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DISTINGUISHED SETS OF IDEALS OF A RING

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1. In any category $\mathfrak{A}$ (for convenience, with a zero element 0) one can define a *filter subcategory* in a quite general manner as a (full) subcategory $\mathfrak{F} \subseteq \mathfrak{A}$ possessing the following two properties:

(i) *Any subobject of an object of $\mathfrak{F}$ belongs to $\mathfrak{F}$.*

(ii) *The class of all subobjects belonging to $\mathfrak{F}$ of an object $A \in \mathfrak{A}$ has a greatest element $(\mathfrak{F}(A), \mu_A)$.*

Furthermore, by a *radical filter subcategory* $\mathfrak{R}$ of $\mathfrak{A}$ one can understand a filter of $\mathfrak{A}$ satisfying the additional property

(iii) *If*

$$(0 \to) \quad \mathfrak{R}(A) \overset{\mu_A}{\to} A \to B \to 0$$

is an exact sequence, then always $\mathfrak{R}(B) - 0$.

Such or similar concepts appear to be useful in some specified categories (see e.g. GABRIEL [2], HELZER [3]). Our intention is to study the filters and radical filters in the category ***Mod R*** of all R-modules (left unital modules over an associative ring R with unity). The latter amounts to the study of certain subsets of the set $\mathscr{L}$ of all proper (i.e. $\neq R$) left ideals of R (see [1] and [2]).

Following the terminology and notation of [1], a subfamily $\mathscr{K}$ of the family $\mathscr{L}$ is called a *Q-set* if

$$\text{(Q)} \qquad K \in \mathscr{K} \ \wedge \ \varrho \in R \setminus K \to K : \varrho \in \mathscr{K} \, .$$

Here, $K : \varrho$ denotes the (right) ideal-quotient of K by ϱ, i.e. the left ideal of all $\chi \in R$ such that $\chi\varrho \in K$. If, besides (Q), the set $\mathscr{K}$ satisfies

$$\text{(E)} \qquad K \subseteq L \wedge K \in \mathscr{K} \ \wedge \ L \in \mathscr{L} \to L \in \mathscr{K}$$

and

$$\text{(I)} \qquad K_1 \in \mathscr{K} \ \wedge \ K_2 \in \mathscr{K} \to K_1 \cap K_2 \in \mathscr{K} \, ,$$

$\mathscr{K}$ is said to be an *F-set* (topological set of [2]). Moreover, an F-set $\mathscr{K}$ is called the *R-set* (idempotent topological set of [2]) if

$$\text{(R)} \qquad L \in \mathscr{L} \wedge \exists K[K \in \mathscr{K} \wedge \forall \varkappa(\varkappa \in K \setminus L \to L : \varkappa \in \mathscr{K})] \to L \in \mathscr{K} .$$

Denoting, for an F-set $\mathscr{K}$, by $\mathfrak{M}_{\mathscr{K}}$ the class of all *$\mathscr{K}$-modules* ($\mathscr{K}$-neglectable modules of [2]), i.e. of all R-modules M such that the order $O(m)$ of every non-zero element $m \in M$ belongs to $\mathscr{K}$, we can express the above mentioned relation between filters in **Mod R** and sets of left ideals of R very simply (cf. [1] and [2]);

(a) *If $\mathscr{K}$ is an F-set, then $\mathfrak{M}_{\mathscr{K}}$ is a filter in* **Mod R**. *On the other hand, if $\mathfrak{F}$ is a filter in* **Mod R**, *then an F-set $\mathscr{K}(\mathfrak{F})$ exists such that $\mathfrak{F} = \mathfrak{M}_{\mathscr{K}(\mathfrak{F})}$ (here, $K \in \in \mathscr{K}(\mathfrak{F}) \leftrightarrow R \bmod K \in \mathfrak{F}$). In particular, a filter in* **Mod R** *is closed under taking quotients, direct sums and inductive limits.*

(b) *If $\mathscr{K}$ is an R-set, then $\mathfrak{M}_{\mathscr{K}}$ is a radical filter in* **Mod R**. *On the other hand, if $\mathfrak{R}$ is a radical filter in* **Mod R**, *then an R-set $\mathscr{K}(\mathfrak{R})$ exists such that $\mathfrak{R} = \mathfrak{M}_{\mathscr{K}(\mathfrak{R})}$.*

