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DISTINGUISHED SETS OF IDEALS OF A RING

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1. In any category \( \mathcal{A} \) (for convenience, with a zero element 0) one can define a filter subcategory in a quite general manner as a (full) subcategory \( \mathcal{F} \subseteq \mathcal{A} \) possessing the following two properties:

(i) Any subobject of an object of \( \mathcal{F} \) belongs to \( \mathcal{F} \).

(ii) The class of all subobjects belonging to \( \mathcal{F} \) of an object \( A \in \mathcal{A} \) has a greatest element \( (\mathcal{F}(A), \mu_A) \).

Furthermore, by a radical filter subcategory \( \mathcal{R} \) of \( \mathcal{A} \) one can understand a filter of \( \mathcal{A} \) satisfying the additional property

(iii) If

\[
(0 \to) \quad \mathcal{R}(A) \xrightarrow{\mu_A} A \to B \to 0
\]

is an exact sequence, then always \( \mathcal{R}(B) = 0 \).

Such or similar concepts appear to be useful in some specified categories (see e.g. GABRIEL [2], HELZER [3]). Our intention is to study the filters and radical filters in the category \( \text{Mod} R \) of all \( R \)-modules (left unital modules over an associative ring \( R \) with unity). The latter amounts to the study of certain subsets of the set \( \mathcal{L} \) of all proper (i.e. \( \neq R \)) left ideals of \( R \) (see [1] and [2]).

Following the terminology and notation of [1], a subfamily \( \mathcal{K} \) of the family \( \mathcal{L} \) is called a \( Q \)-set if

\[ K \in \mathcal{K} \land q \in R \setminus K \to K : q \in \mathcal{K} . \]

Here, \( K : q \) denotes the (right) ideal-quotient of \( K \) by \( q \), i.e. the left ideal of all \( \chi \in R \) such that \( \chi q \in K \). If, besides (Q), the set \( \mathcal{K} \) satisfies

\[ K \subseteq L \land K \in \mathcal{K} \land L \in \mathcal{L} \to L \in \mathcal{K} \]

and

\[ K_1 \in \mathcal{K} \land K_2 \in \mathcal{K} \to K_1 \cap K_2 \in \mathcal{K}, \]

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is said to be an $F$-set (topological set of [2]). Moreover, an $F$-set $\mathcal{K}$ is called the
$R$-set (idempotent topological set of [2]) if

$$(R) \quad L \in \mathcal{L} \land \exists K \in \mathcal{K} \land \forall x \in K \setminus L : x \in \mathcal{K} \to L \in \mathcal{K}.$$

Denoting, for an $F$-set $\mathcal{K}$, by $\mathbb{M}_\mathcal{K}$ the class of all $\mathcal{K}$-modules ($\mathcal{K}$-neglectable modules of [2]), i.e. of all $R$-modules $M$ such that the order $O(m)$ of every non-zero element $m \in M$ belongs to $\mathcal{K}$, we can express the above mentioned relation between filters in $\textbf{Mod} \ R$ and sets of left ideals of $R$ very simply (cf. [1] and [2]):

(a) If $\mathcal{K}$ is an $F$-set, then $\mathbb{M}_\mathcal{K}$ is a filter in $\textbf{Mod} \ R$. On the other hand, if $\mathcal{F}$ is a filter in $\textbf{Mod} \ R$, then an $F$-set $\mathcal{K}(\mathcal{F})$ exists such that $\mathcal{F} = \mathbb{M}_\mathcal{K}(\mathcal{F})$ (here, $K \in \mathcal{K}(\mathcal{F}) \leftrightarrow R \mod K \in \mathcal{F}$). In particular, a filter in $\textbf{Mod} \ R$ is closed under taking quotients, direct sums and inductive limits.

(b) If $\mathcal{K}$ is an $R$-set, then $\mathbb{M}_\mathcal{K}$ is a radical filter in $\textbf{Mod} \ R$. On the other hand, if $\mathcal{R}$ is a radical filter in $\textbf{Mod} \ R$, then an $R$-set $\mathcal{K}(\mathcal{R})$ exists such that $\mathcal{R} = \mathbb{M}_\mathcal{K}(\mathcal{R})$.

