrak{A}$, $A_n \to A$, $\sup_n \|A_n\| = 2t$, $\|A\| = 2s$. Then $\tilde{A}_n \to \tilde{A}$ and the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ has at most $t + s$ points.*

Proof. It is easy to see that $A \subset B$ implies $\tilde{A} \subset \tilde{B}$; whence it follows immediately that $\tilde{A}_n \to \tilde{A}$.

Let us denote $y_1, \ldots, y_s$ the left endpoints of the components of the set $\tilde{A}$ and let H be the set of all these points. Let $x_1 < x_2 < \ldots < x_k$ be arbitrary points of the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n - H$ and let $x_0 = \min H$. Since $|\tilde{A} - \tilde{A}_n| \to \infty$, we can choose such n that $|\tilde{A} - \tilde{A}_n| < |A \cap \langle x_{l-1}, x_l \rangle|$ for $l = 1, \ldots, k$. Hence there exist components $I_1, \ldots, I_k$ of the set $\tilde{A}_n$ lying in the intervals $\langle x_0, x_1 \rangle, \ldots, \langle x_{k-1}, x_k \rangle$ respectively. It follows that $k \leqq t$, and the number of all points of the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ does not exceed $t + s$.

4. Lemma. *Given $Q \subset E_1$, let $\mathfrak{A}_Q$ denote the system of all sets $A \in \mathfrak{A}$ such that $\tilde{A} - Q$ is countable. Then the system $\mathfrak{A}_Q$ is closed.*

Proof. Suppose that $A_n \in \mathfrak{A}_Q$, $A_n \to A$. By the preceding lemma $\tilde{A}_n \to \tilde{A}$ and the set $\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n$ is finite. Then, by the inclusion $\tilde{A} - Q \subset (\tilde{A} - \bigcup_{n=1}^{\infty} \tilde{A}_n) \cup \bigcup_{n=1}^{\infty} (\tilde{A}_n - Q)$, $\tilde{A} - Q$ is countable, i.e. $A \in \mathfrak{A}_Q$.

5. Theorem. *Let G be an open subset of E_1. Then $A \in \boldsymbol{u}(\mathfrak{K}(G))$ if and only if $A \in \mathfrak{A}$ and $\tilde{A} - G$ is countable.*

Proof. a) Using the notation of the preceding lemma we have obviously $\mathfrak{K}(G) \subset \mathfrak{A}_G$ and by that lemma $\boldsymbol{u}(\mathfrak{K}(G)) \subset \mathfrak{A}_G$. This means that $A \in \mathfrak{A}$ and $\tilde{A} - G$ is countable for $A \in \boldsymbol{u}(\mathfrak{K}(G))$.

b) Suppose now that $A \in \mathfrak{A}$ and that $\tilde{A} - G$ is countable. Let us denote $\mathfrak{G}$ the system of all open sets $H \subset E_1$ with the following property: If $B \in \mathfrak{A}$, $\bar{B} \subset H$, then $A \cap B \in \boldsymbol{u}(\mathfrak{K}(G))$. We have:

(i) $G \in \mathfrak{G}$, $E_1 - \tilde{A} \in \mathfrak{G}$. (This is evident.)

(ii) $\bigcup_{H \in \mathfrak{G}_1} H \in \mathfrak{G}$ for $\mathfrak{G}_1 \subset \mathfrak{G}$. (This relation is a consequence of the following assertion: If $B \in \mathfrak{A}$, $\bar{B} \subset \bigcup_{H \in \mathfrak{G}_1} H$, then there exists a finite number of sets $B_i \in \mathfrak{A}$, $i = 1, \ldots, k$, such that $\bigcup_{i=1}^{k} B_i = B$, $\bar{B}_i \subset H_i$ for suitable $H_i \in \mathfrak{G}_1$.)

(iii) If $\alpha < \beta < \gamma$, $(\alpha, \beta) \in \mathfrak{G}$, $(\beta, \gamma) \in \mathfrak{G}$, then $(\alpha, \gamma) \in \mathfrak{G}$. (This is obvious.)

Let us put $H_0 = \bigcup_{H \in \mathfrak{G}} H$. According to (ii), $H_0 \in \mathfrak{G}$ and according to (i), $E_1 - H_0 \subset \subset \tilde{A} - G$. Hence the set $E_1 - H_0$ is a countable closed set without isolated points (see (iii)). It follows that $H_0 = E_1$ whence $A \in \boldsymbol{u}(\mathfrak{K}(G))$.

6. Lemma. *If* $f \in \mathscr{F}$, *then* $\boldsymbol{u}(\mathfrak{A} \cap \mathfrak{M}(f)) = \boldsymbol{u}(\mathfrak{K}(L_f))$.

Proof. The obvious inclusion $\mathfrak{K}(L_f) \subset \mathfrak{A} \cap \mathfrak{M}(f)$ implies $\boldsymbol{u}(\mathfrak{K}(L_f)) \subset \boldsymbol{u}(\mathfrak{A} \cap \cap \mathfrak{M}(f))$. Let $A \in \mathfrak{A} \cap \mathfrak{M}(f)$. Denoting $\langle a_i, b_i \rangle$, $i = 1, 2, \ldots, p$, the components of $\tilde{A}$, we have $(a_i, b_i) \in L_f$, whence $\langle a_i, b_i \rangle \in \boldsymbol{u}(\mathfrak{K}(L_f))$. Since $\boldsymbol{u}(\mathfrak{K}(L_f))$ is a set ring containing all bounded sets M with $|M| = 0$, it follows that $A \in \boldsymbol{u}(\mathfrak{K}(L_f))$. Hence $\boldsymbol{u}(\mathfrak{A} \cap \mathfrak{M}(f)) \subset \boldsymbol{u}(\mathfrak{K}(L_f))$ also holds.