This one-to-one correspondence between filters, or radical filters in **Mod R** and F-sets, or R-sets of left ideals of R, respectively, enables us to investigate the sets of all filters and all radical filters in **Mod R** through the sets of all F- and R-sets. A description of the lattice of all F- and R-sets is derived in the framework of Q-sets and their equivalence classes (see [1]) in the next § 2. In particular, all equivalent F-sets form a lattice with the greatest element, which is a uniquely determined R-set in the respective equivalence class (Theorem 2.7). The example in § 3 shows that this R-set need not be necessarily the greatest element of its equivalence class. Finally, in the last § 4 the results are used to give a simple characterization of the lattice of all R-sets (and thus of all radical filters of modules) in terms of certain sets of prime ideals in the case of a commutative noetherian ring.

2. In what follows, R stands always for a given (fixed) ring and $\mathscr{L}$ for the set of all its proper left ideals. The empty set $\emptyset$ is assumed to be a Q- (as well as, F- and R-) set.

Observing that a (set-theoretical) intersection of Q- or F- or R-sets is again a Q- or F- or R-set, respectively, we deduce immediately the following

Theorem 2.1. *All the Q-sets $\mathscr{K} \subseteq \mathscr{L}$ form (with respect to order by inclusion) a complete sublattice* **Q** *of the lattice* **L** *of all the sets of proper left ideals of R (with the set-theoretical operations $\cap$ and $\cup$, the greatest element $\mathscr{L}$ and the least one $\emptyset$).*

All the F-sets $\mathscr{K} \subseteq \mathscr{L}$ form (with respect to order by inclusion) a complete lattice **F** *with the operations $\bigwedge^{F}_{\omega} \mathscr{K}_\omega = \bigcap_\omega \mathscr{K}_\omega$ and $\bigvee^{F}_{\omega} \mathscr{K}_\omega$, in general different from $\bigcup_\omega \mathscr{K}_\omega$.*

All the R-sets $\mathscr{K} \subseteq \mathscr{L}$ form (with respect to order by inclusion) a complete lattice **R** *with the operations $\bigwedge^{R}_{\omega} \mathscr{K}_\omega = \bigcap_\omega \mathscr{K}_\omega$ and $\bigvee^{R}_{\omega} \mathscr{K}_\omega$, in general different from $\bigcup_\omega \mathscr{K}_\omega$.*

The sets $\mathscr{L}$ and $\emptyset$ are the greatest and the least element of both **F** *and* **R**, *respectively.*

In order to describe the set $\bigvee_{\omega}^{\mathsf{F}} \mathscr{K}_{\omega}$, let us formulate first the following

Lemma 2.2. *Let $\mathscr{K}_{\omega}$ $(\omega \in \Omega)$ be Q-sets. Then the set $\mathscr{K}_{\Omega}^{\vee} \subseteq \mathscr{L}$ defined by*

$$(\vee) \qquad L \in \mathscr{K}_{\Omega}^{\vee} \leftrightarrow L \in \mathscr{L} \wedge L \supseteq \bigcap_{1 \leqq i \leqq n} K_i \wedge K_i \in \bigcup_{\omega} \mathscr{K}_{\omega}$$

is an F-set.

The proof is straightforward and we therefore omit it. Apart from the fact that Lemma may be found useful for constructing new F-sets, we get also immediately (since, obviously, $\mathscr{K}_{\Omega}^{\vee} \subseteq \bigvee_{\omega}^{\mathsf{F}} \mathscr{K}_{\omega}$),

Theorem 2.3. *Let $\mathscr{K}_{\omega}$ $(\omega \in \Omega)$ be F-sets. Then*

$$\mathscr{K}_{\Omega}^{\vee} = \bigvee_{\omega}^{\mathsf{F}} \mathscr{K}_{\omega}.$$

The following theorem establishes a procedure of extending a given F-set.

