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Czechoslovak Mathematical Journal, Vol. 18 (1968), No. 4, 589–594

Persistent URL: `http://dml.cz/dmlcz/100857`

Czechoslovak Mathematical Journal, 18 (93) 1968, Praha

SOME RESULTS ON CERTAIN SETS OF SERIES

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(Received February 27, 1967)

Introduction. Let $\sum_1^\infty u_n$ be conditionally convergent series. Then it is known that

(i) $$u_n \to 0 \quad \text{as} \quad n \to \infty$$

and (ii) the series of positive terms of $\sum_1^\infty u_n$ and the series of absolute values of negative terms of $\sum_1^\infty u_n$ both diverge to $+\infty$.

Any series $\sum_1^\infty u_n$ which satisfies conditions (i) and (ii) will be said, in what follows, to be of type (A).

The authors of the present note have proved [4] that the results of Riemann's theorem [5] on conditionally convergent series all hold for any series of the type (A).

Let $\sum_1^\infty u_n$, which will sometimes be written as $\sum_1^\infty u(n)$, denote any infinite series. To each rearrangement of the series $\sum_1^\infty u(n)$ there corresponds a complex $(x_1, x_2, x_3, \ldots)$ and conversely, where $x_1, x_2, x_3, \ldots$ is a permutation of the set of all positive integers. The set of all such complexes form a metric space E [1] with the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$.

Let $\sum_1^\infty c(n)$ be any conditionally convergent series and let A denote the proper subset of E consisting of all points of E which correspond to convergent rearrangements of $\sum_1^\infty c(n)$. Further, let $A(\alpha)$ denote the proper subset of A consisting of those points of A which correspond to rearrangements of $\sum_1^\infty c(n)$ converging to the given real number α. Then it has been shown by H. M. SEN GUPTA [7] that the sets A and $A(\alpha)$ are dense boundary subsets of E. It immediately follows that the above result holds for any series of the type (A).

The purpose of the present paper is to extend certain results of Sengupta and to determine the cardinal number of sets of certain types of series.

I

The set $A(\lambda, \mu)$. Let $\sum_1^\infty u_n$ be any series of the type (A) and let λ and μ $(\lambda \geqq \mu)$ be any two preassigned real numbers, finite or infinite. We shall denote by $A(\lambda, \mu)$ the set of all points of E which correspond to those rearrangements of $\sum_1^\infty u_n$ for which (S_n, as usual, denoting the n-th partial sum)

$$\overline{\lim}\, S_n = \lambda \quad \text{and} \quad \underline{\lim}\, S_n = \mu\,.$$

It is obvious that $A(\lambda, \lambda) = A(\lambda)$.

Theorem 1. *The set $A(\lambda, \mu)$ is a dense boundary subset of E.*

Proof. Let $\varepsilon > 0$ be given. We can then choose a positive integer N such that $2^{-N} < \varepsilon$. Let $p \equiv (a_1, a_2, \ldots)$ be any point of E. If $q \equiv (b_1, b_2, \ldots) \neq p$ be any point of the sphere $S(p, 2^{-N-1})$, then $b_i = a_i$ for $i = 1, 2, \ldots, N$. Let M be the greatest positive integer such that $b_i = a_i$ for $i = 1, 2, \ldots, M$. Then, clearly $M \geqq N$.

We now consider the series

$$u(b_{M+2}) + u(b_{M+3}) + \cdots$$

which is clearly of the type (A). Hence there exists a rearrangement of this series, say the series

$$u(c_{M+2}) + u(c_{M+3}) + \cdots$$

for which $\overline{\lim}\, S_n = \lambda - \alpha$ and $\underline{\lim}\, S_n = \mu - \alpha$ where $\alpha = u(a_1) + u(a_2) + \ldots + u(a_M) + u(b_{M+1})$.

It follows that for the series

$$u(a_1) + \ldots + u(a_M) + u(b_{M+1}) + u(c_{M+2}) + u(c_{M+3}) + \ldots,$$
$$\overline{\lim}\, S_n = \lambda \quad \text{and} \quad \underline{\lim}\, S_n = \mu\,.$$

Also the point $(a_1, a_2, \ldots, a_M, b_{M+1}, c_{M+2}, c_{M+3}, \ldots)$ lies in $S(p, \varepsilon)$ and is distinct from p. This shows that the set $A(\lambda, \mu)$ is everywhere dense in E. It is easy to show that $A(\lambda, \mu)$ is a boundary set. This completes the proof of Theorem 1.