This one-to-one correspondence between filters, or radical filters in $\textbf{Mod} \ R$ and $F$-sets, or $R$-sets of left ideals of $R$, respectively, enables us to investigate the sets of all filters and all radical filters in $\textbf{Mod} \ R$ through the sets of all $F$- and $R$-sets. A description of the lattice of all $F$- and $R$-sets is derived in the framework of $Q$-sets and their equivalence classes (see [1]) in the next § 2. In particular, all equivalent $F$-sets form a lattice with the greatest element, which is a uniquely determined $R$-set in the respective equivalence class (Theorem 2.7). The example in § 3 shows that this $R$-set need not be necessarily the greatest element of its equivalence class. Finally, in the last § 4 the results are used to give a simple characterization of the lattice of all $R$-sets (and thus of all radical filters of modules) in terms of certain sets of prime ideals in the case of a commutative noetherian ring.

2. In what follows, $R$ stands always for a given (fixed) ring and $\mathcal{L}$ for the set of all its proper left ideals. The empty set $\emptyset$ is assumed to be a $Q$- (as well as, $F$- and $R$-) set.

Observing that a (set-theoretical) intersection of $Q$- or $F$- or $R$-sets is again a $Q$- or $F$- or $R$-set, respectively, we deduce immediately the following

**Theorem 2.1.** All the $Q$-sets $\mathcal{K} \subseteq \mathcal{L}$ form (with respect to order by inclusion) a complete sublattice $Q$ of the lattice $L$ of all the sets of proper left ideals of $R$ (with the set-theoretical operations $\cap$ and $\cup$, the greatest element $\mathcal{L}$ and the least one $\emptyset$).

All the $F$-sets $\mathcal{K} \subseteq \mathcal{L}$ form (with respect to order by inclusion) a complete lattice $F$ with the operations $\bigwedge_\mathcal{K} \mathcal{K} = \bigcap_\mathcal{K} \mathcal{K}$ and $\bigvee_\mathcal{K} \mathcal{K}$, in general different from $\bigcup_\mathcal{K} \mathcal{K}$.

All the $R$-sets $\mathcal{K} \subseteq \mathcal{L}$ form (with respect to order by inclusion) a complete lattice $R$ with the operations $\bigwedge_\mathcal{K} \mathcal{K} = \bigcap_\mathcal{K} \mathcal{K}$ and $\bigvee_\mathcal{K} \mathcal{K}$, in general different from $\bigcup_\mathcal{K} \mathcal{K}$.

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The sets \( L \) and \( \emptyset \) are the greatest and the least element of both \( F \) and \( R \), respectively.

In order to describe the set \( \bigcap_{\omega}^F \mathcal{K} \), let us formulate first the following

**Lemma 2.2.** Let \( \mathcal{K}_{\omega} \) (\( \omega \in \Omega \)) be \( Q \)-sets. Then the set \( \mathcal{K}_{\omega}^\sim \subseteq L \) defined by

\[
(\sim) \quad L \in \mathcal{K}_{\omega}^\sim \leftrightarrow L \in L \land L \supseteq \bigcap_{1 \leq i \leq n} K_i \land K_i \in \bigcup_{\omega} \mathcal{K}_{\omega}
\]

is an \( F \)-set.

The proof is straightforward and we therefore omit it. Apart from the fact that Lemma may be found useful for constructing new \( F \)-sets, we get also immediately (since, obviously, \( \mathcal{K}_{\omega}^\sim \subseteq \bigcap_{\omega}^F \mathcal{K} \)).

**Theorem 2.3.** Let \( \mathcal{K}_{\omega} \) (\( \omega \in \Omega \)) be \( F \)-sets. Then

\[
\mathcal{K}_{\omega}^\sim = \bigcap_{\omega}^F \mathcal{K}_{\omega}^\sim.
\]

The following theorem establishes a procedure of extending a given \( F \)-set.