Theorem 2.4. *Let $\mathscr{K}$ be an F-set. Then the set $\mathscr{K}^*$ defined by*

$$(*) \qquad L \in \mathscr{K}^* \leftrightarrow L \in \mathscr{L} \wedge \exists K[K \in \mathscr{K} \wedge \forall \varkappa(\varkappa \in K \setminus L \rightarrow K : \varkappa \in \mathscr{K})]$$

contains $\mathscr{K}$ and is an F-set, as well. Here, $\mathscr{K} = \mathscr{K}^*$ if and only if $\mathscr{K}$ is an R-set.

Proof. The inclusion $\mathscr{K} \subseteq \mathscr{K}^*$ is obvious (take e.g. $K = L$ in $(*)$). Also, for $\mathscr{K} = \emptyset$ evidently $\mathscr{K}^* = \emptyset$. Thus, assume $\mathscr{K} \neq \emptyset$.

Let $L \in \mathscr{K}^*$ and $\varrho \in R \setminus L$. If $\varrho \in K$ of $(*)$, then $L : \varrho \in \mathscr{K} \subseteq \mathscr{K}^*$. If $\varrho \notin K$, then

$$\forall \varkappa[\varkappa \in (K : \varrho) \setminus (L : \varrho) \rightarrow (L : \varrho) : \varkappa \in \mathscr{K});$$

therefore, $L : \varrho \in \mathscr{K}^*$ again. Hence, $\mathscr{K}^*$ satisfies (Q). The other properties (E) and (I) can be proved in a similar routine manner.

Now, in [1] an equivalence has been defined on **Q** in the following way: Define, for $\mathscr{K} \in \mathbf{Q}$, the "*closure*" $\mathsf{c}(\mathscr{K}) \in \mathbf{Q}$ by

$$(\mathsf{c}) \qquad L \in \mathsf{c}(\mathscr{K}) \leftrightarrow L \in \mathscr{L} \wedge \forall \varrho[\varrho \in R \setminus L \rightarrow \exists \sigma(\sigma \in R \wedge L : \sigma\varrho \in \mathscr{K})].$$

Then, two Q-sets $\mathscr{K}_1$ and $\mathscr{K}_2$ are said to be *equivalent* (in symbol, $\mathscr{K}_1 \approx \mathscr{K}_2$) if

$$\mathsf{c}(\mathscr{K}_1) = \mathsf{c}(\mathscr{K}_2).$$

The equivalence $\approx$ induces, of course, an equivalence (denoted again by $\approx$) on **F** and **R**.

In order to prove the main result of the paper we shall need the following two lemmas.

Lemma 2.5. (i) *For any two Q-sets* $\mathscr{K}_1, \mathscr{K}_2$ *always*

$$\mathbf{c}(\mathscr{K}_1) \cap \mathbf{c}(\mathscr{K}_2) = \mathbf{c}(\mathscr{K}_1 \cap \mathscr{K}_2) .$$

Hence, if $\mathscr{K}'_1 \approx \mathscr{K}_1$ *and* $\mathscr{K}'_2 \approx \mathscr{K}_2$, *then*

$$\mathscr{K}'_1 \cap \mathscr{K}'_2 \approx \mathscr{K}_1 \cap \mathscr{K}_2 .$$

In particular, if $\mathscr{K}_1 \approx \mathscr{K}_2$, *then*

$$\mathscr{K}_1 \cap \mathscr{K}_2 \approx \mathscr{K}_1 .$$

(ii) *For any Q-sets* $\mathscr{K}_\omega (\omega \in \Omega)$ *satisfying* (E), *always*

$$\mathscr{K}_\Omega^\vee \subseteq \mathbf{c}\big(\bigcup_\omega \mathscr{K}_\omega\big) ,$$

i.e.

$$\mathscr{K}_\Omega^\vee \approx \bigcup_\omega \mathscr{K}_\omega .$$

Also, if $\mathscr{K}'_\omega \approx \mathscr{K}_\omega$ $(\omega \in \Omega)$, *then*

$$\bigcup_\omega \mathscr{K}'_\omega \approx \bigcup_\omega \mathscr{K}_\omega ,$$

and hence,

$$\mathscr{K}'^\vee_\Omega \approx \mathscr{K}^\vee_\Omega .$$

In particular, if $\mathscr{K}_\omega$ *and* $\mathscr{K}'_\omega$ *are F-sets, then*

$$\bigvee_\omega^{\mathsf{F}} \mathscr{K}'_\omega \approx \bigvee_\omega^{\mathsf{F}} \mathscr{K}_\omega ;$$

thus, if $\mathscr{K}_\omega \approx \mathscr{K}$ *for all* ω, *then* $\bigvee_\omega^{\mathsf{F}} \mathscr{K}_\omega \approx \mathscr{K}$.