7. Theorem. *Let* $I = \langle a, b \rangle$ *be a compact interval in* E_1.

a) *Let* φ *be an additive continuous function on* $I\mathfrak{A}$. *If we put* $f(x) = \varphi(\langle a, x \rangle)$ *for* $x \in I$, *then the function* f *is continuous on* I.

b) *Conversely, let* f *be a continuous function on* I. *If we put* $\varphi(A) = \sum_{j=1}^{p} (f(b_j) - - f(a_j))$ *for* $A \in I\mathfrak{A}$ *denoting* $\langle a_j, b_j \rangle$, $j = 1, 2, \ldots, p$, *the components of* $\tilde{A}$, *then the function* φ *is additive and continuous on* $I\mathfrak{A}$.

Proof. a) The continuity from the left of f is obvious and the continuity from the right follows from the formula $f(x) = \varphi(\langle a, b \rangle) - \varphi(\langle x, b \rangle)$.

b) The additivity of φ is evident. Suppose that $A_n \to A$, $A \subset I$ and $\sup_n \|A - A_n\| = = 2s$. Let ε be any positive number. There exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon/s$ for $x \in I$, $y \in I$, $|y - x| < \delta$. Further, there exists n_0 such that $|A - A_n| < \delta$ for $n \geqq n_0$. Hence $|\varphi(A) - \varphi(A_n)| = |\varphi(A - A_n)| < s(\varepsilon/s) = \varepsilon$ for $n \geqq n_0$. This proves the continuity of φ.

8. Theorem. *Let* G *be an open subset of* E_1 *and let* I *be a compact interval in* E_1. *Suppose that the set* $I - G$ *is countable. Let* F *and* f *be two functions on* I *such that* F *is continuous on* I *and is a Perron indefinite integral of* f *on each component of* G. *Then* F *is a Perron indefinite integral of* f *on* I.

Proof. Let ε be any positive number. Let (a_n, b_n), $n \in N$, be the components of G and let $I = \langle a, b \rangle$. By the well known theorem on Perron integration there exists the Perron integral $\int_{a_n}^{b_n} f = F(b_n) - F(a_n)$ for each $n \in N$. Let M_n be a majorant of f on $\langle a_n, b_n \rangle$ such that $M_n(b_n) - M_n(a_n) < F(b_n) - F(a_n) + \varepsilon/2^n$. Put $g_n(x) = 0$ for

$x < a_n$, $g_n(x) = M_n(x) - F(x) - M_n(a_n) + F(a_n)$ for $a_n \leqq x \leqq b_n$, $g_n(x) = g_n(b_n)$ for $x > b_n$ and $g = \sum_{n \in N} g_n$. Finally put $h(x) = \varepsilon \sum_k (1/2^k) \operatorname{sgn}(x - s_k)$, where $\{s_1, s_2, \ldots\} = I - G$. Then the function $M = F + g + h$ is a majorant of f on $\langle a, b \rangle$ such that $M(b) - M(a) < F(b) - F(a) + 3\varepsilon$. Using a similar construction we find a minorant m of f on $\langle a, b \rangle$ such that $m(b) - m(a) > F(b) - F(a) - 3\varepsilon$. Hence there exists the Perron integral $\int_a^b f = F(b) - F(a)$.

9. Theorem. *Suppose that $f \in \mathscr{F}$, $I = \langle a, b \rangle$. Then $\beta(f, I)$ exists if and only if there exists the Perron integral $\int_a^b f$ and the set $I - L_f$ is countable. In this case $\beta(f, I) = \int_a^b f$.*

Proof. a) Suppose that $\beta(f, I)$ exists. By the definition of β-integral and by Lemma 6 we have $I \in \boldsymbol{u}(\mathfrak{K}(L_f))$, so that by Theorem 5 the set $I - L_f$ is countable. By Theorem 7 the function F, $F(x) = \beta(f, \langle a, x \rangle)$ for $x \in \langle a, b \rangle$, is continuous on $\langle a, b \rangle$. Now, we can apply Theorem 8 with $G = L_f$. Hence F is an indefinite Perron integral of f on $\langle a, b \rangle$ and $\int_a^b f = F(b) - F(a) = \beta(f, I)$.

b) Conversely, suppose that the Perron integral $\int_a^b f$ exists and the set $I - L_f$ is countable. By Theorem 5 and Lemma 6 we have $I \in \boldsymbol{u}(\mathfrak{K}(L_f)) = \boldsymbol{u}(\mathfrak{A} \cap \mathfrak{M}(f))$. For $A \in I\mathfrak{A}$ let us put $\varphi(A) = \sum_{j=1}^{p} (F(b_j) - F(a_j))$, where $\langle a_j, b_j \rangle$, $j = 1, 2, \ldots, p$, are the components of $\tilde{A}$ and $F(x) = \int_a^x f$. By Theorem 7 the function φ is an additive continuous function on $I\mathfrak{A}$. Since $\varphi(A) = \int_A f$ for $A \in \mathfrak{M}(f) \cap I\mathfrak{A}$, there exists $\beta(f, I) = \varphi(I) = \int_a^b f$.

References

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