**Theorem 2.4.** Let \( \mathcal{K} \) be an \( F \)-set. Then the set \( \mathcal{K}^* \) defined by

\[
(*) \quad L \in \mathcal{K}^* \leftrightarrow L \in L \land \exists K [K \in \mathcal{K}^\sim \land \forall x (x \in K \land L \to K : x \in \mathcal{K})]
\]

contains \( \mathcal{K} \) and is an \( F \)-set, as well. Here, \( \mathcal{K}^* = \mathcal{K}^* \) if and only if \( \mathcal{K} \) is an \( R \)-set.

**Proof.** The inclusion \( \mathcal{K} \subseteq \mathcal{K}^* \) is obvious (take e.g. \( K = L \) in \((*)\)). Also, for \( \mathcal{K} = \emptyset \) evidently \( \mathcal{K}^* = \emptyset \). Thus, assume \( \mathcal{K} \neq \emptyset \).

Let \( L \in \mathcal{K}^* \) and \( q \in R \setminus L \). If \( q \in K \) of \((*)\), then \( L : q \in \mathcal{K} \subseteq \mathcal{K}^* \). If \( q \notin K \), then

\[
\forall x [x \in (K : q) \setminus (L : q) \to (L : q) \land x \in \mathcal{K})
\]

therefore, \( L : q \in \mathcal{K}^* \) again. Hence, \( \mathcal{K}^* \) satisfies \( (Q) \). The other properties \( (E) \) and \( (I) \) can be proved in a similar routine manner.

Now, in [1] an equivalence has been defined on \( Q \) in the following way: Define, for \( \mathcal{K} \in Q \), the "closure" \( \mathcal{c}(\mathcal{K}) \in Q \) by

\[
(c) \quad L \in \mathcal{c}(\mathcal{K}) \leftrightarrow L \in L \land \forall q [q \in R \setminus L \to \exists \sigma (\sigma \in R \land L : \sigma q \in \mathcal{K})]
\]

Then, two \( Q \)-sets \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are said to be equivalent (in symbol, \( \mathcal{K}_1 \approx \mathcal{K}_2 \)) if

\[
\mathcal{c}(\mathcal{K}_1) = \mathcal{c}(\mathcal{K}_2)
\]

The equivalence \( \approx \) induces, of course, an equivalence (denoted again by \( \approx \)) on \( F \) and \( R \).

In order to prove the main result of the paper we shall need the following two lemmas.
Lemma 2.5. (i) For any two Q-sets $\mathcal{X}_1, \mathcal{X}_2$ always

$$c(\mathcal{X}_1) \cap c(\mathcal{X}_2) = c(\mathcal{X}_1 \cap \mathcal{X}_2).$$

Hence, if $\mathcal{X}_1' \approx \mathcal{X}_1$ and $\mathcal{X}_2' \approx \mathcal{X}_2$, then

$$\mathcal{X}_1' \cap \mathcal{X}_2' \approx \mathcal{X}_1 \cap \mathcal{X}_2.$$

In particular, if $\mathcal{X}_1 \approx \mathcal{X}_2$, then

$$\mathcal{X}_1 \cap \mathcal{X}_2 \approx \mathcal{X}_1.$$

(ii) For any Q-sets $\mathcal{X}_\omega(\omega \in \Omega)$ satisfying (E), always

$$\mathcal{X}_\omega \subseteq c(\bigcup_\omega \mathcal{X}_\omega),$$

i.e.

$$\mathcal{X}_\omega' \approx \bigcup_\omega \mathcal{X}_\omega.$$  

Also, if $\mathcal{X}_\omega' \approx \mathcal{X}_\omega (\omega \in \Omega)$, then

$$\bigcup_\omega \mathcal{X}_\omega' \approx \bigcup_\omega \mathcal{X}_\omega,$$

and hence,

$$\mathcal{X}_\omega' \approx \mathcal{X}_\omega.$$  

In particular, if $\mathcal{X}_\omega$ and $\mathcal{X}_\omega'$ are F-sets, then

$$\bigwedge_\omega^{F} \mathcal{X}_\omega' \approx \bigwedge_\omega^{F} \mathcal{X}_\omega;$$

thus, if $\mathcal{X}_\omega \approx \mathcal{X}$ for all $\omega$, then $\bigwedge_\omega^{F} \mathcal{X}_\omega \approx \mathcal{X}$.