(iii) *For an F-set* $\mathscr{K}$, *always* $\mathscr{K}^* \approx \mathscr{K}$.

Proof. (i) The equality $\mathbf{c}(\mathscr{K}_1) \cap \mathbf{c}(\mathscr{K}_2) = \mathbf{c}(\mathscr{K}_1 \cap \mathscr{K}_2)$ follows readily from the definition ($\mathbf{c}$). Thus,

$$\mathbf{c}(\mathscr{K}'_1 \cap \mathscr{K}'_2) = \mathbf{c}(\mathscr{K}'_1) \cap \mathbf{c}(\mathscr{K}'_2) = \mathbf{c}(\mathscr{K}_1) \cap \mathbf{c}(\mathscr{K}_2) = \mathbf{c}(\mathscr{K}_1 \cap \mathscr{K}_2) .$$

(ii) By ($\vee$), for $K \in \mathscr{K}_\Omega^\vee$ there are $K_i \in \bigcup_\omega \mathscr{K}_\omega$, $1 \leqq i \leqq n$ such that $K \supseteq \bigcap_{1 \leqq i \leqq n} K_i$. Hence, for an arbitrary $\varrho \in R \setminus K$, there is either $K : \varrho \supseteq K_n : \varrho$, i.e. $K : \varrho \in \bigcup_\omega \mathscr{K}_\omega$, or

$$K : \sigma_n \varrho \supseteq \bigcap_{1 \leqq i \leqq n-1} K_i : \sigma_n \varrho \quad \text{for} \quad \sigma_n \in (K_n : \varrho) \setminus (K : \varrho) .$$

Proceeding by induction, we can easily find σ such that $K : \sigma\varrho \in \bigcup_\omega \mathscr{K}_\omega$; therefore,

$\check{\mathscr{K}}_\Omega \subseteq \mathbf{c}(\bigcup_\omega \mathscr{K}_\omega)$. Since, on the other hand, $\check{\mathscr{K}}_\Omega \supseteq \bigcup_\omega \mathscr{K}_\omega$ we conclude that

$$\check{\mathscr{K}}_\Omega \approx \bigcup_\omega \mathscr{K}_\omega .$$

The rest of (ii) is trivial.

(iii) Also the assertion of (iii) follows again easily from the definition $(*)$ of $\mathscr{K}^*$.

Lemma 2.6. *Let $\mathscr{K}$ be a Q-set satisfying* (R). *Then, for any $L \in \mathbf{c}(\mathscr{K}) \setminus \mathscr{K}$, there is a proper (left) ideal L_0 of R which contains L and does not belong to $\mathbf{c}(\mathscr{K})$. Thus, in particular, if $\mathscr{K}$ is an R-set, and $\mathscr{K}_0$ an F-set satisfying $\mathscr{K} \subseteq \mathscr{K}_0 \subseteq \mathbf{c}(\mathscr{K})$, then $\mathscr{K} = \mathscr{K}_0$.*

Proof. Define L_0 as follows:

$$(\mathrm{o}) \qquad \varrho \in L_0 \leftrightarrow \varrho \in L \vee L : \varrho \in \mathscr{K} .$$

Clearly, L_0 is a proper left ideal of R containing properly L and, moreover, necessarily $L_0 \notin \mathbf{c}(\mathscr{K})$. For, otherwise, $L_0 \in \mathbf{c}(\mathscr{K})$ implies that there is $\varrho_0 \in R \setminus L_0$ such that $L_0 : \varrho_0 \in \mathscr{K}$ and then, for every $\varkappa \in (L_0 : \varrho_0) \setminus (L : \varrho_0)$, i.e. for every $\varkappa$ such that $\varkappa\varrho_0 \in L_0 \setminus L$,

$$L : \varkappa\varrho_0 = (L : \varrho_0) : \varkappa \in \mathscr{K}$$

in view of (o). Therefore, by (R), $L : \varrho_0 \in \mathscr{K}$ and thus, by (o) again, $\varrho_0 \in L_0$ — a contradiction of our choice of ϱ_0.