Taking $\lambda = \mu = \alpha$ (finite), we get Sengupta's Theorem [7].

It easily follows that each of the sets $A(\infty, \mu)$, $A(\lambda, -\infty)$, $A(\infty, -\infty)$, $A(\infty, \infty)$, $A(-\infty, -\infty)$ is a dense boundary subset of E.

Let $\bar{A}(\lambda)$ denote the set of all points of E which correspond to those rearrangements of $\sum_1^\infty u_n$ for which $\overline{\lim}\, S_n = \lambda$ (whatever be $\underline{\lim}\, S_n$). Then clearly $\bar{A}(\lambda) \supset A(\lambda, \mu)$ and it follows that $\bar{A}(\lambda)$ is also a dense boundary subset of E. Similarly, the set $\underline{A}(\mu)$ consisting of those points of E for which $\underline{\lim}\, S_n = \mu$ is a dense boundary subset of E.

Mapping f.: Let $\sum_1^\infty u_n$ be a series of the type (A) and let B denote the subset of E which consists of those points of E for which $\lambda \equiv \overline{\lim}\, S_n$ is finite. We define a mapping f, of the subspace B onto the real number space E_1, by the following rule

$$f(p) = \overline{\lim}\, S_n \quad \text{for} \quad p \in B\,.$$

The following properties of the mapping f (proofs of which are similar to those of Sengupta's [7] corresponding theorems) may be stated.

1. The function f is everywhere discontinuous over the set B.
2. The map f from B to E_1 is open in the sense that each non-empty subset of B which is open in B is mapped into a non-empty open set in E_1.
3. The map f from B to E_1 is not closed in the sense that there is at least one closed set in B whose image is not closed in E_1.

II

In this section we shall prove certain theorems on the Cardinal number of sets of certain types of series.

Theorem 2. *Let $\sum_1^\infty a_n$ be any series of the type (A) and let λ and μ ($\lambda \geqq \mu$) be any two preassigned numbers. Then the set of all series obtainable from $\sum_1^\infty a_n$ by rearrangement of its terms, for each of which the sequence of partial sums has λ and μ as its upper and lower limits respectively has the power of the continuum.*

Proof. Let $\sum_1^\infty a_n$ be any series of the type (A). Let $p_1, p_2, p_3, \ldots$ denote the non-negative terms and $q_1, q_2, q_3, \ldots$ the negative terms of $\sum_1^\infty a_n$, the terms being taken in the order in which they appear in the original series. It is then possible to decompose

$\sum_1^\infty p_n$ into two subseries $\sum_1^\infty p'_n$ and $\sum_1^\infty p''_n$ and $\sum_1^\infty q_n$ into $\sum_1^\infty q'_n$ and $\sum_1^\infty q''_n$ such that the series $\sum_1^\infty p'_n, \sum_1^\infty p''_n, \sum_1^\infty q'_n, \sum_1^\infty q''_n$ are all properly divergent.

Now, form the series $\sum_1^\infty a'_n$, taking all the terms of $\sum_1^\infty p'_n$ and $\sum_1^\infty q'_n$ (in the order in which they appear in $\sum_1^\infty a_n$). Similarly, we construct the series $\sum_1^\infty a''_n$ with p''_n and q''_n as its terms. From the very manner of construction of the two series $\sum_1^\infty a'_n$ and $\sum_1^\infty a''_n$, it is clear that each is a series of the type (A). Also every term of $\sum_1^\infty a_n$ occurs once and only once in one of $\sum_1^\infty a'_n$ and $\sum_1^\infty a''_n$.

Now, for every real number α, there exist rearrangements $\sum_1^\infty l'_n$ and $\sum_1^\infty l''_n$ of $\sum_1^\infty a'_n$ and $\sum_1^\infty a''_n$ respectively such that

$$\overline{\lim}\, B'_n = \lambda - \alpha\,, \quad \underline{\lim}\, B'_n = \mu - \alpha \quad \text{and} \quad \lim B''_n = \alpha$$

where $B'_n = l'_1 + l'_2 + \ldots + l'_n$, $B''_n = l''_1 + l''_2 + \ldots + l''_n$.