(iii) For an F-set $\mathcal{X}$, always $\mathcal{X}^* \approx \mathcal{X}$.

Proof. (i) The equality $c(\mathcal{X}_1) \cap c(\mathcal{X}_2) = c(\mathcal{X}_1 \cap \mathcal{X}_2)$ follows readily from the definition (c). Thus,

$$c(\mathcal{X}_1 \cap \mathcal{X}_2) = c(\mathcal{X}_1') \cap c(\mathcal{X}_2') = c(\mathcal{X}_1') \cap c(\mathcal{X}_2') = c(\mathcal{X}_1 \cap \mathcal{X}_2).$$

(ii) By (\gamma), for $K \in \mathcal{X}_\omega'$ there are $K_i \in \bigcup_\omega \mathcal{X}_\omega$, $1 \leq i \leq n$ such that $K \supseteq \bigcap_{1 \leq i \leq n} K_i$.

Hence, for an arbitrary $q \in R \setminus K$, there is either $K : q \ni K_n : q$, i.e. $K : q \in \bigcup_\omega \mathcal{X}_\omega$, or

$$K : \sigma_n \ni \bigcap_{i \leq i \leq n-1} K_i : \sigma_n \quad \text{for} \quad \sigma_n \in (K_n : q) \setminus (K : q).$$

Proceeding by induction, we can easily find $\sigma$ such that $K : \sigma \in \bigcup_\omega \mathcal{X}_\omega'$; therefore,
\(\mathcal{H}_\varnothing \subseteq c(\bigcup_{\varnothing} \mathcal{H}_\varnothing)\). Since, on the other hand, \(\mathcal{H}_\varnothing = \bigcup_{\varnothing} \mathcal{H}_\varnothing\) we conclude that

\[\mathcal{H}_\varnothing \cong \bigcup_{\varnothing} \mathcal{H}_\varnothing.\]

The rest of (ii) is trivial.

(iii) Also the assertion of (iii) follows again easily from the definition \((*)\) of \(\mathcal{H}^*\).

**Lemma 2.6.** Let \(\mathcal{H}\) be a \(Q\)-set satisfying (R). Then, for any \(L \in c(\mathcal{H}) \setminus \mathcal{H}\), there is a proper (left) ideal \(L_0\) of \(R\) which contains \(L\) and does not belong to \(c(\mathcal{H})\). Thus, in particular, if \(\mathcal{H}\) is an R-set, and \(\mathcal{H}_0\) an F-set satisfying \(\mathcal{H} \subseteq \mathcal{H}_0 \subseteq c(\mathcal{H})\), then \(\mathcal{H} = \mathcal{H}_0\).

**Proof.** Define \(L_0\) as follows:

\[(o) \quad q \in L_0 \iff q \in L \setminus L : q \in \mathcal{H}.\]

Clearly, \(L_0\) is a proper left ideal of \(R\) containing properly \(L\) and, moreover, necessarily \(L_0 \notin c(\mathcal{H})\). For, otherwise, \(L_0 \in c(\mathcal{H})\) implies that there is \(q_0 \in R \setminus L_0\) such that \(L_0 : q_0 \in \mathcal{H}\) and then, for every \(\kappa \in (L_0 : q_0) \setminus (L : q_0)\), i.e. for every \(\kappa\) such that \(\kappa q_0 \in L_0 \setminus L\),

\[L : \kappa q_0 = (L : q_0) : \kappa \in \mathcal{H}\]

in view of \((o)\). Therefore, by (R), \(L : q_0 \in \mathcal{H}\) and thus, by \((o)\) again, \(q_0 \in L_0\) — a contradiction of our choice of \(q_0\).