Now, the main result of this paragraph follows as a consequence of Lemmas 2.5 and 2.6 and Theorems 2.1 and 2.4:

Theorem 2.7. *For any $\mathscr{K} \in \mathbf{Q}$, the equivalence class $\mathbf{C}(\mathscr{K})$ of all Q-sets equivalent to $\mathscr{K}$ is a convex sublattice of $\mathbf{Q}$ with infinite joins and the greatest element $\mathbf{c}(\mathscr{K})$.*

If $\mathbf{F}_{\mathbf{C}(\mathscr{K})} = \mathbf{C}(\mathscr{K}) \cap \mathbf{F} \neq \emptyset$, then it forms (with respect to order by inclusion) a lattice with meets equal to set-theoretical intersections and with infinite joins; denote the greatest element of $\mathbf{F}_{\mathbf{C}(\mathscr{K})}$ by $\tilde{\mathscr{K}}$.

Since $(\tilde{\mathscr{K}})^* = \tilde{\mathscr{K}}$, $\tilde{\mathscr{K}}$ is an R-set. This means, in particular, that for any F-set, there exists an equivalent R-set.

As a matter of fact, for an R-set $\mathscr{K}$, always $\mathscr{K} = \tilde{\mathscr{K}}$ and hence, $\tilde{\mathscr{K}}$ is the only R-set belonging to $\mathbf{F}_{\mathbf{C}(\mathscr{K})}$.

In this way, a one-to-one correspondence (in fact, a lattice homomorphism) is established between the lattice $\mathbf{R}$ of all R-sets and the lattice of all equivalence classes $\mathbf{C}(\mathscr{K})$ which contain an F-set.

3. In [1], we have proved that $\mathbf{c}(\mathscr{K})$ is an R-set, i.e. that $\tilde{\mathscr{K}} = \mathbf{c}(\mathscr{K})$, provided that $\mathbf{c}(\mathscr{K})$ contains all (left) essential ideals of R. Recall that $L \in \mathscr{L}$ is said to be essential (in R) if the zero ideal is the only left ideal intersecting L trivially. It is therefore quite natural to raise the question whether, for any $\mathscr{K}$ such that $\mathsf{F}_{C(\mathscr{K})} \neq \emptyset$, always $\tilde{\mathscr{K}} =$ $= \mathsf{c}(\mathscr{K})$. The following example will answer the question in negative:

Denote by $\mathscr{S} \subseteq \mathscr{L}$ the set of all *strong* ideals in R, i.e. of all essential idelas L such that

$$\forall \varrho, \sigma(\varrho \in R \setminus L \wedge \sigma \neq o \rightarrow L : \varrho \nsubseteq \{0\} : \sigma).$$

It is easy to check that $\mathscr{S}$ is an F-set. In fact, $\mathscr{S}$ is an R-set. For, assume that $L : \varkappa \in \mathscr{S}$ for all $\varkappa \in K \setminus L$ with $K \in \mathscr{S}$ and yet that elements ϱ_0 and $\sigma_0 \neq 0$ of R exists such that

$$L : \varrho_0 \subseteq \{0\} : \sigma_0 .$$

Then, necessarily $\varrho_0 \notin K$ and $\chi \in K : \varrho_0$ implies either $\chi \in L : \varrho_0$ or $\chi\sigma_0 = 0$. Hence, $K : \varrho_0 \subseteq \{0\} : \sigma_0$, a contradiction of $K \in \mathscr{S}$.