Now, consider the series

$$l'_1 + l''_1 + l'_2 + l''_2 + \ldots$$

which is a rearrangement of the original series $\sum_1^\infty a_n$. If ϱ_n denotes the n-th partial sum of this series, it easily follows that

$$\overline{\lim}\, \varrho_n = \lambda \quad \text{and} \quad \underline{\lim}\, \varrho_n = \mu\,.$$

This shows that the power of our set is $\geqq c$, the power of the continuum. But it is known that the power of the set E is c [6]. This proves the theorem.

Taking $\lambda = \mu = \alpha$ we get the following theorem.

Theorem 3. *Let $\sum_1^\infty a_n$ be any series of the type (A) and let α be any preassigned number. Then the set of all series, obtainable from $\sum_1^\infty a_n$ by rearrangement of its terms, which converge to α has the power of the continuum.*

It is easy to see that Theorem 2 remains valid in each of the following cases also.

I. $\lambda = +\infty$, μ is finite,

II. λ is finite, $\mu = -\infty$,

III. $\lambda = +\infty$, $\mu = -\infty$,

IV. $\lambda = \mu = \pm\infty$.

Theorem 4. *Let* $\sum_1^\infty a_n$ *be any series of the type* (A) *and let* α *be any number given in advance. Then the set of all subseries of* $\sum_1^\infty a_n$ *which converges to* α *has the power of the continuum.*

Proof. Let us suppose that $\alpha > 0$. Let x denote any number in $0 < t \leqq 1$. Then, using the same notation as in Theorem 2, there exists [3] a subseries $\sum_1^\infty p_{n_K}$ of $\sum_1^\infty p_n$ and a subesries $\sum_1^\infty q_{m_K}$ of $\sum_1^\infty q_n$ such that

$$\sum_1^\infty p_{n_K} = \alpha + x \quad \text{and} \quad \sum_1^\infty q_{m_K} = -x .$$

The series

$$p_{n_1} + q_{m_1} + p_{n_2} + q_{m_2} + \dots$$

clearly converges absolutely to α and by means of a suitable rearrangement of this series we obtain a subseries of $\sum_1^\infty a_n$, the sum of which is α. It follows that the set of all such series has power $\geqq c$, but it is easy to see that the set of all subseries of a given series has the power c. Similar proof applies to the case when $\alpha \leqq 0$. The theorem is thus proved.

Theorem 5. *Let* $\sum_1^\infty a_n$ *be a divergent series of positive terms where* $a_n \to 0$ *as* $n \to \infty$ *and let* s *be any positive number chosen in advance. Then the set of all subseries of* $\sum_1^\infty a_n$ *which converges to* s *has the power* c.

Proof. We decompose $\sum_1^\infty a_n$ into two subseries $\sum_1^\infty b_n$ and $\sum_1^\infty c_n$ such that the series $\sum_1^\infty b_n$ and $\sum_1^\infty c_n$ are both divergent.

Let x denote any number in $0 < t < \frac{1}{2}s$. Then it is possible to find [3] subseries $\sum_1^\infty b_{n_K}$ and $\sum_1^\infty c_{m_K}$ of $\sum_1^\infty b_n$ and $\sum_1^\infty c_n$ respectively such that

$$\sum_1^\infty b_{n_K} = \tfrac{1}{2}s + x \quad \text{and} \quad \sum_1^\infty c_{m_K} = \tfrac{1}{2}s - x .$$

The series of positive terms

$$b_{n_1} + c_{m_1} + b_{n_2} + c_{m_2} + \dots$$

converges to s and by means of a suitable rearrangement of this series we obtain a subseries of $\sum_1^\infty a_n$, the sum of which is s. It follows that the required power is c.

It easily follows that the set of all subseries of a divergent series $\sum_1^\infty a_n$ of positive terms (where $a_n \to 0$) which themselves are divergent has the power c. It also follows that the set of all divergent subseries $\sum_1^\infty a_{n_K}$ of $\sum_1^\infty a_n$, for which $n_{K+1} - n_K \to \infty$ as $K \to \infty$ [2] has the power c.

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