Now, the main result of this paragraph follows as a consequence of Lemmas 2.5 and 2.6 and Theorems 2.1 and 2.4:

**Theorem 2.7.** For any \(\mathcal{H} \in \mathbb{Q}\), the equivalence class \(C(\mathcal{H})\) of all \(Q\)-sets equivalent to \(\mathcal{H}\) is a convex sublattice of \(\mathbb{Q}\) with infinite joins and the greatest element \(c(\mathcal{H})\).

If \(F_{C(\mathcal{H})} = C(\mathcal{H}) \cap F \neq \emptyset\), then it forms (with respect to order by inclusion) a lattice with meets equal to set-theoretical intersections and with infinite joins; denote the greatest element of \(F_{C(\mathcal{H})}\) by \(\tilde{\mathcal{H}}\).

Since \((\tilde{\mathcal{H}})^* = \mathcal{H}\), \(\tilde{\mathcal{H}}\) is an R-set. This means, in particular, that for any F-set, there exists an equivalent R-set.

As a matter of fact, for an R-set \(\mathcal{H}\), always \(\mathcal{H} = \tilde{\mathcal{H}}\) and hence, \(\tilde{\mathcal{H}}\) is the only R-set belonging to \(F_{C(\mathcal{H})}\).

In this way, a one-to-one correspondence (in fact, a lattice homomorphism) is established between the lattice \(R\) of all R-sets and the lattice of all equivalence classes \(C(\mathcal{H})\) which contain an F-set.

3. In [1], we have proved that \(c(\mathcal{H})\) is an R-set, i.e. that \(\tilde{\mathcal{H}} = c(\mathcal{H})\), provided that \(c(\mathcal{H})\) contains all (left) essential ideals of \(R\). Recall that \(L \in \mathcal{L}\) is said to be essential (in \(R\)) if the zero ideal is the only left ideal intersecting \(L\) trivially. It is therefore quite
natural to raise the question whether, for any $\mathcal{X}$ such that $F_{\mathcal{X}}(af) + 0 > \text{yt} = \sum c(c \mathcal{X})$. The following example will answer the question in negative:

Denote by $\mathcal{S} \subseteq \mathcal{L}$ the set of all strong ideals in $R$, i.e. of all essential ideals $L$ such that

$$\forall \sigma \in R \setminus L \land \sigma = 0 \rightarrow L : \sigma \neq [0] : \sigma.$$ 

It is easy to check that $\mathcal{S}$ is an $F$-set. In fact, $\mathcal{S}$ is an $R$-set. For, assume that $L : \chi \in \mathcal{S}$ for all $\chi \in K \setminus L$ with $K \in \mathcal{S}$ and yet that elements $\varrho_0$ and $\sigma_0 \neq 0$ of $R$ exists such that

$$L : \varrho_0 \subseteq [0] : \sigma_0.$$ 

Then, necessarily $\varrho_0 \notin K$ and $\chi \in K : \varrho_0$ implies either $\chi \in L : \varrho_0$ or $\chi \sigma_0 = 0$. Hence, $K : \varrho_0 \subseteq [0] : \sigma_0$, a contradiction of $K \in \mathcal{S}$.

Consider the ring $R^*$ of all pairs $(n, r)$ of $n \in \mathbb{Z}$ (integers) and $r \in \mathbb{Q}$ (rational numbers) with the component-wise addition and the multiplication defined by

$$(n_1, r_1) (n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1).$$ 

The subset $R^0$ of $R^*$ of all pairs $(0, r)$, $r \in \mathbb{Q}$, is obviously an ideal of $R^*$. The ideals of $R^*$ which are contained in $R^0$ are in one-to-one correspondence $\varphi$ to the subgroups $G$ of the additive group of all rational numbers:

$$\varphi(G) = I_G = \{(0, r)\}_{r \in G}.$$ 

All the remaining ideals of $R^*$ contain $R^0$ and are in one-to-one correspondence $\psi$ to non-zero subgroups $\langle k \rangle$, $k > 0$, $k \in \mathbb{Z}$ of the additive group of integers:

$$\psi(\langle k \rangle) = I_k = \{(n, r)\}_{n \in \langle k \rangle \setminus \{0\}, r \in \mathbb{Q}}.$$ 