Consider the ring R^* of all pairs (n, r) of $n \in Z$ (integers) and $r \in Q$ (rational numbers) with the component-wise addition and the multiplication defined by

$$(n_1, r_1)(n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1).$$

The subset R^0 of R^* of all pairs $(0, r)$, $r \in Q$, is obviously an ideal of R^*. The ideals of R^* which are contained in R^0 are in one-to-one correspondence φ to the subgroups G of the additive group of all rational numbers:

$$\varphi(G) = I_G = \{(0, r)\}_{r \in G} .$$

All the remaining ideals of R^* contain R^0 and are in one-to-one correspondence ψ to non-zero subgroups $\langle k \rangle$, $k > 0$, $k \in Z$ of the additive group of integers:

$$\psi(\langle k \rangle) = I_k = \{(n, r)\}_{n \in \langle k \rangle, r \in Q} .$$

Hence, any non-zero ideal is essential in R^*. There are only two annihilator ideals, viz. $\{0\}$ and R^0. Consequently, $\mathscr{S} = \{I_k\}_{k > 1, k \in Z}$. Furthermore,

$$\mathsf{c}(\mathscr{S}) = \mathscr{L} \setminus \{\{0\}, R^0\} \neq \tilde{\mathscr{S}} = \mathscr{S} .$$

For, if $(0, r) \notin I_G$, then

$$I_G : (0, r) = I_k \in \mathscr{S} ,$$

where k is the least natural member such that $kr \in G$. And, for $(n, r) \in R^*$ with $n \neq 0$, there is $s \in Q$ such that $ns \notin G$ and

$$(I_G : (n, r)) : (0 : s) = I_G : (0, ns) \in \mathscr{S}$$

again. Finally, $\{0\} : (0, r) = R^0$ for every $r \neq 0$, and $R^0 : (n, r) = R^0$ for every $n \neq 0$.

4. In this final paragraph, we are going to establish – in the case of a commutative noetherian ring R – a simple characterization of R-sets in terms of prime ideals.

Let, for a moment, R be an arbitrary ring and

$$\mathscr{P} = \{P_\omega \mid \omega \in \Omega\}$$

the set of all proper two-sided (strictly) prime ideals, i.e. the set of all ideals P_ω such that

$$P_\omega = P_\omega : \varrho \quad \text{for any} \quad \varrho \in R \setminus P .$$

Let us observe that an intersection $K = P_1 \cap P_2$ with $P_i \neq K$, $P_i \in \mathscr{P}$ $(i = 1, 2)$ no longer belongs to $\mathscr{P}$ (cf. next Lemma 4.1 (c)); this follows from the fact that $K : \varrho = P_i : \varrho = P_i$ for any $\varrho \in K \setminus P_i$. We shall call a subset $\mathscr{Q}$ of $\mathscr{P}$ a *filter subset* if

$$P_1 \in \mathscr{Q} \wedge P_2 \in \mathscr{P} \wedge P_1 \subseteq P_2 \to P_2 \in \mathscr{Q} .$$

Furthermore, for any Q-set $\mathscr{K}$ define the Q-subset $p(\mathscr{K})$ by

$$p(\mathscr{K}) = \mathscr{K} \cap \mathscr{P} .$$

Lemma 4.1. (a) *Let $\mathscr{K}_1 \approx \mathscr{K}_2$ be two equivalent Q-sets. Then $p(\mathscr{K}_1) = p(\mathscr{K}_2)$.*

(b) *If $\mathscr{K}$ satisfies (E), – in particular, if it is an F-set, then $p(\mathscr{K})$ is a filter subset of $\mathscr{P}$.*

(c) *If $\mathscr{Q}$ is a filter subset of $\mathscr{P}$, then the F-set $\mathscr{Q}^\vee$ (defined in Lemma 2.2) satisfies*

$$p(\mathscr{Q}^\vee) = p(\mathscr{Q}) = \mathscr{Q} .$$

(d) *As a consequence, for any filter subset $\mathscr{Q}$ of $\mathscr{P}$ there exists an R-set $\mathscr{Q}^r$ $(= \widetilde{\mathscr{Q}}^\vee)$ such that*

$$p(\mathscr{Q}^r) = p(\mathscr{Q}) = \mathscr{Q} .^{1})$$

Proof. (a) Let $K \in p(\mathscr{K}_1) = \mathscr{K}_1 \cap \mathscr{P}$. Then, for a suitable $\varrho \in R \setminus K$, $K : \varrho \in \mathscr{K}_2$. Also, $K : \varrho = K$. Hence, $K \in p(\mathscr{K}_2)$, as required.

(b) Trivial.