Hence, any non-zero ideal is essential in $R^*$. There are only two annihilator ideals, viz. $\{0\}$ and $R^0$. Consequently, $\mathcal{S} = \{I_k\}_{k > 1, k \in \mathbb{Z}}$. Furthermore,

$$c(\mathcal{S}) = \mathcal{L} \setminus \{[0], R^0\} = \mathcal{F} = \mathcal{S}.$$ 

For, if $(0, r) \notin I_G$, then

$$I_G : (0, r) = I_k \in \mathcal{S},$$ 

where $k$ is the least natural member such that $kr \in G$. And, for $(n, r) \in R^*$ with $n \neq 0$, there is $s \in \mathbb{Q}$ such that $ns \notin G$ and

$$(I_G : (n, r)) : (0 : s) = I_G : (0, ns) \in \mathcal{S}$$ 

again. Finally, $\{0\} : (0, r) = R^0$ for every $r \neq 0$, and $R^0 : (n, r) = R^0$ for every $n \neq 0$.

4. In this final paragraph, we are going to establish — in the case of a commutative noetherian ring $R$ — a simple characterization of $R$-sets in terms of prime ideals.
Let, for a moment, \( R \) be an arbitrary ring and
\[
\mathcal{P} = \{ P_\omega \mid \omega \in \Omega \}
\]
the set of all proper two-sided (strictly) prime ideals, i.e. the set of all ideals \( P_\omega \) such that
\[
P_\omega = P_\omega : q \quad \text{for any} \quad q \in R \setminus P.
\]
Let us observe that an intersection \( K = P_1 \cap P_2 \) with \( P_i \neq K \), \( P_i \in \mathcal{P} \) \((i = 1, 2)\) no longer belongs to \( \mathcal{P} \) (cf. next Lemma 4.1 (c)); this follows from the fact that
\[
K : q = P_1 : q = P_1 \quad \text{for any} \quad q \in K \setminus P_1.
\]
We shall call a subset \( \mathcal{L} \) of \( \mathcal{P} \) a filter subset if
\[
P_1 \in \mathcal{L} \land P_2 \in \mathcal{P} \land P_1 \preceq P_2 \rightarrow P_2 \in \mathcal{L}.
\]
Furthermore, for any \( Q \)-set \( \mathcal{K} \) define the \( Q \)-subset \( p(\mathcal{K}) \) by
\[
p(\mathcal{K}) = \mathcal{K} \cap \mathcal{P}.
\]

Lemma 4.1. (a) Let \( \mathcal{K}_1 \approx \mathcal{K}_2 \) be two equivalent \( Q \)-sets. Then \( p(\mathcal{K}_1) = p(\mathcal{K}_2) \).
(b) If \( \mathcal{K} \) satisfies (E), — in particular, if it is an \( F \)-set, then \( p(\mathcal{K}) \) is a filter subset of \( \mathcal{P} \).
(c) If \( \mathcal{L} \) is a filter subset of \( \mathcal{P} \), then the \( F \)-set \( \mathcal{L}^\prec \) (defined in Lemma 2.2) satisfies
\[
p(\mathcal{L}^\prec) = p(\mathcal{L}) = \mathcal{L}.
\]
(d) As a consequence, for any filter subset \( \mathcal{L} \) of \( \mathcal{P} \) there exists an \( R \)-set \( \mathcal{L}' \) (\( = \mathcal{L}^\prec \)) such that
\[
p(\mathcal{L}') = p(\mathcal{L}) = \mathcal{L}.\)