(c) Only the proof of $p(\mathscr{Q}^\vee) \subseteq \mathscr{Q}$ is needed. Let $K \in \mathscr{Q}^\vee$, i.e.

$$K \supseteq \bigcap_{1 \leq i \leq n} K_i \quad \text{with} \quad K_i \in \mathscr{Q} .$$

If $K \supseteq K_i$ for a suitable i, then evidently $K \in \mathscr{Q}$ whenever $K \in p(\mathscr{Q}^\vee)$. Otherwise, there is $2 \leq m \leq n$ such that

$$K \supseteq \bigcap_{1 \leq i \leq m} K_i \quad \text{and} \quad K \not\supseteq \bigcap_{1 \leq i \leq m-1} K_i .$$

And then, for $\varrho \in K_m \setminus K$,

$$K : \varrho \supseteq \bigcap_{1 \leq i \leq m} (K_i : \varrho) = \bigcap_{1 \leq i \leq m-1} (K_i : \varrho) = \bigcap_{1 \leq i \leq m-1} K_i .$$

Hence, $K : \varrho \neq K$, i.e. $K \notin \mathscr{P}$.

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[1]) Here, with some additional conditions imposed on $\mathscr{Q}$ we can assert that $\mathscr{Q}^r \approx \mathscr{Q}$; e.g. this is the case when $\mathscr{Q}$ consists of two-sided ideals of R maximal (as left ideals) in R.

(d) The assertion follows immediately from Theorem 2.7 and (a) of this lemma.

Now, in the remaining part of the paper, let R stand for a commutative noetherian ring. One of the important features of such a ring is that, for any proper ideal L of R, there exists $\varrho \in R \setminus L$ such that $L : \varrho$ is prime. Hence, we can formulate the following

Lemma 4.2. *For any Q-set $\mathscr{K}$, always*

$$\mathscr{K} \approx p(\mathscr{K}) .$$

Consequently, the equality $p(\mathscr{K}_1) = p(\mathscr{K}_2)$ for two Q-sets $\mathscr{K}_1$ and $\mathscr{K}_2$ implies that $\mathscr{K}_1 \approx \mathscr{K}_2$.

The characterization of the set of all R-sets then reads as follows (cf. the case of integers in [1]):

Theorem 4.3. *Let R be a commutative noetherian ring. Then, for any filter subset $\mathscr{Q}$ of $\mathscr{P}$, there exists a unique R-set $\mathscr{Q}^r$ such that*

$$p(\mathscr{Q}^r) = \mathscr{Q} .$$

In fact, $\mathscr{Q}^r = \tilde{\mathscr{Q}}^{\vee}$ and thus, $\mathscr{Q}^r \approx \mathscr{Q}^{\vee}$. Moreover, if

$$\mathscr{Q}_1 \subseteq \mathscr{Q}_2 \subseteq \mathscr{P}$$

are two filter subsets, then

$$\mathscr{Q}_1^r \subseteq \mathscr{Q}_2^r .$$

As a consequence, there is a one-to-one correspondence between all R-sets (and thus, all radical filters in **Mod R**) *and all filter subsets of prime ideals; more precisely, the lattice of all R-sets* **R** *and the complete sublattice* **P** *of* **L** *(with set-theoretical operations) of all filter subsets of prime ideals are isomorphic. In addition, the set of all minimal R-sets (atoms of* **R**) *corresponds to the set of all singletons $\mathscr{Q} = \{P\} \in$* **P**, *where P is a maximal ideal of R. In this case, as well as in the more general case when $\mathscr{Q}$ consists of maximal ideals of R only, $\mathscr{Q}^r \approx \mathscr{Q}$.*

Proof. Lemma 4.1 yields the existence and Lemma 4.2 together with Theorem 2.7 the uniqueness of $\mathscr{Q}^r$. Also, if $\mathscr{Q}_1 \subseteq \mathscr{Q}_2$, then $\mathscr{K} = \mathscr{Q}_1^r \cap \mathscr{Q}_2^r$ is an R-set such that $p(\mathscr{K}) = \mathscr{Q}_1$ and, thus, $\mathscr{K} = \mathscr{Q}_1^r$, i.e. $\mathscr{Q}_1^r \subseteq \mathscr{Q}_2^r$, as required. The final assertion $\mathscr{Q}^r \approx \mathscr{Q}$ follows immediately from Lemma 2.5 (ii), because $\mathscr{Q}$ satisfies in this case (E).

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