Proof. (a) Let \( K \in p(\mathcal{K}_1) = \mathcal{K}_1 \cap \mathcal{P} \). Then, for a suitable \( q \in R \setminus K \), \( K : q \in \mathcal{K}_2 \).
Also, \( K : q = K \). Hence, \( K \in p(\mathcal{K}_2) \), as required.
(b) Trivial.
(c) Only the proof of \( p(\mathcal{L}^\prec) \subseteq \mathcal{L} \) is needed. Let \( K \in \mathcal{L}^\prec \), i.e.
\[
K \supseteq \bigcap_{1 \leq i \leq n} K_i \quad \text{with} \quad K_i \in \mathcal{L}.
\]
If \( K \supseteq K_i \) for a suitable \( i \), then evidently \( K \in \mathcal{L} \) whenever \( K \in p(\mathcal{L}^\prec) \). Otherwise, there is \( 2 \leq m \leq n \) such that
\[
K \supseteq \bigcap_{1 \leq i \leq m} K_i \quad \text{and} \quad K \notin \bigcap_{1 \leq i \leq m-1} K_i.
\]
And then, for \( q \in K_m \setminus K \),
\[
K : q \supseteq \bigcap_{1 \leq i \leq m} (K_i : q) = \bigcap_{1 \leq i \leq m-1} (K_i : q).
\]
Hence, \( K : q \neq K \), i.e. \( K \notin \mathcal{P} \).

1) Here, with some additional conditions imposed on \( \mathcal{L} \) we can assert that \( \mathcal{L}' \approx \mathcal{L} \); e.g. this is the case when \( \mathcal{L} \) consists of two-sided ideals of \( R \) maximal (as left ideals) in \( R \).
(d) The assertion follows immediately from Theorem 2.7 and (a) of this lemma.

Now, in the remaining part of the paper, let $R$ stand for a commutative noetherian ring. One of the important features of such a ring is that, for any proper ideal $L$ of $R$, there exists $q \in R \setminus L$ such that $L : q$ is prime. Hence, we can formulate the following

**Lemma 4.2.** For any $\mathcal{Q}$-set $\mathcal{K}$, always

$$\mathcal{K} \cong p(\mathcal{K}).$$

Consequently, the equality $p(\mathcal{K}_1) = p(\mathcal{K}_2)$ for two $\mathcal{Q}$-sets $\mathcal{K}_1$ and $\mathcal{K}_2$ implies that $\mathcal{K}_1 \cong \mathcal{K}_2$.

The characterization of the set of all $R$-sets then reads as follows (cf. the case of integers in [1]):

**Theorem 4.3.** Let $R$ be a commutative noetherian ring. Then, for any filter subset $\mathcal{D}$ of $\mathfrak{P}$, there exists a unique $R$-set $\mathcal{I}$ such that

$$p(\mathcal{I}) = \mathcal{D}. \tag{1}$$

In fact, $\mathcal{I} = \mathcal{D}^-$ and thus, $\mathcal{I} \cong \mathcal{D}^-$. Moreover, if

$$\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \mathfrak{P}$$

are two filter subsets, then

$$\mathcal{D}_1 \subseteq \mathcal{D}_2. \tag{2}$$

As a consequence, there is a one-to-one correspondence between all $R$-sets (and thus, all radical filters in $\text{Mod} \ R$) and all filter subsets of prime ideals; more precisely, the lattice of all $R$-sets $\mathcal{R}$ and the complete sublattice $\mathfrak{P}$ of $\text{L}$ (with set-theoretical operations) of all filter subsets of prime ideals are isomorphic. In addition, the set of all minimal $R$-sets (atoms of $R$) corresponds to the set of all singletons $\mathcal{D} = \{P\} \in \mathfrak{P}$, where $P$ is a maximal ideal of $R$. In this case, as well as in the more general case when $\mathcal{D}$ consists of maximal ideals of $R$ only, $\mathcal{I} \cong \mathcal{D}$.

**Proof.** Lemma 4.1 yields the existence and Lemma 4.2 together with Theorem 2.7 the uniqueness of $\mathcal{I}$. Also, if $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then $\mathcal{K} = \mathcal{D}_1 \cap \mathcal{D}_2$ is an $R$-set such that $p(\mathcal{K}) = \mathcal{D}_1$ and, thus, $\mathcal{K} = \mathcal{D}_1^-$, i.e. $\mathcal{D}_1 \subseteq \mathcal{D}_2^-$, as required. The final assertion $\mathcal{I} \cong \mathcal{D}$ follows immediately from Lemma 2.5 (ii), because $\mathcal{D}$ satisfies in this case (E).

**Bibliography**


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